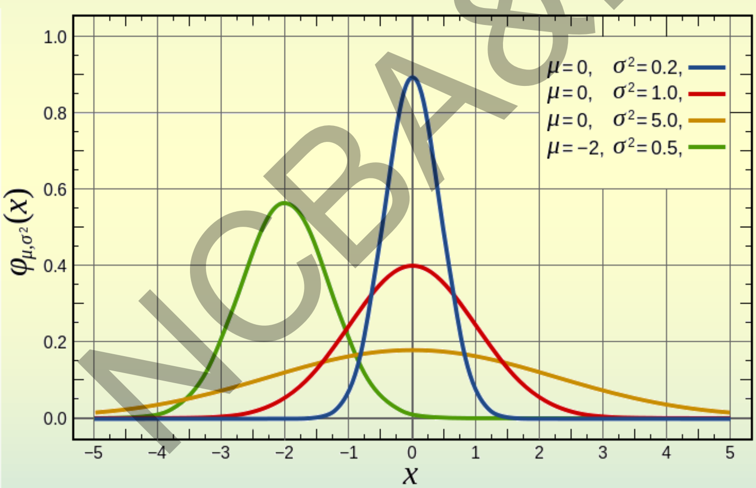
CONTRIBUTION TO STATISTICS

ISBN 978-969-8858-21-6

MUNIR AHMAD



ISOSS Publications
44-A, Civic Centre, Sabzazar
Multan Road, Lahore (Pakistan)
URL: http://www.isoss.net

CONTRIBUTION TO STATISTICS

Dr. Munir Ahmad

ISBN 978-969-8858-21-6

Published by: ISOSS Publications, Lahore, Pakistan.
Printed in Pakistan

2017

CONTRIBUTION TO STATISTICS

Prof. Dr. Munir Ahmad
Professor (Rtd.) Statistics
Rector, National College of Business Administration & Economics
Patron, Islamic Countries Society of Statistical Sciences
Chief Editor, Pakistan Journal of Statistics

ISOSS Publication
44-A, Civic Centre, Sabzazar
Multan Road, Lahore (Pakistan)

Dedicated to my

Wife Furrukh Sultan,

Son Dr. Aftab Ahmad,

Daughter Dr. Alia Ahmed

&

Grand Children.

PREFACE

As discussed at different fora, I consider that Pakistan is almost a statistically advanced country as it meets most of the well-defined criteria of advancement in statistics. The only deficiencies we have; are delayed holding of population censuses, not many statistics journals, not many statistical societies, and lack of research activities in government establishments. The main objective of this publication is to show that Pakistan is trying to fill in these gaps.

This book is an indication that academia has recently been very active and many academic professionals are publishing research papers and is having a good deal of share in the statistical literature otherwise unknown to the world.

Contribution to Statistics contains publication, in theoretical and applied statistics. These publications are papers published in Pakistan Journal of Statistics (PJS) from 1985 till 2016.

There are two main reasons for writhing this look. The first is my opinion that students in particular and researchers in general will have a familiarity with the recent areas of research. The second is my maxim and precept that students and researchers would know the specific areas of research being conducted in Pakistan.

This book is a blend of my work and should be studied properly. I hope this work will trigger further theoretical research and offer handy tools that may generate further fruitful research. The work has shortages of applications and students and researchers may pick up more new areas of applications.

I am indebted to Mr. Muhammad Iftikhar, Mr. Muhammad Imtiaz and Mr. Ahsan Qureshi for compiling and typesetting of the material as many papers have to be retyped. Almost all retyping has been done by Mr. M. Imtiaz for which I am grateful to him.

Dr. Munir Ahmad
April, 2017

CONTENTS

Chapter-4: DISCRETE DISTRIBUTION

Chapter-5: SAMPLING

Chapter-6: QUALITY CONTROL AND ACCEPTANCE SAMPLING

ON THE MOMENTS OF BERNSTEIN RELIABILITY MODELS*

Munir Ahmad, M.H. Kazi and **A.K. Sheikh**
University of Petroleum & Minerals
Dhahran, Saudi Arabia

ABSTRACT

Exact expressions are obtained for the moments of the positive integral powers of a normal random variable in terms of Hermite polynomials. The saddle point method has been employed to obtain the asymptotic expressions for moments of the positive integral powers of the reciprocals of the normal variates with mean μ and variance σ^2. The method is used to derive the asymptotic expressions for the higher moments of the maximum likelihood estimator of the reciprocal of mean μ and compared with the approximate moments obtained by Srivastava and Bhatnagar[1.] and Zellner[2].

KEYWORDS

Consistent estimator, diffuse prior, efficiency, Hermite polynomials, moments of inverse of mean, relative bias, saddle points and steepest descent method.

I. INTRODUCTION

The problems of estimation of the reciprocals often arise in many situations, for instance, in econometrics, biological sciences and engineering sciences [1,2,3.]. Moments of the powers of reciprocals have not been investigated in the literature, though expectation and variance of the reciprocals have been approximated by Srivastava and Bhatnagar[1] and Zellner[2]. The moments of the reciprocals of normal random variable do not exist[4]. However, Srivastava and Bhatnagar[1] have given some estimators which possess finite moments. In this paper, we obtain exact expressions for the moments of the positive integral powers of a normal random variable in terms of Hermite polynomials and use saddle point method to obtain asymptotic expressions for moments of the positive integral powers of the reciprocals of the normal variates. The saddle point method is also used to derive asymptotic expressions for higher moments of the maximum likelihood estimator of the reciprocal of mean and the results are compared with those obtained by Srivastava and Bhatnagar[1] and Zellner[2].

2. EXACT MOMENTS OF THE POSITIVE INTEGRAL POWERS OF A NORMAL RANDOM VARIABLE

Gusev and Roshchin[5] have obtained the following expressions for the exact moments of order r for positive odd integral powers of a normal random variable X in terms of Hermite polynomials

* Published in Pak. J. Statist. (1987), Vol. 3(1).

$$\mu_r' = \left(\frac{\sigma}{\sqrt{2i}}\right)^{mr} H_{mr}\left(\frac{i\mu}{\sqrt{2\sigma}}\right) \tag{2.1}$$

where μ and σ^2 are the mean and variance of the normal random variable, m is a positive odd integer, $i = \sqrt{-1}$, and $H_k(.)$ is a Hermite polynomial of degree k. These moments are required to evaluate the life and the scatter of lives of machine components with random loading. These evaluations are used in solving important practical problems such as the number of spare parts needed in any particular time, and the optimal machine overhaul periods.

It will be of some interest to find the expression for moments of even powers of normal random variable in terms of Hermite polynomials.

Consider $m = 2n(n > 0)$. The probability density function of $Y = X^{2n}$ is $-\frac{2n-1}{2n}$

$$g(y) = \frac{1}{\sqrt{2\pi}} \frac{y}{n\sigma} \exp\left[-\frac{1}{2}\left(\frac{y^{\frac{1}{2n}} - \mu}{\sigma}\right)^2\right], y > 0 \tag{2.2}$$

The rth moments about origin is

$$\mu_r' = \frac{1}{\sqrt{2\pi}n\sigma} \int_0^\infty y^{r-\frac{2n-1}{2n}} \exp\left[-\frac{1}{2}\left(\frac{y^{\frac{1}{2n}} - \mu}{\sigma}\right)^2\right] dy.$$

Let

$$\frac{y^{\frac{1}{2n}} - \mu}{\sigma} = t, \text{ then } y = (\mu + \sigma t)^{2n} \text{ and } dy = 2n\sigma(\mu + \sigma t)^{2n-1} dt,$$

whence

$$\mu_r' = \frac{2}{\sqrt{2\pi}} \int_{-\mu/\sigma}^{\infty} (\mu + \sigma t)^{2nr} e^{-\frac{1}{2}t^2} dt \tag{2.3}$$

Introducing $i = \sqrt{-1}$ in (2.3), we have

$$\begin{aligned}
\mu_r' &= \sqrt{\frac{2}{\pi}}\left(\frac{\sigma}{i}\right)^{2nr} \int_{-\mu/\sigma}^{\infty} \left(it + \frac{i\mu}{\sigma}\right)^{2nr} e^{-\frac{1}{2}t^2} dt \\
&= 2\sqrt{\frac{2}{2\pi}}\left(\frac{\sigma}{i}\right)^{2nr} \left[\int_{-\infty}^{\infty} \left(it + \frac{i\mu}{\sigma}\right)^{2nr} e^{-\frac{1}{2}t^2} dt - \int_{-\infty}^{-\mu/\sigma} \left(it + \frac{i\mu}{\sigma}\right)^{2nr} e^{-\frac{1}{2}t^2} dt\right]
\end{aligned}$$

$$= 2\left(\frac{\sigma}{i}\right)^{2nr}\left[H_{2nr}\left(\frac{i\mu}{\sigma}\right) - \frac{2}{\sqrt{2\pi}}\sum_{j=0}^{2nr}\binom{2nr}{j}\left(\frac{i\mu}{\sigma}\right)^{2nr-j}\int_{-\infty}^{-\mu/\sigma}(it)^j e^{-\frac{1}{2}t^2}dt\right]$$

$$= 2\left(\frac{\sigma}{i}\right)^{2nr}\left[H_{2nr}\left(\frac{i\mu}{\sigma}\right) - \frac{1}{\sqrt{2\pi}}\sum_{j=0}^{2nr}\binom{2nr}{j}\left(\frac{\mu}{\sigma}\right)^{2nr-j} i^{2nr}\int_{\mu/\sigma}^{\infty} t^j e^{-\frac{1}{2}t^2}dt\right]$$

After integration and simplification, we obtain

$$\mu_r' = 2\left(\frac{\sigma}{i}\right)^{2nr} H_{2nr}\left(\frac{i\mu}{\sigma}\right) - \sqrt{\frac{2}{\pi}}\sum_{j=o}^{2nr}\binom{2nr}{j}2^{\frac{j-1}{2}}(-\sigma)^j(\mu)^{2nr-j}\Gamma_a\left(\frac{j+1}{2}\right) \quad (2.4)$$

where

$$a = \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2, \Gamma_a(b) = \int_a^{\infty} x^{b-1}e^{-x}dx$$

and

$$H_k(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(it+x)^k e^{-\frac{1}{2}t^2}dt$$

We note that if X is normal with mean μ and variance σ^2, then X^2/σ^2 is a non-central chi-square distribution with one degree of freedom and μ^2/σ^2 is non-centrality parameter. The equation (2.4) also gives *r*th moment of the *n*th power of non-central chi-square random variate.

3. ASYMPTOTIC EXPRESSIONS FOR MOMENTS OF THE POWERS OF RECIPROCALS

Consider a random variable $Y = X^{-m}$ where X is $N(\mu, \sigma^2)$ and m is any positive integer. The probability density function of Y and its properties are discussed by Gusev and Roshchin[5]. The *r*th moment about origin of the random variable Y is

$$\mu_r' = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}u(x)dx \quad (3.1)$$

where

$$\mu(x) = x^{-mr}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad (3.2)$$

The function $u(x)$ appears to have a singularity at $x = 0$ and the integral (3.1) is divergent as such, but we can find asymptotic expression for the integral for small values of σ using the steepest descent method which enables us to pick up the dominant contribution to the integral from the neighborhood of the saddle point. For the details of the saddle point method with applications to statistics, reference may be made to Daniel[6].

We confine ourselves to describe only the salient features of the method.

Consider the integral

$$1 = \int_c g(z) e^{ph(z)} dz. \tag{3.3}$$

where c is the path of integration in the complex z - plane along the real axis and the functions $g(z)$ and $h(z)$ are functions of the complex variables z not necessarily analytic, which as a special case may involve only real values of z. In order to evaluate the integral asymptotically for large values of ρ, the path of integration is deformed to satisfy the following conditions:

i) the path passes through a zero z_0 (called saddle point) of $h'(z)$.

ii) the imaginary part of $h(z)$ is constant on the path.

If we write $h(z) = h_1 + ih_2$ where h_1 and h_2 are real functions, h_2 is constant on a path of steepest descent, then the dominant part of the asymptotic expansion arises from the part of the path near the highest saddle-point. If the path c is deformed to pass through the saddle point, then the integral will be obtained in the neighborhood of the saddle point. The saddle point is obtained by solving $\frac{dh}{dz} = 0$ and the path of integration (3.3) will be the locus of the points determined by the equation

$$h(z) = h(z_0) - s^2, \quad -\infty < s < \infty. \tag{3.4}$$

The saddle point corresponds to the value $s = 0$. The integral (3.3) taken over c is now replaced by the integral of the same integrand over the new path of integration given by the equation (3.4) which transforms z to s given by $\phi(s) \equiv g(z)\frac{dz}{ds}$ and the dominant contribution to the integral now stems from the vicinity of the saddle point.

The integral (3.3) is written as

$$I = \int_{-\infty}^{\infty} e^{\rho\left[h(z_0) - s^2\right]} \phi(s)\, ds$$

$$= e^{\rho h(z_0)} \int_{-\infty}^{\infty} e^{-\rho s^2} \phi(s)\, ds \tag{3.5}$$

For large values of ρ, only small values of s will contribute significantly to the integral. Expanding $\phi(s)$ in a series of powers of s, substituting in (3.5), and integrating over s and using the formula

$$\int_{-\infty}^{\infty} s^m e^{-\rho s^2} ds = \begin{cases} 0, & \text{When } m \text{ is odd} \\ \sqrt{2\pi}\,\dfrac{m!\left(\sqrt{2\rho}\right)^{-m-1}}{2^{m/2}\left(m/2\right)!}, & \text{When } m \text{ is even} \end{cases}$$

We obtain the following asymptotic expansion of the integral for large values of ρ :

$$I = \exp\left[\rho h(z_0)\right]\left(\pi/\rho\right)^{\frac{1}{2}}\left[\phi(0) + \frac{1}{4\rho}\phi^{(2)}(0) + ...\right] \tag{3.6}$$

where

$$\phi^{(k)}(0) = \frac{d^k}{ds^k}\left[\phi(s)\right]_{s=0}, \quad k = 0,1,2,.....$$

In case of the integral (3.1), we have

$$g(z) = \frac{z^{-mr}}{\sqrt{2\pi}\sigma},$$

$$h(z) = -\frac{1}{2}(z-\mu)^2,$$

and

$$\rho = \frac{1}{\sigma^2}.$$

For small values of σ, ρ is large. The saddle point is $z_0 = \mu$ and also $h(z_0) = 0$ The path of integration is (3.4) is given by

$$z = (\mu + \sqrt{2}s),$$

and the function $\phi(s)$ is given by

$$\phi(s) = \frac{1}{\sqrt{\pi}\sigma}\left(\mu + \sqrt{2}s\right)^{-mr}$$

Using the saddle point method and substituting these values in (3.6) we have

$$\mu_r' = \mu^{-mr}\left[1 + \sum_{j=1}^{\infty}\frac{(mr)_{2j}}{2^j\,(j!)}\left(\frac{\sigma}{\mu}\right)^{2j}\right] \tag{3.7}$$

where

$$(a)_k = a(a+1)....(a+k-1).$$

4. ESTIMATION OF THE INVERSE OF MEAN

As an illustration of the usage of the method we consider the estimation of the inverse of mean and compare our results with those obtained by Srivastava and Bhatnagar[1] who consider a similar problem.

The maximum likelihood estimate of $1/\mu$ is $1/\bar{x}$ which does not possess finite moments. Srivastava and Bhatnagar[1], Zellner[2] and others have recently discussed the estimation of $1/\mu$. Srivastava and Bhatnagar[1] considered the estimator

$$t_k = n\bar{x}\Big/\left(n\bar{x}^2 + ks^2\right) \quad \text{for } k > 0 \tag{4.1}$$

where $\bar{x}$ and s^2 are unbiased estimators of population mean μ and variance σ^2 respectively of a normal population. They obtained $E\left(t_k\right)$ and $E\left(t_k^2\right)$. The moments of t_k, exist for $k > 0$ and for small values of k or large values of n. t_k is an approximate estimate of $1/\mu$. Zellner[2] obtained a minimum expected loss (MELO) estimation of $1/\mu$ using the relative squared loss function and the observation model $y_i = \mu + \mu_i, i = 1, 2, ...n$ where y_i is the *i*th observation, μ is the common mean of the observations and μ_i is the *i*th disturbance or error term. The MELO estimate for $1/\mu$ is given by

$$\frac{1}{\bar{y}}\left[1 + \frac{vs^2}{n(v-2)\bar{y}^2}\right]^{-1}, \qquad v = n - 1 > 2 \tag{4.2}$$

which is identical to the S-B estimator (4.1) when $k = \dfrac{v}{v-2}$ Zellner[2] showed that the MELO estimator has finite moments but has not found their explicit expressions. Srivastava and Bhatnagar[1] found the first two moments of their estimators. If $k = v/(v-2)$, we have the first two moments for the Zellner[2] MELO estimator of $1/\mu$. Following Srivastava and Bhatnagar[1] notations, we find explicit expressions for the *r*th moment of S-B and Zellner estimators, when (i) $r = 2m$ and (ii) $r = 2m - 1$

$$\mu'_{2m} = \frac{1}{\mu^{2m}} \frac{\left(n/2\theta\right)^m e^{-\frac{n}{2\theta}}}{\Gamma\left(\frac{n-1}{2}\right)} \sum_{\alpha=0}^{\infty} \sum_{j=0}^{\infty} \frac{\binom{2m+\alpha-1}{\alpha}\left(1 - \frac{k}{n-1}\right)^{\alpha} \Gamma(a)\beta(b,c)}{\Gamma\left(j + \frac{1}{2}\right) j!} \left(\frac{n}{2\theta}\right)^j$$

where

$$a = j - 2m + \frac{n}{2},\ b = j + 2m + \frac{3}{2} \text{ and } c = \alpha + \frac{n}{2} - \frac{1}{2} \tag{4.3}$$

$$\mu'_{2m+1} = \frac{1}{\mu^{2m+1}} \frac{\left(n/2\theta\right)^m e^{-n/2\theta}}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$\sum_{\alpha=0}^{\infty} \sum_{j=0}^{\infty} \binom{2m+\alpha-2}{\alpha}\left(1-\frac{k}{n-1}\right)^{\alpha} \frac{\Gamma\left(a-1\right)\beta\left(b-2,c\right)}{\Gamma\left(j+\frac{3}{2}\right)j!}\left(\frac{n}{2\theta}\right)^{j} \quad (4.4)$$

If $m=1$, we obtain S- B expressions for $E\left(t_k\right)$ and $E\left(t_k^2\right)$ and if $k=v/\left(v-2\right)$, we have the first two moments of the Zeillner MELO estimator of $1/\mu$.

The asymptotic expressions for the first two moments of the S- B estimator for a normal population are

$$E\left(t_k\right) \simeq \frac{1}{\mu}\left[1+\left(1-k\right)\frac{\theta}{n}+\left(k^2-6k+3\right)\frac{\theta^2}{n^2}\right] \quad (4.5)$$

$$Var\left(t_k\right) \simeq \frac{1}{\mu^2}\left[\frac{\theta}{n}+\left(k^2-8k+9\right)\frac{\theta^2}{n^2}\right] \quad (4\ 6)$$

It is possible to find asymptotic expressions for the moments of the MLE estimator of $1/\mu$ for a normal population when σ^2 is a known quantity. Similar results can be obtained when the variance σ^2 is unknown. If σ^2 is unknown then σ^2 is replaced by its unbiased estimator s^2. The rth moment of the random variable $\left(\bar{X}\right)^{-1}$ can be evaluated asymptotically for large n using the formula (3.6).

Here

$$g\left(\bar{x}\right) = \frac{\left(\bar{x}\right)}{\sqrt{2\pi\sigma^2}},$$

$$h\left(\bar{x}\right) = -\frac{1}{2\sigma^2}\left(\bar{x}-\mu\right)^2,$$

and

$$\rho = n .$$

The saddle point is $\bar{x}_0 = \mu$ and also $h\left(\bar{x}_0\right) = 0$, The transformation from $\bar{x}$ to s is given by

$$\bar{x} = \left(\mu + \sqrt{2}s\right)$$

and the function $\phi\left(s\right)$ is given by

$$\phi(s) = \frac{1}{\sqrt{\pi\sigma}}\left(\mu + \sqrt{2}s\right)^{-r}$$

Now we have for large n

$$\mu'_r = \mu^{-r}\left[1 + \sum_{j=1}^{\infty} \frac{(r)_{2j}}{2^j n^j}\left(\frac{\sigma}{\mu}\right)^{2j}\right] \qquad (4.7)$$

where $(a)_k = a(a+1)......(a+k-1)$. The rth moment about origin is not finite unless n is large. Using the first two terms of (4.7), we have

$$\mu'_r \sim \mu^{-r}\left[1 + \frac{r(r+1)}{2n}\left(\frac{\sigma}{\mu}\right)^2 + \frac{r(r+1)(r+2)(r+3)}{8n^2}\left(\frac{\sigma}{\mu}\right)^4\right] \qquad (4.8)$$

If $r = 1$ and 2, the results are identical to the S-B estimator when $k = 0$.

If μ and σ^2 are unknown, σ/μ can be replaced either by their unbiased estimators or by the consistent estimator of σ/μ which is the coefficient of variation of the observations.

REFERENCES

1. Sarivastava, V.K. and Bhatnagar, S. (1981). Estimation of the Inverse of Mean. *J. Statist. Planning and Inference*, 5, 329-334.
2. Zellner, A. (1978). Estimation of Functions of Population Means and Regression Coefficients Including Structural Coefficients: A Minimum Loss (MELO) Approach. *J. Econometrica*, 8, 127-158.
3. Box, G.E.P. and Tio, G.C. (1973). *Bayesian Inference in Statistical Analysis*, Addison Wesley Publ. Co. Inc., p. 87.
4. Feller, W. (1971). *An Introduction to Probability Theory and its Applications*, Vol. 2, John Wiley & Sons, N.Y. New York.
5. Gusev, A.S. and Roshchin, I.M. (1975). Calculation of the Scatter of Fatigue Lives of Machine Components with Random Loading. *Vestnik Mashinostroeniya*, 55(10), 3-6. (U.D.C. 620.178.3)
6. Daniel, H.E. (1954). Saddle Point Approximations in Statistics. *Ann. Mathematical Statist.*, 25, 631-650.

A NOTE ON THE ESTIMATION OF VARIANCE OF A NORMAL POPULATION*

Munir Ahmad
Department of Mathematical Sciences, University of Petroleum and Minerals
Dhahran, Saudi Arabia

ABSTRACT

In this paper we obtain analytical expressions for the exact moments of a larger class of estimators of variance of a normal population than that of Pandey and Singh (1981) and derive formulae for their efficiency and relative bias.

KEYWORDS

Consistent estimator, efficiency, mean square error and relative bias.

1. INTRODUCTION

Pandey and Singh (1981) proposed the following two estimators of variance of a normal population with unknown mean μ and unknown variance σ^2 :

$$\hat{\hat{\sigma}}^2 = \frac{n\bar{x}^2 s^2}{n\bar{x}^2 + s^2} \text{ and } \sigma^{*2} = k\hat{\hat{\sigma}}^2 \tag{1}$$

where $k = \left[\frac{2s^4}{n(n-1)\bar{x}^4} + \frac{4s^4}{n\bar{x}^2} + 1\right]^{-1}$, $\bar{x}$ and s^2 are unbiased estimators of population mean μ and variance σ^2, respectively. Pandey and Singh (1981) have developed these estimators from the estimators of μ^2 suggested by Govindarajuln and Sahai (1972) and Das (1975) and have shown for large n that the estimators in (1) are more efficient than s^2. They have studied the efficiency and relative bias of these estimators numerically.

In this paper, we generalize the Pandey and Singh (1981) estimator and obtain analytical expressions of exact moments of these estimators and find their relative bias and efficiency.

2. EXACT EXPRESSIONS FOR. MOMENTS

Consider the following class of estimators characterized by a scalar t

$$\hat{\hat{\sigma}}_t^2 = \frac{n\bar{x}^2 s^2}{n\bar{x}^2 + ts^2}(t > 0), \quad \sigma_t^{*2} = k\hat{\hat{\sigma}}_t^2 \tag{2}$$

* Published in Pak. J. Statist. (1993), Vol. 9(2).

We note that for $t=1$, we obtain the Pandey and Singh (1981) estimators in (1) which have shown to be more efficient than usual estimator of s^2

The, mth moment about origin of the estimator in (2) is, by definition

$$E\left(\hat{\sigma}_t^2\right)^m = \int_{-\infty}^{\infty}\int_0^{\infty}\left(\frac{n\bar{x}^2 s^2}{n\bar{x}^2+ts^2}\right)^m f\left(\bar{x},s^2\right)ds^2 d\bar{x} \tag{3}$$

where

$$f\left(\bar{x},s^2\right)=\sqrt{\frac{n}{2\pi}}\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}\left[\sigma^3\Gamma\left(\frac{n-1}{2}\right)\right]^{-1}\left(s^2/\sigma^2\right)^{\frac{n-3}{2}}$$
$$\times\exp\left[-\frac{1}{2\sigma^2}\left\{n\left(\bar{x}-\mu\right)^2+\left(\left(n-1\right)s^2\right)\right\}\right]$$

Pandey and Singh (1981) evaluated the integral (3) numerically for $t=1$ and computed the relative bias and efficiency by generating 1200 random samples of size 5 from $N(10,\ 4)$, $N(10, 64)$ and $N(10,\ 100)$ by employing the 9-point Gauss-Laguerre quadrature formula on the inner integral and the 9-point Gauss-Hermite quadrature formula on the outer integral.

In this paper, we obtain analytical expressions for exact moments of these estimators and evaluate their relative bias and efficiency.

Following Srivastava and 13 Bhatnagar (1981), we write $Z=\frac{\sqrt{n}\bar{X}}{\sigma}, \theta=\sigma^2/\mu^2$ and $V=\left(n-1\right)s^2/\sigma^2$. The random variable Z follows a normal distribution with mean $\sqrt{\frac{n}{\theta}}$ and variance 1, while the random variable V follows a chi-square distribution with $n-1$ degrees of freedom. The random variables Z and V are independent.
The integral in (3) can be written as

$$E\left(\hat{\sigma}_t^2\right)^m = c\int_{-\infty}^{\infty}\int_0^{\infty}\left(\frac{z^2 v}{z^2+\frac{t}{n-1}v}\right)^m v^{\frac{n-3}{2}} e^{-\frac{1}{2}\left[\left(z-\sqrt{n/\theta}\right)^2+v\right]}dv dz \tag{4}$$

where

$$c=\left[\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)2^{n/2}\right]^{-1}\left(\frac{\sigma^2}{n-1}\right)^m.$$

We now rewrite $\left(\hat{\sigma}_t^2\right)^m$ as

$$\left(\frac{z^2 v}{z^2+v}\right)^m \left[1-\left(1-\frac{t}{n-1}\right)\frac{v}{z^2+v}\right]^{-m}$$

$$=\left(\frac{z^2 v}{z^2+v}\right)^m \sum_{\alpha=0}^{\infty}\binom{m+\alpha+1}{\alpha}\left(1-\frac{t}{n-1}\right)^{\alpha}\frac{v^{\alpha}}{\left(z^2+v\right)^{\alpha}}$$

and

$$e^{-\frac{1}{2}\left[\left(z-\sqrt{\frac{n}{\theta}}\right)^2+\nu\right]} \text{ as } e^{-\frac{1}{2}\left(z^2+v+\left(n/\theta\right)\right)}\sum_{j=0}^{\infty}\frac{\left(\sqrt{n/\theta}\right)^2}{h!}z^j .$$

Substituting these values in (4), we get

$$E\left(\hat{\sigma}_t^2\right)^m = ce^{-n/2\theta}\sum_{\alpha=0}^{\infty}\binom{m+\alpha-1}{\alpha}\left(1-\frac{t}{n-1}\right)^{\alpha}\sum_{j=0}^{\infty}\frac{\left(n/\theta\right)^{j/2}}{j!}$$

$$\times\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{z^{2m+j}v^{\alpha+m+\frac{n-3}{2}}}{\left(z^2+v\right)^{m+\alpha}}e^{-\frac{1}{2}\left(z^2+v\right)}dvdz \qquad (5)$$

For odd integral values of j the integrand in (5) vanishes and the equation (5) reduces to

$$E\left(\hat{\sigma}_t^2\right)^m = ce^{-n/2\theta}\sum_{\alpha=0}^{\infty}\sum_{j=0}^{\infty}\binom{m+\alpha-1}{\alpha}\left(1-\frac{t}{n-1}\right)^{\alpha}\frac{\left(n/\theta\right)^j}{(2j)!}$$

$$\times\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{z^{2m+2j}\nu^{\alpha+m+\frac{n-3}{2}}}{\left(z^2+\nu\right)^{m+\alpha}}e^{-\frac{1}{2}\left(z^2+\nu\right)}d\nu dz$$

Applying the transformation

$$z^2 = y_1 y_2 \qquad 0 \le y_1 \le \infty$$

and

$$v = y_1\left(1-y_2\right) \qquad 0 \le y_2 < 1$$

and the duplication formula $\sqrt{\pi}\left(2j\right)! = 2^{2j}\, j!\,\Gamma\left(j+\frac{1}{2}\right)$

We get after some effort

$$E\left(\hat{\sigma}_t^{*2}\right)^m = \left(\frac{2\sigma^2}{n-1}\right)^m \frac{e^{-n/2\theta}}{\Gamma\left(\frac{n-1}{2}\right)} \sum_{\alpha=0}^{\infty} \sum_{j=0}^{\infty} \binom{m+\alpha-1}{\alpha}\left(1-\frac{t}{n-1}\right)^{\alpha} \times \frac{\left(n/2\theta\right)^j \Gamma\left(j+m+\frac{n}{2}\right)\beta\left(m+j+\frac{1}{2}, \alpha+m+\frac{n-1}{2}\right)}{j!\,\Gamma\left(j+\frac{1}{2}\right)} \tag{6}$$

where $\beta(a,b)$ and $\Gamma(c)$ are the Beta and Gamma functions respectively.

A similar procedure is followed for $\hat{\sigma}_t^{*2}$ and an exact expression for the *mth* moment is obtained by writing *k* as

$$k = \frac{\sigma^2}{n-1}\frac{z^2 v}{\left(z^2+v\right)^3}\left[\left\{1-\left(1-\frac{t}{n-1}\right)\frac{v^2}{z^2+v}\right\}\left\{1-2\left(1-\frac{2}{n-1}\right)\frac{z^2v^2}{\left(z^2+v\right)^2}\right\} \times \left(1-\left(1-\frac{t}{n-1}\right)\frac{v}{z^2+v}\right)-\left(1-\frac{2}{\left(n-1\right)^3}\right)\frac{v^2}{\left(z^2+v\right)^2}\right]^{-1}$$

$$E\left(\hat{\sigma}_t^{*2}\right)^m = \left(\frac{\sigma^2}{2\left(n-1\right)}\right)^m \frac{e^{-n/2\theta}}{\Gamma\left(\frac{n-1}{2}\right)} \sum_{\alpha,\beta,\gamma,j=0}^{\infty} \frac{\left(n/2\theta\right)^{2j}}{j!}\binom{m+\alpha-1}{\alpha}\binom{m+\beta-1}{\beta} \times \binom{m+\beta+\gamma-1}{\gamma}\left(1-\frac{t}{n-1}\right)^{\alpha}\left(1-\frac{2}{\left(n-1\right)^3}\right)^{\beta}\left(\frac{2\left(n-3\right)}{n-1}\right)^{\gamma}\frac{\Gamma\left(\alpha\right)\beta\left(b,c\right)}{\Gamma\left(j+\frac{1}{2}\right)} \tag{7}$$

where

$a = k+m+\frac{n}{2}+\frac{1}{2}$, $b = m+\gamma+j+1$ and

$c = m+\alpha+2\beta+\gamma+\frac{n}{2}-\frac{1}{2}$ and $\sum$ is the summation over all α, β, γ and j.

When we use **m =** 1 and 2 in (6), the results are

$$E\left(\hat{\hat{\sigma}}_t^{*2}\right)=\frac{2\sigma^2}{(n-1)}\frac{e^{-n/2\theta}}{\Gamma\left(\frac{n-1}{2}\right)}\sum_{\alpha=0}^{\infty}\sum_{j=0}^{\infty}\left(1-\frac{t}{n-1}\right)^{\alpha}$$
$$\times\frac{\Gamma\left(j+\frac{n}{2}+1\right)\beta\left(j+\frac{3}{2},\alpha+\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(j+\frac{1}{2}\right)j!}\left(\frac{n}{2\theta}\right)^{j}$$

and

$$E\left(\hat{\hat{\sigma}}_t^{*2}\right)^2=\left(\frac{2\sigma^2}{(n-1)}\right)^2\frac{e^{-n/2\theta}}{\Gamma\left(\frac{n-1}{2}\right)}\sum_{\alpha=0}^{\infty}\sum_{j=0}^{\infty}(\alpha+1)\left(1-\frac{t}{n-1}\right)^{\alpha}$$
$$\times\frac{\Gamma\left(j+\frac{n}{2}+2\right)\beta\left(j+\frac{5}{2},\alpha+\frac{n}{2}+\frac{3}{2}\right)}{\Gamma\left(j+\frac{1}{2}\right)j!}\left(\frac{n}{2\theta}\right)^{j}$$

If we define $I_1=\frac{E\left(\hat{\hat{\sigma}}_t^2\right)}{\sigma^2}$ and $I_2=\frac{E\left(\hat{\hat{\sigma}}_t^2\right)^2}{\sigma^4}$, the efficiency $eff\left(\hat{\hat{\sigma}}_t^2\right)=2\left[(n-1)\left(I_2-2I_1+1\right)\right]^{-1}$ and relative bias $\hat{\hat{\sigma}}^2=I_1-1$ cab be evaluated.

ACKNOWLEDGEMENT

The author is indebted to the referee for helpful comments and to the King Fahd University of Petroleum and Minerals for excellent research facilities.

REFERENCES

1. Das, H. (1975). Estimation of μ^2 in normal population. *Calcutta Statist. Assoc. Bull.* 24, 135-140.
2. Govindarajulu, Z. and Sahai, H. (1972). Estimation of the parameters of a normal distribution with known coefficient, of variation. *Rep. Statist. Appl.* JUSE, 19, 85-98.
3. Pandey, M. and Singh, *J.* (1981). Estimation of variance of a normal population. *Comm. Statist. - Theor. Meth.*, A 10(18), 1879-1884.
4. Srivastava. V.K. and Bhatnagar, S. (1981). Estimation of the inverse of mean. *J. Statist. Planning and Inference*, 5, 329-334.

ESTIMATION OF PARAMETERS OF BURR PROBABILITY MODEL USING FRACTIONAL MOMENTS*

A.M. A1-Marzoug and **Munir Ahmad**
Department of Mathematical Sciences, University of Petroleum and Minerals
Dhahran, Saudi Arabia

ABSTRACT

In this paper, we define fractional moments of a random variable and use lower fractional moments to estimate the parameters of Burr probability model. We compare it with the maximum likelihood and moment methods of estimation.

KEY WORDS

Maximum likelihood, moment method, Pearson system of distributions, fractional moments.

1. INTRODUCTION

Burr (1942) introduced a family of distributions with the basic properties of cumulative functions covering the curve-shape characteristics for three main Pearson System Types I, IV and VI, as well as many transitional types such as the Type **III** or gamma distributions. The Burr cumulative function defined by

$$F\left(x\right)=\left[1+e^{-\int g\left(x\right)dx}\right]^{-1}. \tag{1.1}$$

where g(x) is a positive function for $0 \leq F(x) \leq 1$, appeared *in* a book by Bierens de Haan (1939) but no statistical properties were discussed. Burr (1942, 1968), Burr and Cislak (1968), Khalique (1971, 1983), Austin (1973), and Ahmad (1983, 1984) derived many properties of a special cumulative probability function

$$F\left(x\right)=1-\frac{1}{\left(1+x^{a}\right)^{\beta}}, \qquad x>0; \qquad a,\beta>0. \tag{1.2}$$

and estimated its parameters. Hatke (1949) showed that the Burr function could be used to graduate several observed distributions classified as Pearson types including the three main types I, IV, and VI by using one symbolic form whereas the Pearson curves required several different expressions of various complexity and involved approximate integrations.

In this paper we use a fractional moment method to estimate Burr parameters and compare its biases with moment and maximum likelihood estimates.

---

* Published in Pak. J. Statist. (1985), 1(1).

2. MAXIMUM LIKELIHOOD AND MOMENT ESTIMATION

Consider a random sample $\left(x_1,\ ...,\ x_n\right)$ of size n from a Burr probability model

$$f\left(x\right)=\alpha\beta x^{\alpha-1}\left(1+x^{\alpha}\right)^{-\beta-1},\alpha,\beta,x>0.$$

The maximum likelihood estimates of a and β are given by

$$\hat{\beta}=n\left[\sum_{i=1}^{n}ln\left(1+x_i^{\hat{a}}\right)\right]^{-1}$$

and

$$\sum_{i=1}^{n}lnx_i=\left(\hat{\beta}+1\right)\sum_{i=1}^{n}\left[x_i^{\hat{\alpha}}ln\,x_i\left(1+x_i^{\hat{\alpha}}\right)^{-1}\right]-n\left(\hat{\alpha}\right)^{-1}.$$

These MLEs can be solved numerically. The asymptotic variance-covariance matrix of $\hat{\alpha}$ and $\hat{\beta}$ can also be easily derived

$$v\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix}=\frac{1}{n}\begin{pmatrix}1/\alpha^2+\left(\beta+1\right)A\left(\alpha,\beta\right) & \left(1+\alpha\beta\right)/(1+\beta)\\ \left(1+\alpha\beta\right)/\left(1+\beta\right) & 1/\beta^2\end{pmatrix}^{-1}.$$

where

$$A\left(\alpha,\beta\right)=E\,\frac{x^{\alpha}\left(ln\,x\right)^2\left(1+x^{\alpha}\right)-\left(x^{\alpha}ln\,x\right)^2}{\left(1+x^{\alpha}\right)^2}$$

and

$$B=E\left(ln\ x\right).$$

The $rt\underline{h}$ moment of Burr random variable is given by

$$\mu_r'=E\left(X^r\right)=\frac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma\left(\beta-\frac{r}{\alpha}\right)}{\Gamma\left(\beta\right)}$$

By equating sample moments and population moments, (if these exist) and replacing $\left(\alpha,\beta\right)$ by $\left(\tilde{\alpha},\tilde{\beta}\right)$, the moment estimators of α and β are given by

$$m_1'=\frac{\Gamma\left(\frac{1}{\tilde{\alpha}}+1\right)\Gamma\left(\tilde{\beta}-\frac{1}{\tilde{\alpha}}\right)}{\Gamma\left(\tilde{\beta}\right)} \qquad (2.1)$$

and

$$m_2'=\frac{\Gamma\left(\frac{2}{\tilde{\alpha}}+1\right)\Gamma\left(\beta-\frac{2}{\tilde{\alpha}}\right)}{\Gamma\left(\tilde{\beta}\right)} \qquad (2.2)$$

where $m_1' = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $m_2' = \frac{1}{n}\sum_{i=1}^{n} x_{i,}^2$ $\tilde{\alpha}$ and $\tilde{\beta}$ are the moment estimators of α and β. The moment estimating equations (2 .1.1) and (2.1.2) can be solved numerically. For this purpose, we have tabulated the values of m_1' and m_2' for some values $\tilde{\alpha}$ of $\tilde{\beta}$ andas shown in Table 1.

The asymptotic variance-covariance matrix of moment estimators of α and β is given by

$$v\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix} = \begin{pmatrix} \dfrac{k_4^2 v_1 + k_2^2 v_2 - 2k_2 k_4 v_{12}}{v^2} & \dfrac{-k_3 k_4 v_1 - k_1 k_2 v_2 + \left(k_1 k_2 + k_2 k_3\right) v_{12}}{v^2} \\ \dfrac{-k_3 k_4 v_1 - k_1 k_2\, v_2 + \left(k_1 k_4 + k_2 k_3\right) v_{12}}{v^2} & \dfrac{k_3^2 v_1 + k_1^2 v_2 - 2k_1 k_2\;\, v_{12}}{v^2} \end{pmatrix}$$

where

$$a + c = k_1 \;\; and \;\;\; b - d = k_2 .\; a_1 + c_1 = k_3 \;\; and \;\; b_1 - d_1 = k_4 .$$

Table 1

Values of m_1' and m_2' from above to below respectively

β \ α	1	2	3	4	5	6	7	8	9	10
1	- -	1.5708 -	1.2092 2.4184	1.1107 1.5708	1.0690 1.3213	1.0472 1.2092	1.0344 1.1481	1.0262 1.1107	1.0206 1.0861	1.0166 1.0690
2	1.0000 -	0.7854 1.0000	0.8061 0.8061	0.8330 0.7854	0.8552 0.7928	0.8727 0.8061	0.8866 0.8201	0.8979 0.8330	0.9072 0.8447	0.9150 0.8552
3	0.5000 1.0000	0.5891 0.5000	0.6718 0.5374	0.7289 0.5891	0.7697 0.6342	0.7999 0.6718	0.8233 0.7029	0.8418 0.7289	0.8568 0.7509	0.8692 0.7697
4	0.3333 0.3333	0.4909 0.3333	0.5971 0.4180	0.6682 0.4909	0.7183 0.5497	0.7555 0.5971	0.7841 0.6360	0.8067 0.6682	0.8251 0.6952	0.8403 0.7183
5	0.2500 0.1667	0.4295 0.2500	0.5474 0.3483	0.6264 0.4295	0.6824 0.4947	0.7240 0.5474	0.7561 0.5905	0.7815 0.6264	0.8022 0.6566	0.8193 0.6824
6	0.2000 0.1000	0.3866 0.2000	0.5109 0.3019	0.5951 0.3866	0.6551 0.4551	0.6999 0.5109	0.7345 0.5568	0.7620 0.5951	0.7843 0.6275	0.8029 0.6551
7	0.1667 0.0667	0.3544 0.1667	0.4825 0.2683	0.5703 0.3544	0.6333 0.4248	0.6805 0.4825	0.7170 0.5303	0.7461 0.5702	0.7698 0.6042	0.7895 0.6333
8	0.1429 0.0476	0.3290 0.1429	0.4595 0.2428	0.5499 0.3290	0.6152 0.4005	0.6643 0.4595	0.7024 0.5086	0.7328 0.5499	0.7576 0.5850	0.7782 0.6152
9	0.1250 0.0357	0.3085 0.1250	0.4404 0.2226	0.5327 0.3085	0.5998 0.3805	0.6504 0.4404	0.6898 0.4905	0.7213 0.5327	0.7471 0.5688	0.7684 0.5998
10	0.1111 0.2778	0.2913 0.1111	0.4241 0.2061	0.5179 0.2913	0.5865 0.3636	0.6384 0.4241	0.6789 0.4749	0.7113 0.5179	0.7378 0.5547	0.7599 0.5865

$$\frac{1}{(\mu_1')^2}\operatorname{var}(m_1') = v_1, \quad \frac{1}{(\mu_2')^2}\operatorname{var}(m_2') = v_2$$

$$\frac{1}{\mu_1'\mu_2'}\operatorname{cov}(m_1', m_2') = v_{12} \quad \textit{and} \quad k_1k_4 - k_2k_3 = v$$

$$a = \frac{\partial}{\partial\alpha}\left[ln\Gamma\left(\frac{1}{\alpha}+1\right)\right], \; a_1 = \frac{\partial}{\partial\alpha}\left[ln\Gamma\left(\frac{2}{\alpha}+1\right)\right]$$

$$b = \frac{\partial}{\partial\alpha}\left[ln\Gamma\left(\beta-\frac{1}{\alpha}\right)\right], \; b_1 = \frac{\partial}{\partial\alpha}\left[ln\Gamma\left(\beta-\frac{2}{\alpha}\right)\right],$$

$$c = \frac{\partial}{\partial\beta}\left[ln\Gamma\left(\beta-\frac{1}{\tilde{\alpha}}\right)\right], \; c_1 = \frac{\partial}{\partial\beta}\left[ln\Gamma\left(\tilde{\beta}+\frac{2}{\alpha}\right)\right],$$

$$d = \frac{\partial}{\partial\beta}\left[\ln\Gamma(\beta)\right] \quad \textit{and} \quad d_1 = \frac{\partial}{\partial\beta}\left[\ln\Gamma(\beta)\right].$$

3. FRACTIONAL MOMENT METHOD OF ESTIMATION

In view of the fact that the lower moments are considered to be more efficient than higher moments, recently some authors have employed fractional moments of order less than one in estimation of parameters of some probability functions instead of integral moments of higher order. The fractional moments are restricted to positive random variables only. Wolfe (1975) derived moments of probability distribution functions for positive random variables.

The rth fractional moment of a random variables X with density function $f(x;\theta)$ (where θ may be a vector) is defined as

$$u_r = \int_{-\infty}^{\infty} x^r f(x;\theta)\,dx, \quad 0 < r < 1.$$

The corresponding empirical rth fractional moment from a random sample $x_1,...x_n$ can be defined as

$$m(r) = \frac{1}{n}\sum_{i=1}^{n} x_i^r, \quad 0 < r < 1.$$

The method of fractional moments will consist of equating $m(r)$ with u_r for some values of r in $0 < r < 1$. Khaliq (1983) estimated θ for which

$$1_n\left(\tilde{\theta}_n\right) = \frac{\min}{\theta}\int_0^1 \left[m(r) - u_r\right]^2 dH(r)$$

where $H(r)$ is a suitably chosen weight function and $\tilde{\theta}_n$ is the estimator of θ. Khalique (1983) has used $dH(r) = e^{-r^2}$ following Paulson *et al.* (1975). Khalique (1983) used it to find fractional moments of gamma and Burr probability functions. We used standard moment method technique for estimating parameters of the Burr distribution and compared our results with those of Khalique (1983)'s. However, Khalique (1983)'s results are based on simulation, whereas our results are based on simultaneous solution of actual estimating equations. We need two fractional moments to estimate the two parameters of the Burr probability function. Suppose we take two arbitrary values of r say $r_1, 0 < r_1, < 1$ and $r_2, 0 < r_2 < 1.$.

The fractional moment estimating equations are:

$$m_{r_1} = \frac{\Gamma\left(\frac{r_1}{\tilde{\alpha}} + 1\right)\Gamma\left(\beta - \frac{r_1}{\tilde{\alpha}}\right)}{\Gamma\left(\tilde{\beta}\right)} \qquad (3.1.1)$$

and

$$m_{r_2} = \frac{\Gamma\left(\frac{r_2}{\tilde{\alpha}} + 1\right)\Gamma\left(\beta - \frac{r_2}{\tilde{\alpha}}\right)}{\Gamma\left(\tilde{\beta}\right)} \qquad (3.1.2)$$

where $m_r = \frac{1}{n}\sum x_i^r$ and $x_1, x_2, ..., x_n$ is a random sample.

We solve these equations for $\tilde{\alpha}$ and $\tilde{\beta}$ by iteration, using the secant method and the Hooke and Jeeves optimization routine (Kauster *et al.*, 1973). We compute m_r for three typical values of $r\left(i.e., r = \frac{1}{4}, \frac{1}{2}\, and\, \frac{3}{4}\right)$ andfor various values of $\tilde{\alpha}$ and $\tilde{\beta}$ which are given in Table 2. In order to obtain the estimates of $\tilde{\alpha}$ and $\tilde{\beta}$ we use inverse interpolation.

We follow the same procedure as in (2.1.3) to derive the asymptotic variance-covariance matrix of fractional moment estimators of the Burr parameters.

The asymptotic variance –covariance matrix of FM estimators of α and β is given by

$$v\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} = \begin{pmatrix} \frac{k_4^2 v_1 + k_2 v_2 - 2k_2 k_4 v_{12}}{v^2} & \frac{-k_3 k_4 v_1 - k_1 k_2 v_2 + \left(k_1 k_2 + k_2 k_3\right) v_{12}}{v^2} \\ \frac{-k_3 k_4 v_1 - k_1 k_2 v_2 + \left(k_1 k_2 + k_2 k_3\right) v_{12}}{v^2} & \frac{k_3^2 v_1 + k_1 v_2 - 2k_1 k_2 v_{12}}{v^2} \end{pmatrix}$$

where $k_1, k_2, k_3, k_4, v_1, v_2$ and v_{12} are the same as in (2.1.3) except that r_1 and r_2 are fractions instead of integers.

SOME FRACTIONAL MOMENT FUNCTIONS

Table 2
Values of $m_{1/4}$, $m_{1/2}$ and $m_{3/4}$ from above to below respectively

β \ α	1	2	3	4	5	6	7	8	9	10
1	1.1107 1.5708 3.3322	1.0262 1.1107 1.2752	1.0115 1.0472 1.1107	1.0065 1.0262 1.0603	1.0041 1.0166 1.0380	1.0029 1.0115 1.0262	1.0021 1.0084 1.0191	1.0016 1.0065 1.0146	1.0013 1.0051 1.0115	1.0010 1.0041 1.0093
2	0.8330 0.7854 0.8330	0.8979 0.8330 0.7970	0.9272 0.8727 0.8330	0.9436 0.8979 0.8615	0.9539 0.9150 0.8823	0.9611 0.9272 0.8979	0.9663 0.9364 0.9099	0.9703 0.9436 0.9195	0.9735 0.9493 0.9272	0.9760 0.9539 0.9336
3	0.7289 0.5891 0.5207	0.8418 0.7289 0.6475	0.8886 0.7999 0.7289	0.9141 0.8418 0.7807	0.9301 0.8692 0.8161	0.9411 0.8886 0.8418	0.9491 0.9030 0.8612	0.9552 0.9141 0.8764	0.9599 0.9229 0.8886	0.9638 0.9301 0.8986
4	0.6682 0.4909 0.3905	0.8067 0.6682 0.5666	0.8639 0.7555 0.6682	0.8950 0.8067 0.7319	0.9146 0.8403 0.7753	0.9280 0.8639 0.8067	0.9378 0.8815 0.8304	0.9452 0.8950 0.8490	0.9511 0.9058 0.8639	0.9558 0.9146 0.8761
5	0.6264 0.4295 0.3173	0.7815 0.6264 0.5135	0.8459 0.7240 0.6264	0.8810 0.7815 0.7976	0.9031 0.8193 0.7462	0.9183 0.8459 0.7815	0.9294 0.8657 0.8082	0.9378 0.8810 0.8291	0.9444 0.8932 0.8459	0.9498 0.9031 0.8597
6	0.5951 0.3866 0.2697	0.7620 0.5951 0.4750	0.8318 0.6999 0.5951	0.8700 0.7620 0.6714	0.8941 0.8029 0.7239	0.9107 0.8318 0.7620	0.9228 0.8534 0.7909	0.9392 0.8700 0.8136	0.9392 0.8833 0.8318	0.9451 0.8941 0.8468
7	0.5703 0.3544 0.2360	0.7461 0.5703 0.4453	0.8203 0.6805 0.5703	0.8610 0.7461 0.6505	0.8867 0.7895 0.7058	0.9043 0.8203 0.7461	0.9173 0.8432 0.7768	0.9271 0.8610 0.8008	0.9349 0.8751 0.8203	0.9411 0.8867 0.8362
8	0.5499 0.3290 0.2107	0.7328 0.5499 0.4214	0.8105 0.6643 0.5499	0.8533 0.7328 0.6330	0.8803 0.7782 0.6906	0.8990 0.8105 0.7328	0.9126 0.8346 0.7649	0.9230 0.8533 0.1901	0.9311 0.8682 0.8105	0.9378 0.8803 0.8273
9	0.5327 0.3085 0.1909	0.7213 0.5327 0.4016	0.8021 0.6504 0.5327	0.8466 0.7213 0.6182	0.8748 0.7685 0.6777	0.8943 0.8021 0.7213	0.9085 0.8271 0.7546	0.9194 0.8466 0.7809	0.9279 0.8621 0.8021	0.9348 0.8748 0.8195
10	0.5179 0.2913 0.1750	0.7113 0.5179 0.3849	0.7946 0.6384 0.5179	0.8407 0.7113 0.6053	0.8700 0.7599 0.6664	0.8901 0.7946 0.7113	0.9049 0.8206 0.7456	0.9162 0.8407 0.7727	0.9250 0.8568 0.7946	0.9322 0.8700 0.8127

4. COMPARISON OF FRACTIONAL MOMENT ESTIMATORS WITH OTHER ESTIMATORS

In this section we compare fractional moment estimators with moment, maximum likelihood and Khalique fractional moment estimators of the Burr parameters. Khalique (1983) used a simulation technique to find the FM estimators of α and β based on various sample sizes with one hundred replications. The estimates of α and β along with the results of Khalique (1983) are given in Table 3. Our FM estimators seem to be better than Khalique FM estimators, but the direct comparison is not possible because his results are based on simulation. Khalique (1983)'s method cannot be applied to a given set of experimental data, whereas it is much easier to obtain estimates based on experimental data using' Table 2.

We have tried three combinations of fractional order of moments viz.

$$\left(r_1 = \frac{1}{4}, r_2 = \frac{1}{2}\right), \left(r_1 = \frac{1}{4}, r_2 = \frac{3}{4}\right) \text{ and } \left(r_1 = \frac{1}{2}, r_2 = \frac{3}{4}\right) \text{ and find that } r_1 = \frac{1}{4}, r_2 = \frac{3}{4}$$

give the best estimates in the sense of smallest bias. Khalique (1983) showed the superiority of FM over MM. Khalique (1983)'s results showed that FM estimates were much superior to MM estimates, but comparison with ML was hard to make. The reduction of sampling variability by lowering the order of moment is well reflected in the FM estimates. MM and ML methods are more general whereas FM method can be applied to positive random variables only.

We have also given the variances and covariances of fractional moments but we did not compute them as these were not directly comparable with those given by Khalique (1983).

Table 3
Estimates by Different Methods in Burr Distribution a = 3, 6 = 2,
(Replication = 100) (MM, ML and FM are taken from Khalique, 1983)

Sample Size	Method	Mean $\tilde{\alpha}$	Bias	Mean $\tilde{\beta}$	Bias
10	MM	3.4682	0.4682	2.3113	0.3113
	ML	3.2999	0.2999	2.2723	0.2723
	FM	3.3942	0.3942	2.2984	0.2984
	(Khalique, 1983)				
25	MM	3.1604	0.1604	2.1168	0.1168
	ML	3.0668	0.668	2.0942	0.0942
	FM	3.1077	1.1077	2.1066	0.1066
	(Khalique, 1983)				
50	MM	3.1323	0.1323	2.0475	0.0853
	ML	3.0483	0.0483	2.0313	0.0801
	FM	3.0839	0.0839	2.0400	0.0841
	(Khalique, 1983)				
For any size	FM: $r_1 = 1/4$, $r_2 = 3/4$	2.9930	0.0070	1.9972	0.0028
	$r_1 = 1/4$, $r_2 = 1/2$	2.9806	0.0094	1.9940	0.0060
	$r_1 = 1/2$, $r_2 = 3/4$	2.9878	0.0122	1.9972	0.0028

ACKNOWLEDGEMENTS

The authors are indebted to the University of Petroleum and Minerals for excellent research facilities. The authors thank the referees for helpful comments.

REFERENCES

1. Ahmed, M. (1983). Estimation of parameters of Burr distribution based on ordered statistics. To appear in *Punjab Journal of Mathematics.*
2. Ahmed, M. (1984). Estimating the parameters of Burr population from some ordered statistics. To appear in *Studia Centeriwn Hungarica.*
3. Austin, J.A., Jr. (1973). Control chart constants for largest and smallest in sampling from a normal distribution using the generalized Burr distribution. *Technometrics,* 15, 931-933.
4. Bierens de Haan (1939). *Referrred to in Burr* (1942).
5. Burr, I.W. (1972). Cumulative frequency functions. *Ann. Math. Statist.*, 13, 215-232.
6. Burr, I.W. (1968). On a general system of distribution III, The Sample Range. *J. Amer. Statist. Assoc.,* 63, 636-643.
7. Burr, I.W. and Cislak, P.J. (1968). On a general system of distributions, its curve-shape characteristics-II, The Sample Median. *J. Amer. Statist. Assoc.*, 63, 627-635.
8. Hatke, M.A. (1949). A certain cumulative probability function. *Ann. Math. Statist.*, 20, 461-463.
9. Q. Khalique, A. (1973). On Burr distributions. Unpublished M.Phil. Thesis, Department of Statistics, University of Karachi, Karachi, Pakistan.
10. Khalique, A. (1983). Estimation by fractional moments. *Proc. First Sandi Symposium on Statistics and its Applications,* 553-562.
11. Kaester, James L. and Mize, Joe H. (1973). Optimization *Techniques with Fortran.* McGraw-Hill Book Co., New York.
12. El-Marzoug, A. (1985). Estimation of parameters of some reliability models using fractional moments. Unpublished Thesis, Department of Mathematical Sciences, University of Petroleum and Minerals, Dhahran, Saudi Arabia.
13. Paulson, A., Holcomb, E.W. and Leitch, R.A. (1975). The estimation of the parameters of the stable laws. *Biametrika,* 62, 1, 163-170.
14. Wolfe, S.J. (1975). On moments of probability distribution functions, In fractional calculus and its applications, B. Ross (ed.), *Lecture Notes in Mathematics,* No. 457, Springer-Verlag, Berlin, 306-316.

ESTIMATION OF PARAMETERS OF THE GAMMA DISTRIBUTION BY THE METHOD OF FRACTIONAL MOMENTS*

Masood Rafiq[1], **Munir Ahmad**[2] and **Faqir Muhammad**[3]
[1] The Institute of Child Health, Lahore, Pakistan
[2] Islamic Society of Statistical Sciences, Lahore, Pakistan
[3] Department of Mathematics and Statistics
University of Agriculture, Faisalabad, Pakistan

ABSTRACT

The method of moments has been widely used for estimating the parameters of a distribution. Usually lower order moments are used to find the parameter estimates as they are known to have less sampling variability. With this approach in mind some authors (Wolfe (1975), Min (1977), Marx (1980), Khalique (1983) and Almarzoug and Ahmad (1985)) used the method of fractional moments to estimate the parameters of certain distributions.

The gamma distribution is quite widely used as a lifetime model. The gamma distribution does fit a wide variety of lifetime data adequately. However, there are failure process models which lead to it. The gamma distribution arises mathematically as the sum of independently distributed exponential random variables.

In the present study we use the method of fractional moments to estimate parameters of the gamma distribution and obtain their asymptotic variances and a comparison is made with those of the moment estimators and ML estimators. We also minimize the determinant of the var-cov matrix w.r.t r to obtain the values of r, In terms of asymptotic variances the FM method is far better than the method of moments and is almost equal to the method of ML. Small sample properties of the three methods are also discussed and FM method is found to be slightly better than the method of ML.

KEYWORDS

Gamma distribution, factorial moments, lower order moments, asymptotic variances.

1. INTRODUCTION

In the conventional method of moments lower order moments arc used to estimate the parameters of distributions as it is known that sampling variability of the moments increases as their order is increased. Therefore, the order of moments may be reduced to less than 1 with order of moments variable over (0,1) However, to be more general, the order of moments may be taken as variable over $(-\infty,\infty)$. Then the rth fractional moment of a non-negative random variable X with density function $f(x,\theta)$ is defined as

$$\mu_r' = \int_0^\infty X^r f(x;\theta)\,dx \qquad r > 0$$

* Published in Pak. J. Statist. (1996), Vol. 12(3).

The corresponding sample rth fractional moment from a random sample $X_1, X_2, ...X_n$, can be defined as

$$m'_r = \frac{1}{n}\sum_{i=0}^{n} X_i^r \qquad r > 0$$

The method of fractional moments will consist of obtaining as many values of m'_r, as the no. of parameters to be estimated and equating m'_r, with μ'_r, for some values of $r, 0 < r < 8$.

2. ESTIMATION BY THE METHOD OF MOMENTS AND ML

The pdf $f(x)$ of the gamma distribution is given by

$$f(x) = \frac{\lambda^\beta}{\Gamma\beta} x^{\beta-1} e^{-(\lambda x)} \qquad x > 0$$

$$\lambda, \beta > 0$$

The moment estimates of λ and β arc

$$\hat{\lambda} = \frac{m'_1}{m'_2 - m_1'^2} \qquad \hat{\beta} = \frac{m_1'^2}{m'_2 - m_1'^2}$$

and the asymptotic variance-covariance matrix of the moment estimators of λ and β is

$$V\begin{pmatrix} \hat{\lambda} \\ \hat{\beta} \end{pmatrix} = \frac{1}{n}\begin{pmatrix} \frac{\lambda^2}{\beta}(2\beta+3) & 2\beta(\beta+1) \\ 2\beta(\beta+1) & 2\beta(2\beta+1) \end{pmatrix}$$

The estimating equations to find the maximum likelihood estimates of λ and β arc

$$\tilde{\lambda} = \left(\frac{n\tilde{\beta}}{\sum x_i}\right) \tag{1}$$

$$n \ln\left(\frac{n\tilde{\beta}}{\sum x_i}\right) - n\psi\left(\tilde{\beta}\right) + \sum \ln x_i = 0 \tag{2}$$

(2) can be solved numerically to obtain the maximum likelihood estimates of β and then the value of λ can be obtained from equation (1).

The asymptotic variance-covariance matrix of MLEs of λ and β is given by

$$V\begin{pmatrix}\hat{\lambda}\\ \hat{\beta}\end{pmatrix} = \frac{\lambda^2}{n^2\left[\beta\psi'(\beta)-1\right]}\begin{pmatrix} n\psi'(\beta) & n/\lambda \\ n/\lambda & n\beta/\lambda \end{pmatrix}$$

The determinant of V is

$$|V| = \frac{\lambda^2}{n^2\left[\beta\psi'(\beta)-1\right]}$$

3. ESTIMATION BY THE METHOD OF FRACTIONAL MOMENTS

The rth fractional moment of the gamma distribution is given by

$$\mu_r = \frac{1}{\lambda^r \Gamma\beta}\Gamma(\beta+r)$$

The gamma distribution has two parameters λ and β, we therefore need two fractional moments to estimate these two parameters. We take two arbitrary values of r say $r_1, 0 < r_1 < \infty$ and $r_2, 0 < r_2 < \infty$

Then the population fractional moments are

$$\mu_{r_1} = \frac{1}{\lambda^{r_1}\Gamma\beta}\Gamma(\beta+r_1)$$

$$\mu_{r_2} = \frac{1}{\lambda^{r_2}\Gamma\beta}\Gamma(\beta+r_2)$$

The corresponding sample fractional moments or the estimating equations are

$$m_{r_1} = \frac{1}{\hat{\lambda}^{r_1}\Gamma\hat{\beta}}\Gamma\left(\hat{\beta}+r_1\right) \qquad (3)$$

$$m_{r_2} = \frac{1}{\hat{\lambda}^{r_2}\Gamma\hat{\beta}}\Gamma\left(\hat{\beta}+r_2\right) \qquad (4)$$

Equations (3) and (4) can be solved for $\hat{\lambda}$ and $\hat{\beta}$ numerically to obtain the fractional moment estimates of λ and β

4. ASYMPTOTIC VARIANCE-COVARIANCE MATRIX OF FM ESTIMATORS

Let V be the asymptotic variance covariance matrix of the fractional moment estimators of λ and β. Then the elements of V are given by

$$Var\left(\hat{\lambda}\right)=\frac{\hat{\lambda}}{n\left(a_1r_2-a_2r_1\right)^2}\left[a_2^2\left(g_{r_1}-1\right)\right]+a_1^2\left(g_{r_2}-1\right)-2a_1a_2\left(g_{r_{12}}-1\right)$$

$$Var\left(\hat{\beta}\right)=\frac{1}{n\left(a_1r_2-a_2r_1\right)^2}\left[r_2^2\left(g_{r_1}-1\right)\right]+r_1^2\left(g_{r_2}-1\right)-2r_1r_2\left(g_{r_{12}}-1\right)$$

$$Cov\left(\hat{\lambda},\hat{\beta}\right)=\frac{\lambda}{n\left(a_1r_2-a_2r_1\right)^2}\left[a_2r_2\left(g_{r1}-1\right)+a_1r_1\left(g_{r2}-1\right)-\left(a_1r_2+a_2r_1\right)\left(g_{r12}-1\right)\right]$$

where

$$a_1=\frac{\Gamma'\left(r_1+\beta\right)}{\Gamma\left(r_1+\beta\right)}-\frac{\Gamma'\beta}{\Gamma\beta}=\psi\left(r_1+\beta\right)-\psi\left(\beta\right)$$

$$a_2=\psi\left(r_2+\beta\right)-\psi\left(\beta\right) \qquad g_{r_1}=\frac{\Gamma\beta\,\Gamma\left(2r_1+\beta\right)}{\left[\Gamma\left(r_1+\beta\right)\right]^2},$$

$$g_{r_2}=\frac{\Gamma\beta\,\Gamma\left(2r_2+\beta\right)}{\left[\Gamma\left(r_2+\beta\right)\right]^2} \qquad g_{r_{12}}=\frac{\Gamma\left(r_1+r_2+\beta\right)}{\Gamma\left(r_1+\beta\right)\Gamma\left(r_2+\beta\right)}$$

Let D be the determinant of the variance covariance matrix of fractional moment estimators of the gamma distribution then

$$D=\frac{\lambda^2}{n^2\left(a_1r_2-a_2r_1\right)^2}\left[\left(g_{r_1}-1\right)\left(g_{r_2}-1\right)-\left(g_{r_{12}}-1\right)\right] \qquad \text{(A)}$$

D is minimized w.r.t. r_1 and r_2 for given values of λ and β. It has been observed that the values of r_1 and r_2 which minimize D are almost invariant for shape and scale parameters, therefore rounded values of r_1 and r_2 $\left(r_1=0.001 \text{ and } r_2=1.0\right)$ have been used. The comparison of the FM estimators with those of the ML estimators and moment estimators in terms of their asymptotic variances is given in Table-I. The asymptotic efficiencies of FM estimators and moment estimators w.r.t. MLEs arc also given in Table-II. FM and ML estimates arc equally efficient whereas moment estimates arc 50% as efficient as MLEs.

To investigate the small sample properties of the three estimation methods 100 samples of different sizes were generated, for $\lambda=1$ and $\beta=2$, using MINITAB and the results are given in Table-III and Table-1V. For small samples method of FM is more efficient than the methods of moment and ML.

5. EXAMPLE

The rainfall data for the month of January from the years 1960 to 1994 was obtained from the Meteorological Dept. University of Agriculture, Faisalabad. The gamma distribution was fitted to the data and the estimates of the parameters were obtained using the three estimation methods mentioned in the previous sections. The results of this example arc given below.

<u>January</u>

36.322, 15.240, 30.226, 6.350, 12.954, 6,604, 7.620, 0.762, 2.794, 12.446, 1.270, 12.700, 5.500, 7.000, 33.600, 16.600, 16.000, 2.200, 1.900, 2.000. 8.300, 3.500. 30.000, 20.210, 30.000, 6.500, 6.500.

Histogram

Midpoint	**Count**
0	5 *****
5	8 ********
10	3 ***
15	5 *****
20	1 *
25	0
30	3 ***
35	2**

	MOM	**MLE**	**FM**
$\hat{\lambda}$	0.109204	0.101224	0101254
$V\left(\hat{\lambda}\right)$	(0.05024776)	(0.02467382)	(0.02468685)
$\hat{\beta}$	1.355340	1.256290	1.256660
$V\left(\hat{\beta}\right)$	(6.38457303)	(2.54428451)	(2.54590773)

Table 1
Asymptotic Variances of the Estimators of Shape and Scale Parameter of the Gamma Distribution

	λ =**1** β = **2**	λ =**2** β = **5**	λ =**7** β =**1**
MOM			
$V(\hat{\lambda})$	3.5	10.40	245.00
$V(\hat{\beta})$	12.0	60.00	4.00
MLE			
$V(\hat{\lambda})$	2.224922273	8.303650749	124.976758740
$V(\hat{\beta})$	6.899689014	46.897813433	1.550546097
FM			
$V(\hat{\lambda})$	2.224922865	8.303650316	124.976845278
$V(\hat{\beta})$	6.899689146	46.897814476	1.550547863

Table 2
Relative Efficiency of Different Estimating Methods for the Gamma Distribution

	λ =**1** β = **2**	λ =**2** β = **5**	λ =**7** β =**1**
MOM			
$Eff(\hat{\lambda})$	0.63569	0.79843	0.51011
$Eff(\hat{\beta})$	0.57497	0.78163	0.38764
Overall eff.	0.57497	0.78163	0.38764
FM			
$Eff(\hat{\lambda})$	1.00	1.00	1.00
$Eff(\hat{\beta})$	1.00	1.00	1.00
Overall cff.	1.00	1.00	1.00

Table 3
n = 10

	MOM	**MLE**	**FM**
$E(\hat{\lambda})$	1.363947	1.235152	1.235021
$V(\hat{\lambda})$	0.472254	0.3593874	0.358687
$MSE(\hat{\lambda})$	0.604711	0.414684	0.413922
$E(\hat{\lambda})$	2.646975	2.421351	2.421137
$V(\hat{\lambda})$	1.072720	0.894802	0.892764
$MSE(\hat{\beta})$	1.491296	1.072338	1.070121

Table 4
n = 15

	MOM	**MLE**	**FM**
$E(\hat{\lambda})$	1.331982	1.257851	1.257091
$V(\hat{\lambda})$	0.236694	0.173416	0.173322
$MSE(\hat{\lambda})$	0.346906	0.239903	0.239418
$E(\hat{\lambda})$	2.514766	2.392305	2.392800
$V(\hat{\lambda})$	0.535597	0.518820	0.511433
$MSE(\hat{\beta})$	0.800581	0.672723	0.665725

REFERENCES

1. Ahmad, M. and Al-Marzoug, A.M. (1985). Estimation of parameters of Burr probability model using fractional moments. *Pak. J. Statist*., 1(1), 67-77.
2. AI-Marzoug, A.M. (1985). Estimation of parameters of some reliability models using fractional moments. Unpublished thesis, Dept. of Mathematical Sciences. University of Petroleum and Minerals, Dhahran, Saudi Arabia.
3. Khalique, A. (1983). Estimation by fractional moments. Proc. *First Saudi Symposium on Statistics and its Applications*. 553-562.
4. Marx, K. and Lauve, G. (1980). Remarks on the relation between fractional moments and fractional derivatives of characteristic function. *Journal of Applied Probability*, 17, 456-466.
5. Min, K.W. (1977). On the application of fractional moments in determining average molecular weights. *Journal of Applied Polymer Science*, 22, 589-591.
6. Rain, M. (1995). Estimation of the parameters of certain probability distributions by the method of fractional moments. Unpublished thesis, Dept. of. Math. & Stat. University of Agriculture. Faisalabad. Pakistan.
7. Wolfe, S.J. (1975). On moments of probability distribution functions. In fractional calculus and its applications. Ed. by Ross. Lecture Notes in Mathematics. No. 457, Springer Verlog, Berlin, 306-316.

ESTIMATION OF A COMMON CORRELATION COEFFICIENT AND TESTING OF EQUALITY OF $k(\geq 2)$ INDEPENDENT CORRELATION COEFFICIENTS*

Masood Anwar[1] and **Munir Ahmad**[2]

[1] Government Gordon College, Rawalpindi, Pakistan
Email: msdnwr@yahoo.com

[2] National College of Business Administration & Economics
Lahore, Pakistan. Email: drmunir@brain.net.pk

ABSTRACT

An estimator of a common population correlation coefficient ρ from $k(\geq 2)$ independent random samples drawn from bivariate normal populations is derived. Mean and MSE of the proposed estimator are computed using an approximate method and standard score estimator (Donner and Rosner, 1980) of ρ. The relative efficiency of the proposed MLE to standard score estimator is one for equal sample sizes and close to one for unequal sample sizes. The proposed estimator is also compared numerically with standard score estimator and MLE. Finally, following the Samiuddin's statistic, proposed estimator is used to test the equality of several independent correlation coefficients. An example is given to illustrate the calculations.

KEYWORDS

Bivariate normal distribution; Maximum likelihood estimator; Standard score estimator; Samiuddin's statistic; Taylor's expansion.

1. INTRODUCTION

In a Monte Carlo study, Donner and Rosner (1980) compare four estimators of a common correlation coefficient ρ from $k \geq 2$ independent random samples drawn from bivariate normal populations. These are r_F, based on Fisher's (1921) Z-transformation, r_H, based on modification of r_F (Hotelling, 1953), r_S, based on averaging the simple correlations (Donner and Rosner, 1980) and r_M, the maximum likelihood estimator (Pearson, 1933). As Donner and Rosner's study is limited to the case of equal sample size, Paul (1988, 1989) develops two new estimators based on Hotelling's adjusted Z-statistic, which are applicable to both equal and unequal sample sizes.

In this paper we develop an estimator based on Pearson (1933) equation. The proposed MLE is compared theoretically and numerically with other methods. Samiuddin (1970) considers a statistic for testing an assigned value of correlation coefficient ρ in a bivariate normal population. In this paper we propose a statistic based on Samiuddin's statistic to test the equality of $k(\geq 2)$ independent correlation coefficients.

* Published in Pak. J. Statist. (2006), Vol. 22 (3).

2. ESTIMATION OF A COMMON CORRELATION

Consider k independent random samples (X_{ij}, Y_{ij}), $j = 1, 2, ..., n_i$, $i = 1, 2, ..., k$ drawn from a bivariate normal population having means (μ_{1i}, μ_{2i}), standard deviations $(\sigma_{1i}, \sigma_{2i})$ and common correlation ρ. The most frequently recommended procedure for estimating ρ is based on Fisher's (1921) Z-transformation and is given by $r_F = \left[\exp(2\bar{Z}_F) - 1\right] \Big/ \left[\exp(2\bar{Z}_F) + 1\right]$, where $\bar{Z}_F = \sum_{i=1}^{k}(n_i - 3)Z_i \Big/ \sum_{i=1}^{k}(n_i - 3)$ and $Z_i = 2^{-1}\ln\left[(1+r_i)/(1-r_i)\right]$ as r_i is the product moment correlation coefficient for the ith sample n_i. Hotelling (1953) proposes an adjustment to $\bar{Z}_F$ by $\bar{Z}_H = \bar{Z}_F - r_F/(2n - 4.5)$ and estimating ρ by $r_H = \tanh \bar{Z}_H$, when all n_i's are equal. When n_i's are not equal, Paul (1988, 1989) develops two estimators based on Hotelling's (1953) adjusted Z-statistic for bias. Donner and Rosner (1980) compute the estimator r_S of ρ by standard score method as $r_S = \sum_{i=1}^{k}(n_i - 1)r_i \Big/ \sum_{i=1}^{k}(n_i - 1)$ [See also Paul (1988)]. The maximum likelihood estimator, $\hat{\rho} = r_M$ of ρ is the solution to the equation [See David (1938)]

$$\sum_{i=1}^{k} n_i \left(r_i - \hat{\rho}\right) \Big/ \left(1 - r_i\hat{\rho}\right) = 0 . \qquad (2.1)$$

Though the solution to (2.1) can numerically be found, yet we expand $\left(1 - r_i\hat{\rho}\right)^{-1}$ as powers of $\hat{\rho}$ and obtain

$$\sum_{i=1}^{k} n_i \left(r_i - \hat{\rho}\right)\left(1 + r_i\hat{\rho}\right) \cong 0 , \qquad (2.2)$$

as $|r_i| \le 1$ and $|\hat{\rho}| < 1$ and $|r_i\hat{\rho}| << 1$. If $r_i = 0$ then $\hat{\rho} = 0$; if $r_i = \pm 1$, then $\hat{\rho} = \pm 1$. The equation (2.2) is equivalent to $\hat{\rho}^2 + b\hat{\rho} - 1 \cong 0$, where $b = \sum_{i=1}^{k} n_i\left(1 - r_i^2\right) \Big/ \sum_{i=1}^{k} n_i r_i$. Solving the quadratic equation, we get an estimate r_Q (say), a function of r_i, $i = 1, 2, ..., k$ of ρ as

$$r_Q = \frac{1}{2}\left(-b \pm \sqrt{b^2 + 4}\right). \qquad (2.3)$$

If $r_i = \pm 1$, $b = 0$ and then $r_Q = \pm 1$. Since the numerator of b is positive, the sign of b depends on $\sum_{i=1}^{k} n_i r_i$. If each $r_i > 0$, then $\sum_{i=1}^{k} n_i r_i > 0$, and b > 0 and we take $r_Q^{+} = \frac{1}{2}\left(-b + \sqrt{b^2 + 4}\right)$. If each $r_i < 0$, then $\sum_{i=1}^{k} n_i r_i < 0$, and b < 0 and we take

$r_Q^- = \frac{1}{2}\left(b - \sqrt{b^2 + 4}\right)$. If each $r_i = 0$, then $\sum_{i=1}^{k} n_i r_i = 0$, and from (2.2), $r_Q = 0$. If some $r_i\,'s$ are positive and some $r_i\,'s$ are negative such that $\sum_{i=1}^{k} n_i r_i$ is either positive or negative. Then in case of $\sum_{i=1}^{k} n_i r_i > 0$, r_Q^+ holds and in case of $\sum_{i=1}^{k} n_i r_i < 0$, r_Q^- holds.

3. MEANS AND MSE'S OF r_Q AND r_S

We find the means and MSEs of r_Q in all the cases. For the case, $\sum_{i=1}^{k} n_i r_i > 0$, b > 0 and $n_1 \neq n_2 \neq n_3 \cdots\cdots \neq n_k$, we expand r_Q^+ in Taylor's expansion about ρ and get

$$r_Q^+ = \rho + \sum_{i=1}^{k} \left|\frac{\partial r_Q}{\partial r_i}\right|_{r_i=\rho} \left(r_i - \rho\right) + O\left(n^{-1}\right) \tag{3.1}$$

The mean of r_Q^+ is

$$E\left(r_Q^+\right) = \rho + \sum_{i=1}^{k} \left|\frac{\partial r_Q}{\partial r_i}\right|_{r_i=\rho} E\left(r_i - \rho\right). \tag{3.2}$$

Now

$$\left|\frac{\partial r_Q^+}{\partial r_i}\right|_{r_i=\rho} = \frac{n_i}{N}, \text{ where } \sum_{i=1}^{k} n_i = N \tag{3.3}$$

We have, using Hotelling (1953) relation,

$$E\left(r_i - \rho\right) = \frac{-\rho\left(1-\rho^2\right)}{2(n_i - 1)} \text{ to } O\left(n_i^{-2}\right) \tag{3.4}$$

Putting (3.3) and (3.4) in equation (3.2), we get

$$E\left(r_Q^+\right) = \rho\left[1 - \frac{\left(1-\rho^2\right)}{2N} \sum_{i=1}^{k} \left(\frac{n_i}{n_i - 1}\right)\right], \tag{3.5}$$

as r_Q^+ is a biased estimator of ρ and $Bias = \frac{-\rho\left(1-\rho^2\right)}{2N} \sum_{i=1}^{k} \left(\frac{n_i}{n_i - 1}\right)$. Bias = 0 as $n_i \to \infty$.

Now

$$MSE\left(r_Q\right) = E\left(r_Q^+ - \rho\right)^2. \tag{3.6}$$

As $r_1, r_2, ..., r_k$ are independent and all having the same location parameter ρ, therefore $E\left[\left(r_i - \rho\right)\left(r_j - \rho\right)\right] = 0$ for all $i \neq j$.

$$MSE\left(r_Q^+\right) = \sum_{i=1}^{k}\left(\left|\frac{\partial r_Q}{\partial r_i}\right|_{r_i=\rho}\right)^2 E\left(r_i - \rho\right)^2, E(r_i) \neq \rho. \quad (3.7)$$

Using Hotelling's (1953) relation

$$E(r_i - \rho)^2 = \frac{\left(1-\rho^2\right)^2}{n_i - 1}\left[1 + \frac{23\rho^2}{4(n_i - 1)}\right], \text{ to } O\left(n_i^{-3}\right), \quad (3.8)$$

and using (3.3) in (3.7) we get,

$$MSE\left(r_Q^+\right) = \frac{\left(1-\rho^2\right)^2}{N^2}\left[\sum_{i=1}^{k}\frac{n_i^2}{\left(n_i - 1\right)} + \frac{23\rho^2}{4}\sum_{i=1}^{k}\frac{n_i^2}{\left(n_i - 1\right)^2}\right]. \quad (3.9)$$

When $n_i = n$ (say), $i = 1, \ 2, \cdots, k$, we have

$$E\left(r_Q^+\right) = \rho\left[1 - \frac{(1-\rho^2)}{2(n-1)}\right], \quad Bias = \frac{-\ \rho\ (1-\rho^2)}{2(n-1)}$$

and

$$MSE\left(r_Q^+\right) = \frac{(1-\rho^2)^2}{(N-k)}\left[1 + \frac{23\rho^2}{4(n-1)}\right]. \quad (3.10)$$

In case of $\sum_{i=1}^{k} n_i r_i < 0$, b < 0 and $n_1 \neq n_2 \neq \cdots\cdots \neq n_k$, we have (2.5)

Then $r_Q^- = -\rho$, $\left|\frac{\partial r_Q}{\partial r_i}\right|_{r_i=\rho} = \frac{-n_i}{N}$,

$$E\left(r_Q^-\right) = -\rho\left[1 - \frac{(1-\rho^2)}{2N}\sum_{i=1}^{k}\left(\frac{n_i}{n_i - 1}\right)\right]. \quad (3.11)$$

$MSE\left(r_Q^-\right)$ remains the same, as in equation (3.9).

The mean and MSE of r_S for unequal sample sizes are

$$E(r_S) = \rho\left[1 - \frac{k\left(1-\rho^2\right)}{2\left(N-k\right)}\right] \text{ and } MSE(r_S) = \frac{\left(1-\rho^2\right)^2}{\left[N-k\right]^2}\left[\left(N-k\right) + \frac{23\rho^2 k}{4}\right],$$

respectively, where $Cov(r_i, r_j) = 0$ for all $i \neq j$ as $r_1, r_2, \cdots, r_k$ are independent and all having same location parameter ρ. For equal sample sizes the mean and variance of r_S are

$$E(r_S) = \rho \left\{ 1 - \frac{\left(1-\rho^2\right)}{2(n-1)} \right\}$$

and

$$MSE(r_S) = \frac{\left(1-\rho^2\right)^2}{(N-k)} \left[1 + \frac{23\rho^2}{4(n-1)} \right], \text{ respectively.}$$

4. EFFICIENCY COMPARISON

When $n_i = n$ (say), $i = 1, 2, \cdots, k$, the relative efficiency of r_Q to r_S is one, showing r_Q and r_S are equally efficient. When samples are not equal, then the relative efficiency of r_Q to r_S is given by

$$E_f = \frac{MSE\left(r_s\right)}{MSE\left(r_Q\right)} = \frac{N^2}{(N-k)^2} \frac{\left[(N-k) + \dfrac{23\rho^2 k}{4} \right]}{\left[\displaystyle\sum_{i=1}^{k} \frac{n_i^2}{(n_i-1)} + \frac{23\rho^2}{4} \sum_{i=1}^{k} \frac{n_i^2}{(n_i-1)^2} \right]},$$

which is more than one for all values of ρ.

5. NUMERICAL COMPARISON OF r_S, r_Q, r_M

In Table 1 we compute r_S, r_Q and r_M for different values of k, n_i and r_i, $i = 1, 2, \cdots, k$, where r_M is obtained by solving (2.1) iteratively using Newton Raphson method. Table 1 shows that for equal sample sizes, r_S, r_Q and r_M are identical and for unequal sample sizes, $r_S < r_Q < r_M$. It is observed that r_S gives better results for small values of ρ and r_M for large values of ρ whereas r_Q is better than the two for moderate values of ρ.

Table 1
Numerical Values of r_S, r_Q and r_M for Different Values of
k, n_i and r_i, $i = 1, 2, \cdots, k$.

$k=2$			$k=3$			$k=4$			$k=5$		
n_1	r_1	r_S	n_1	r_1	r_S	n_1	r_1	r_S	n_1	r_1	r_S
n_2	r_2	r_Q	n_2	r_2	r_Q	n_2	r_2	r_Q	n_2	r_2	r_Q
		r_M	n_3	r_3	r_M	n_3	r_3	r_M	n_3	r_3	r_M
						n_4	r_4		n_4	r_4	
									n_5	r_5	
10	0.10	0.085	10	0.065	0.0853	10	0.82	0.84	10	0.32	0.27
10	0.07	0.085	10	0.089	0.0853	10	0.86	0.84	10	0.28	0.27
		0.085	10	0.102	0.0853	10	0.78	0.84	10	0.19	0.27
						10	0.90		10	0.21	
									10	0.37	
25	0.27	0.23	30	0.422	0.460	25	0.56	0.64	25	0.53	0.532
25	0.19	0.23	30	0.388	0.462	25	0.73	0.64	25	0.55	0.532
		0.23	30	0.569	0.464	25	0.61	0.64	25	0.51	0.532
						25	0.65		25	0.49	
									25	0.58	
18	0.45	0.5016	10	0.30	0.4193	20	0.41	0.5044	10	0.65	0.8546
16	0.56	0.5029	15	0.40	0.4197	30	0.60	0.5057	25	0.79	0.8552
		0.5038	25	0.49	0.4206	40	0.51	0.5066	50	0.91	0.8630
						50	0.48		75	0.87	
									100	0.85	
10	0.48	0.647	24	0.850	0.8277	10	0.24	0.3236	25	0.25	0.2582
25	0.71	0.649	32	0.780	0.8285	15	0.19	0.3245	25	0.19	0.2583
		0.664	31	0.860	0.8310	20	0.34	0.3302	50	0.23	0.2592
						25	0.42		50	0.30	
									100	0.27	
98	0.78	0.8095	24	0.520	0.6666	30	0.89	0.7023	75	0.47	0.5067
95	0.84	0.8099	29	0.560	0.6777	25	0.77	0.7102	75	0.49	0.5071
		0.8117	32	0.870	0.7062	50	0.56	0.7113	100	0.50	0.5080
						40	0.70		150	0.48	
									200	0.55	

6. TESTING OF EQUALITY OF SEVERAL CORRELATION COEFFICIENTS

If $r_1, r_2, ..., r_k$ are independent product moment correlation coefficients with r_i based on a sample of n_i observations from a bivariate normal population having correlation coefficient ρ_i, there are several approaches to the problem of testing equality of several correlation coefficients $\left(H_0 : \rho_i = \rho, i = 1, 2, \cdots, k \quad vs \quad H_1 : \rho_i \neq \rho_j \quad \text{some} \quad i \neq j\right)$. The most common is based on the normality assumption of Fisher's Z-transform of the sample correlation coefficients r_i (David, 1938). Paul (1989) derives two $C(\alpha)$ statistics and the likelihood-ratio statistic for testing the equality of several correlation coefficients and also the asymptotic relationship of these statistics is established. No attempt in the literature has been made to test the hypothesis that $H_0: \rho_i = \rho_0, i = 1, 2, \cdots, k \quad vs. \quad H_1: \rho_i \neq \rho_0$.

Samiuddin (1970) considers a test statistic

$$t = (r - \rho)\sqrt{n-2} \Big/ \sqrt{\left(1-r^2\right)\left(1-\rho^2\right)}, \tag{6.1}$$

and found that the statistic (6.1) has an exact student's t-distribution for $\rho = 0$ and has an asymptotic t-distribution for $\rho \neq 0$. We use Samiuddin's statistic and propose a test statistic to test the equality of several correlation coefficients for $k \geq 2$ independent random samples drawn from bivariate normal populations. Therefore, a test of significance for r_Q is

$$t_Q = \left(r_Q - \rho_0\right)\sqrt{(N-k-1)} \Big/ \sqrt{\left(1-r_Q^2\right)\ \left(1-\rho_0{}^2\right)}, \tag{6.2}$$

which under the null hypothesis of equal correlations follows student's t-distribution with (N-k-1) degrees of freedom for $\rho_0 = 0$ and has an asymptotic t-distribution for $\rho_0 \neq 0$. The rationale for the degrees of freedom (d.f) of this test is that each of the k samples contributes ($n_i - 2$) d.f and that there are (k-1) d.f among the individual r_i's, which reflect sampling variation. Thus the statistic r_Q has associated with it a total of $\sum_{i=1}^{k}\left(n_i - 2\right) + \left(k-1\right) = (N-k-1)$ d.f.

7. EXAMPLE

As an example, we use Tishler et al. (1977) data on the familial aggregation of blood pressure in children. The relationship of interest is between diastolic blood pressure and weight. It is relevant to investigate the correlations between blood pressure and weight separately in age-groups 6-8, 9-11, and 12-14. The simple correlations r_i in samples of 30 boys from each of these age-groups are given by 0.422, 0.388 and 0.569, respectively. Under the assumption that these estimates are homogeneous, we test $H_0: \rho_i = 0.4$ against $H_1: \rho_i \neq 0.4$, $i = 1, 2, 3$. We have $t_Q = 0.707$ for $r_Q = 0.462$, $N = 90$ and $k = 3$. Thus the t_Q test gives strong evidence that $\rho = 0.4$.

8. ACKNOWLEDGEMENT

The authors are indebted to the editor and referees for their suggestions, which improved the first version of the paper.

REFERENCES

1. David, F.N. (1938). *Tables of the ordinates and probability integral of the distribution of the correlation coefficient in small samples*. London Cambridge University Press, VII-XXXIV.
2. Donner, A. and Rosner, B. (1980). On Inferences concerning a common correlation coefficient. *Applied Statistics,* 29, 69-76.
3. Fisher, R.A. (1921). On the probable error of a coefficient of correlation deduced from a small sample. *Metron*, 1, 1-32.
4. Hotelling, Harold. (1953). New light on correlation coefficient and its transformation. *J. Roy. Statist. Soc.,* Series B, 15 (2), 193-232.
5. Paul, S.R. (1988). Estimation of and testing significance for a common correlation coefficient. *Comm. Statistics. Theory and Methods,* A 17, 39-53.
6. Paul, S.R. (1989). Test for the equality of several correlation coefficients. *The Can. J. Statist*, 17(2), 217-227.
7. Pearson, K. (1933). On a method of determining whether a sample of size *n* supposed to have been drawn from a parent population having a known probability integral has probably been drawn at random. *Biometrika,* 25, 379-410.
8. Samiuddin, M. (1970). On a test for an assigned value of correlation in a bivariate normal distribution. *Biometrika,* 57, 461-464.
9. Tishler, P., Donner, A., Kass, E. and Taylor, J. (1977). Familial aggregation of blood pressure in very young children. *CVD Epidemiology Newsletter*, 22, 45.

ON SOME PROPERTIES OF SESHADRI FAMILY OF DISTRIBUTIONS*

Saleha Naghmi Habibullah[1] and **Munir Ahmad**[2]

[1] Kinnaird College for Women, Lahore, Pakistan
Email: salehahabibullah@hotmail.com

[2] National College of Business Administration & Economics
Lahore, Pakistan. Email: drmunir@brain.net.pk

ABSTRACT

During the past three decades, inverted probability distributions have attracted the attention of a fairly large number of researchers, both in the theoretical and the applied areas. Seshadri (1965) has focused on distributions the functional form of which remains unchanged under the reciprocal transformation. He has presented some properties of such distributions. In this paper, we present some additional properties of this class of distributions.

KEY WORDS

Inverted probability distributions, reciprocal transformation, life-testing, negative moments.

INTRODUCTION

By the term 'inverted distributions' we mean those distributions each of which is obtained by applying the transformation $Y = 1/X$ to a distribution $f(x)$. During the past three decades, inverted probability distributions have attracted the attention of a fairly large number of researchers. Inverted distributions find applications in a variety of real life situations including Econometrics, Survey Sampling, Biological and Engineering Sciences, Life-testing, etc. [See Vysokovskii (1966, 1970), Pronikov (1973) and Kordonsky and Friedman (1976)]. Various authors have derived a variety of inverted distributions including the inverted gamma, inverted beta, inverted Weibull, inverted normal, inverted chi square, inverted Dirichlet and inverted multivariate t distributions, and a considerable amount of work has been done to obtain the basic properties of various inverted distributions. [See Bartholomew (1957),Ahmad and Sheikh (1983, 1984), Sheikh and Ahmad (1982, 1983), Ahmad (1985), Habibullah (1987) and Ahmad (1995)]. Also, a number of researchers have worked on the negative or inverse moments of various discrete and continuous distributions. [See Stephan (1945), Grab and Savage (1954), Mendenhall and Lehmann (1960), Rider (1962), Chao and Strawderman (1972), Ahmad, Munir, A.K. Sheikh and A.K.A. Kattan (1998), Roohi (2002), and Ahmad and Roohi (2004)].

Cobb (1980) has developed Pearson type differential equation that includes some inverted distributions. However, Seshadri (1965) has discussed distributions whose

* Published in Pak. J. Statist. (2008), Vol. 24(3).

functional form remains unchanged under the reciprocal transformation, and Habibullah and Ahmad (2006) have presented a functional form which yields Seshadri-type distributions over the domain $\left(a, \frac{1}{a}\right)$, $0 < a < 1$.

2. PROPERTIES OF THE SESHADRI DISTRIBUTIONS

Seshadri (1965) defined a family of distributions that remain unchanged under the reciprocal transformation $f(x) = \frac{1}{x^2} f\left(\frac{1}{x}\right)$, $x > 0$. Seshadri (1965) also discussed the family and derived some of its properties. We present some additional properties of the Seshadri (1965) family:

Theorem 2.1:

If $f(x)$ represents the Seshadri (1965) pdf, then $E\left[g\left(X^{-1}\right)\right] = E\left[g(X)\right]$, where $g(X)$ is any function of X.

The proof is trivial.

Some of the applications of Theorem 2.1 are given below:

i) If $g(X) = X^{-r}$, $r = 1, 2, ...$ then $E\left(X^{-r}\right) = E\left(X^{r}\right)$, if $E\left(X^{-r}\right)$ exists.

ii) If $r = 1$, then $HM = E^{-1}\left(\frac{1}{X}\right) = E^{-1}(X) = (AM)^{-1}$.

[See also Ahmad and Sheikh (1981)].

iii) If $g(X) = e^{itX^{-1}}$, then the Characteristic Function $E\left[e^{\frac{it}{X}}\right] = E\left[e^{itX}\right]$.

iv) The geometric mean, $GM(x) = 1$, $x > 0$.

Proof:

By definition, the geometric mean of any random variable is given by $GM(x) = e^{E[\ln x]}$ where $GM(x)$ denotes the geometric mean. Since $E\left[\ln X^{-1}\right] = E\left[\ln X\right]$, hence $GM(X) = e^{-E[\ln X]} = \frac{1}{GM(X)}$ or $\left[GM(X)\right]^2 = 1 \Rightarrow GM(X) = 1$ as X is a positive random variable.

Combining Seshadri's (1965) result with property (iv), we have $\tilde{X} = GM(x) = 1$ where $\tilde{X}$ …… is the median of the random variable X. Also, properties (ii) and (iv) lead to the following:

The inequality $\dfrac{1}{E\left(X^{-1}\right)} < 1 < E\left(X\right)$ holds, provided that $E\left(X^{-1}\right)$ exists.

Theorem 2.2:

Let X_1 and X_2 be identically and independently distributed random variables that are invariant under the reciprocal transformation. Then, the distribution of the product X_1X_2 is the same as the distribution of the ratio X_1 / X_2 (as well as the distribution of the ratio X_2 / X_1), and is also closed under inversion.

The proof is simple.

ACKNOWLEDGEMENT

We are indebted to the referees for improving the present version of the paper.

REFERENCES

1. Ahmad, M. (1985). Inverted Class of Distributions. *Proceedings of the 45th Session of the International Statistical Institute*, 1-3, ISI, Netherlands.
2. Ahmad, M. (1995). Inverted Class of Distributions. *Invited paper to Second Iranian Conference*. Mashhad, Iran.
3. Ahmad, M. and Roohi, A. (2004). Characterization of the Hyper-Poisson probability function using First Order Negative Moments. *Proceedings of the International Conference on Distribution Theory*. Order Statistics and Inference, Spain, June 16-18, 2004.
4. Ahmad, M. and Sheikh, A.K. (1983). Some Estimation Problems of a Class of Inverted Distributions. Contributed Paper, *44th Session of the International Statistical Institute*. Madrid, Espana.
5. Ahmad, M. and Sheikh, A.K. (1984). Bernstein Reliability Model: Derivation and Estimation of Parameters. *Reliability Engineering*, 8, 131-148.
6. Bartholomew, A.J. (1957). A Problem in Life Testing. *JASA*, 52, 350-351.
7. Chao, M.T. and Strawderman, W.E. (1972), Negative Moments of Positive Random Variables. *J. Amer. Statist. Assoc*. 67, 429-431.
8. Cobb, L. (1980). The Multimodal Exponential Families of Statistical Catastrophe Theory. *Statistical Distributions in Scientific Work*, 4, 69-90.
9. Grab, E.L. and Savage, I.R. (1954). Tables of the Expected Value of $1/X$ for positive Bernoulli and Poisson variables. *J. Amer. Statist. Assoc.* 49, 169-177.
10. Habibullah, S.N. (1987). On the Modes of the Inverted Bivariate Normal Distribution. *Pak. J. Statist.* 3(1), Series A, 49-62.
11. Habibullah, S.N. and Ahmad, M. (2006). On a New Class of Univariate Continuous Distributions that are Closed Under Inversion. *Pak. J. Statist. and Oper. Res.*, 2(2), 151-158.
12. Kordonsky, K.B. and Fridman, Y.M. (1976). Certain Questions Concerning the Probabilistic Description of Fatigue Longevity (survey). (in Russian, *Zavodskaya Laboratoriya*, 7, 829-847), (in English: *Industrial Laboratory*, INDLAP 42(7), 1001-1186, Consultants Bureau, New York).

13. Mendenhall, W. and Lehmann, E. H. Jr. (1960). An Approximation to the Negative Moments of the Positive Binomial useful in Life-testing. *Technometrics*, 2,227-241.
14. Pronikov, A.S. (1973). Calculation of Reliability Indices in Sequential (wear based) Failures. *Nedezhnosti Kontrol Kachestra*, No. 2 (in Russian).
15. Rider, P.R. (1962). Expected Value and Standard Deviations of the Reciprocal of a Variable from a Decapitated Negative Binomial Distribution. *J. Amer. Statist. Assoc*. 57, 439-445.
16. Roohi, A. (2002). *Negative Moments of Non-Negative Random Variables*. Unpublished M.Phil. Dissertation, The University of Lahore, Lahore. Pakistan.
17. Seshadri, V. (1965). On Random Variables which have the Same Distribution as their Reciprocals. *Canadian Mathematical Bulletin*, 8(6), 819-824.
18. Sheikh, A.K. and Ahmad, M. (1982). A Reliability Model for a Non-linear Damage Process. *19th Annual Meeting of the Society of Engineering Sciences*, Rolla, Missouri.
19. Sheikh, A.K. and Ahmad, M. (1983). Material Damage Processes and Reliability Models. Contributed Paper, *44th Session of the International Statistical Institute*, Madrid, Espana.
20. Stephan, F.F. (1945). The Expected Value and Variance of the Reciprocal and Other Negative Powers of a Positive Bernoullian Variate. *Ann. Math. Statist.*, 16, 50-61.
21. Vysokovskii, E.S. (1966). Reliability of Tools Used on Semi Automatic Lathes. *Russian Engineering Journal*, XLVI(6), 46-50.
22. Vysokovskii, E.S. (1970). Reliability of Cutting Tools in Automated Production. *Russian Eng. J.* 3, 63-67.

AN F-TYPE TEST BASED ON THE RATIO OF THE LARGEST TO THE SMALLEST RANGES FROM INDEPENDENT SAMPLES*

Sajeela Tazeen Dara[1], **Muhammad Hanif**[1], **Abdul Khalique**[2] and **Munir Ahmad**[1]

[1] National College of Business Administration & Economics, Lahore, Pakistan.
Email: sajeelaqm@hotmail.com; hanif@lums.edu.pk; drmunir@brain.net.pk

[2] National University of Computer & Emerging Sciences, Lahore, Pakistan.
Email: khalique15@hotmail.com

ABSTRACT

A statistical problem arises in estimating a variance component in an analysis of variance in various experimental designs when the estimate of the variance component becomes negative. Various authors [Hartely (1950), Link (1950), Patnaik (1950), Leslie and Brown (1966), Barlow et al. (1969), Pearson (1932, 1966) and Ahmad (1974, 1978)] have devised alternative methods to estimate variance that guarantee non-negative estimate of variance components. However, the distribution of the ratio of two ranges is restricted to two samples of unequal sizes.

In this paper, an attempt has been made to extend Leslie and Brown (1966) *F*-test (F_{LB}) based on k samples of unequal sizes. The sampling distribution of ranges is derived based on k-samples of unequal sizes from a standard normal population. Since the integration is difficult to compute the critical regions, Monte Carlo procedure has been adopted. An example is given to compare it with Snedecor *F*-test (F_S) and F_{LB}.

1. INTRODUCTION

A number of applications of range, maxima, minima and other order statistics for solutions of various problems arising in industrial quality control, floods, drought predictions, climatology, engineering, medical science etc. have recently been discussed by various authors [See Benjamini et al. (2004), Carbajal (2003), Fuentes (2002), Liu (2001), Meinshausen and Buhlmann (2005), Petricoin et al. (2002) and Stephenson and Tawn (2005)]. Among the known tests, Hartely (1950) used the ratio of maximum variances to minimum variances in a set of k independent sample variances in place of *F*-test. Link (1950) derived the distribution of ratio of two ranges from two independent samples of unequal sizes and computed values of ratios for all combinations of sample sizes at different levels of significance. McKay and Pearson (1933) derived distributions of the range from normal population whereas Daly (1946), Lord (1947), Gupta et al. (1964), Harter (1959) and others derived density of ranges. Leslie and Brown (1966) obtained the density of the ratio of maximum range to the minimum range of k samples each of equal size independently drawn from a normal population, proposed a test based on the ratio and computed tables for this purpose. Pearson (1966) gave Monte Carlo results on the three tests of heterogeneity of variance, including one of Leslie and Brown (1966). Later Ahmad (1974, 1978) obtained the probability function of the ratio of maximum range to minimum range of k samples of varying sizes drawn independently

* Published in Pak. J. Statist. (2009), Vol. 25(3).

from normal populations and proposed an alternative F-type (F_W say) test based on the ratio, but no table was prepared for computing critical regions.

In this paper, the distribution of the proposed F_W statistic has been used to obtain its values for different values of n_i at 5% significance level.

2. DISTRIBUTION OF ORDER STATISTICS

Let X be a statistic in a sample of size n_j and its distribution function be $G_j(x)$ with density $g_j(x)dx = dG_j(x)$. The distribution of i^{th} order statistic, Y_i of a sample of size k drawn from each of $G_j(x), j = 1, 2, ...k$ is given by

$$H_{Y_i}(y) = \int_{-\infty}^{y} \sum_{m=1}^{M} \prod_{r=i+1}^{k} \left[1 - G_{mr}(t)\right] d\left[\prod_{r=1}^{i} G_{mr}(t)\right], \qquad (2.1)$$

where $\sum_{m=1}^{M}$ extends over those values of m for which $m_i \neq m_j$ for $i \neq j$. There are $\binom{k}{i} = M$ terms in the summation. In particular, if $i = 1$ the distribution of Y_1 is $P(Y_1 \le y) = 1 - \prod_{r=1}^{k} \left[1 - G_r(y)\right]$. The joint density function of r^{th} and s^{th} order statistics is given by $h(y_r, y_s)dy_r\, dy_s = \sum_{m=1}^{M} \left[\prod_{i=1}^{r-1} G_{mi}(y_r) \prod_{i=r+i}^{s-1} \{G_{mi}(y_s) - G_{mi}(y_r)\} \prod_{i=s+1}^{k} \{1 - G_{mi}(y_r)\} dG_{mr}(y_r) dG_{mr}(y_s)\right]$ where $\sum_{m=1}^{M}$ is over m such that $m_i \neq m_j, i \neq j$. The joint density of the smallest and the largest observations is

$$H(y_1, y_k) = \prod_{i=1}^{k} G_i(y_k) - \prod_{i=1}^{k} \left[G_i(y_k) - G_i(y_1)\right].$$

The distribution function of range $W_i = Y_{n_i} - Y_{n_1}$, is

$$P(W_i \le w) = n\int_{-\infty}^{\infty} \left[G(t+w) - G(t)\right]^{n_i} dG(t) \text{ for } w \ge 0.$$

3. A TEST BASED ON THE RATIO OF RANGES

Leslie and Brown (1966) proposed a test based on $W = W_{\max}/W_{\min}$ where $W_{\max}$ is the maximum of ranges and $W_{\min}$ is the minimum of ranges among k samples of equal sizes. Suppose the sample sizes are unequal. The joint distribution of $W_{\max}$ and $W_{\min}$ [see

Ahmad (1974, 1978)] is $H\left(W_{\max}, W_{\min}\right) = \prod_{j=r}^{k} G_j\left(W_{\max}\right) - \prod_{j=1}^{k}\left[G_j\left(W_{\max}\right) - G_j\left(W_{\min}\right)\right]$,
and the cumulative distribution function of W is

$$H(w) = \int_{0}^{w} \int_{-\infty}^{\infty} h\left(x + W_{\min}, W_{\min}\right) dW_{\min}\ dx .$$

The test based on W is F_W where $F_W = 1 - H(w) - \alpha$,

If $n_j = n$ for each j, $G_j(x) = G(x)$ reduces $H(w)$ to an equation of Leslie and Brown (1966). We have evaluated $H(w)$ based on random samples each containing $n_i = 2(1)10, i = 1, 2, ...10$ observations drawn from a standard normal population and F_W values for upper 5% have been computed. [See Dara (2007) at website www.pakjs.com].

4. COMPARISON AND A NUMERICAL EXAMPLE

The Snedecor F_S, Leslie and Brown, F_{LB} and F_W values computed at α = 0.05 for two samples of equal sizes, are given in Table 1.

Table 1
Comparison of *F*-values for $n_1 = n_2$

Sample Sizes/ Degree of freedom	F_S	F_{LB}	F_W
(3,3)	9.28	9.392	6.308
(4,4)	6.39	6.371	3.973
(5,5)	5.05	5.149	3.161
(6,6)	4.28	4.487	2.769
(7,7)	3.79	4.070	2.510
(8,8)	3.44	3.781	2.336
(9,9)	3.18	3.568	2.216
(10,10)	2.98	3.404	2.112

The F_W values are smaller than the F_S and F_{LB} values. F_S is based on ratio of mean square of treatments to error mean square whereas F_{LB} and F_W depend on ranges.

Example: [Dougherty (1990)].

Suppose an engineer tests the abilities of three robots numbered 1, 2, and 3 to locate a box of machine parts. Each robot is tested five times. The observations are given below:

Robot	Observations					W_i	Mean
1	93	56	82	104	45	59	76.0
2	42	64	112	73	100	70	78.2
3	30	55	60	98	85	68	65.6

W_i is the range of the ith Robot giving $W_{\max} = 70$ and $W_{\min} = 59$, then $W = 70/59 = 1.68$.

The ANOVA gives $F_S = 227/705 = 0.32$ and $F_{.05}(2,12) = 3.89$. Therefore F_S does not reject H_0.

Now, $F_{LB}(3,5) = 4.018$ and $F_W(5,5,5) = 4.016$. The $MS(E) = 705$ is much larger than the $MS(Robots) = 227$ giving negative values of Robot variance component and as such F_S may not be a good test statistic. Either F_W or F_{LB} is applicable however with both of them not rejecting H_0 at $\alpha = 0.05$.

REFERENCES

1. Ahmad, M. (1974). On an alternative F-distribution. *Proceedings of Pakistan Statistical Association*, 18, 81-87.
2. Ahmad, M. (1978). *Distribution of some order statistics and its applications in testing of variance*. Proceedings of Pakistan Statistical Association, 20, 112-116.
3. Barlow, R.E., Gupta, S.S. and Panchapakesan, S. (1969). On the distribution of maximum and minimum of ratios of order statistics. *Ann. Math. Statist.*, 40(3), 918-934.
4. Benjamini, Y., Krieger, A.M. and Yekutieli, D. (2004). Adaptive linear step-up procedures that controls the false discovery rate. *Biometrika*, 2004 (revised).
5. Carbajal, Ricardo; Veerapen, Soocramanien; Couderc, Sophie; Jugie, Myriam and Ville, Yves (2003) Analgesic effect of breast feeding in term neonates: randomised controlled trial. *British Medical Journal*, 326(7379), 13.
6. Daly, J.F. (1946). On the use of the sample range in an analogue of Student's t-test. *Ann. Math. Statist.*, 17(1), 71-74.
7. Dara, Sajeela Tazeen (2007). *The F-type test based on ratio of maximum ranges to minimum ranges*. Unpublished M.Phil. dissertation, National College of Business Administration & Economics, Lahore, Pakistan.
8. Dougherty, E.R. (1990). *Probability Statistic for the Engineering, Computing and Physical Sciences*. Printic-Hall Int. Edition, Englewood Cliffs. N.J. USA.
9. Fuentes, M. (2002). Spectral methods of non-stationary spatial processes. *Biometrika*, 89(3), 197-210.
10. Gupta, R.P. and Singh, C. (1981). Distribution of the ratio of two ranges of samples from non normal populations. *Commun. Stat. Simul. Comput.*, B 10(6), 629-637.

11. Gupta, S.S., Pillai, K.C. and Steck, G.P. (1964). On the distribution of linear functions and ratios of linear functions of ordered correlated normal random variables with emphasis on range. *Biometrika*, 51(1/2), 143-151.
12. Harter, H. Leon (1963). Percentage points of the ratio of two ranges and power of the associated test. *Biometrika*, 50, 187-94.
13. Harter, H.L. (1959). The use of sample quasi-range in estimating population standard deviation. *Ann. Math. Statist.*, 30(4), 980-990.
14. Hartley, H.O. (1950). Use of range in analysis of variance. *Biometrika*, 37(3/4), 271-280.
15. Leslie, R.T. and Brown, B.M. (1966). Use of range in testing heterogeneity of variance. *Biometrika*, 53(1/2), 221-225.
16. Link, R.F. (1950). The sampling distribution of the ratio of two ranges from independent samples. *Ann. Math. Statist.*, 21(1), 112 -116.
17. Liu, J.S. (2001). *Monte Carlo Strategies in Scientific Computing*. Springer Verlag, N.Y. USA.
18. Lord, E. (1947). The use of range in place of standard deviation in the t-test. *Biometrika*, 34(1/2), 41-67.
19. McKay, A.T. and Pearson, E.S. (1933). A note on the distribution of range in samples from a normal population. *Biometrika*, 25 (3/4), 415-420.
20. Meinshausen, M. and Buhlmann, P. (2005). Lower bounds for the number of false hypotheses for multiple testing of association under general dependence structures. *Biometrika*, 92(3), 893-907.
21. Patnaik, P.B. (1950). The use of mean range as an estimator of variance in statistical tests. Biometrika, 37(1/2), 78-87.
22. Pearson, E.S. (1932). The percentage limits for the distribution of range in samples from a normal population. *Biometrika*, 24(3/4), 404-417.
23. Pearson, E.S. (1966) Alternate tests for heterogeneity of variance; some Monte Carlo results. *Biometrika*, 53(1/2), 229-234.
24. Petricoin, E.F., Ardekani, A.M. and Hitt, B.A. (2002). Use of protomic patterns in serum to identify ovarian cancer. *The Lancet*, 359, 572-577.
25. Stephenson, A.L. and Tawn, J. (2005). Exploiting occurrence time in likelihood inference for component wise maxima. *Biometrika*, 92(4), 794-807.

THE GAMMA-WEIBULL DISTRIBUTION*

Serge B. Provost[1], Abdus Saboor[2] and **Munir Ahmad[3]**

[1] Department of Statistical and Actuarial Sciences, The University of Western Ontario London, Ontario, Canada. Email: sp@uwo.ca

[2] Department of Mathematics, Kohat University of Science and Information Technology, Kohat, Pakistan. Email: saboorhangu@gmail.com

[3] National College of Business Administration & Economics, Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

An extension of the Weibull distribution which involves an additional shape parameter is being proposed. Interestingly, the additional parameter acts somewhat as a location parameter while the support of the distribution remains the positive half-line. Since the gamma distribution is a particular case of this distribution, the latter is referred to as a gamma-Weibull distribution. The gamma-Weibull distribution is in fact a reparameterization of the generalized gamma distribution, which has received little attention in recent years. Some parameters of the gamma-Weibull model have a more straightforward interpretation than those associated with the generalized gamma distribution. Moreover, the gamma-Weibull distribution does not contain a threshold parameter. Accordingly, it readily lends itself to various estimation methodologies and exhibits regular asymptotics. Numerous distributions such as the Rayleigh, half-normal and Maxwell distributions can also be obtained as special cases. The moment generating function of a gamma -Weibull random variable is derived by making use of the inverse Mellin transform technique and expressed in terms of generalized hypergeometric functions. This provides computable representations of the moment generating functions of several of the distributions that were identified as particular cases. Other statistical functions such as the cumulative distribution function of a gamma-Weibull random variable, its moments, hazard rate and associated entropy are also given in closed form. The proposed reparametrization is utilized to model two data sets. The gamma–Weibull distribution provides a better fit than the two parameter Weibull model or its shifted counterpart, as measured by the Anderson-Darling and Cramer-von Mises statistics.

KEYWORDS

Weibull distribution; Gamma distribution; Moment generating function; Inverse Mellin transform; Hazard rate; Entropy; Moments; Parameter estimation; Goodness-of-fit statistics.

1. INTRODUCTION

The Weibull distribution has been originally defined by the Swedish physicist Waloddi Weibull. He made use of it in Weibull (1939) in connection with the breaking strength of materials. Many applications in industrial quality control are discussed in Berrettoni (1964). Various distributional aspects of this distribution have been

* Published in Pak. J. Statist. (2011) Vol. 27(2).

investigated in several recent papers. For instance, products and ratios of Weibull random variables were studied by Nadarajah and Kotz (2006). Exact coverage probabilities of approximate prediction intervals for the number of failures to be observed in a future inspection of a sample were evaluated in Nordman and Meeker (2002). Hirose and Lai (1997) constructed confidence intervals for the parameters, including a location parameter, for the case of grouped data. A certain generalization of the Weibull distribution is described in Mudholkar et al. (1996) and applied to survival data. For the basic distributional properties of the Weibull distribution, techniques for the estimation of its parameters as well as numerous applications, the reader is refered to Johnson and Kotz (1976).

A reparameterization of the generalized gamma distribution called the gamma-Weibull distribution is introduced in Section 2. Unlike shifted models whose asymptotics are not regular, the maximum likelihood estimators of the gamma-Weibull model have a normal asymptotic distribution whose covariance matrix can be obtained in terms of the partial derivatives of the log likelihood function. For various results in connection with the generalized gamma distribution and some of its asymptotic properties, the reader is referred to Prentice (1974), Farewell and Prentice (1977), Smith and Naylor (1987), Evans et al. (1993), Cheng and Traylor (1995), and Meeker and Escobar (1998). Several particular cases of interest are enumerated in Section 3. Techniques for determining maximum likelihood estimates are discussed in Section 4, and the proposed model is applied to two data sets in Section 5.

The remainder of this section is devoted to the inverse Mellin transform technique, which is central to the derivation of the moment generating function of the gamma-Weibull distribution. First, the Mellin transform of a function and its inverse are defined.

If $f(x)$ is a real piecewise smooth function that is defined and single valued almost everywhere for $x > 0$ and such that $\int_0^\infty x^{k-1}\left|f(x)\right|dx$ converges for some real value k, then $M_f(s) = \int_0^\infty x^{s-1} f(x)dx$ is the Mellin transform of $f(x)$. Whenever $f(x)$ is continuous, the corresponding inverse Mellin transform is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} M_f(s)ds \tag{1.1}$$

which, together with $M_f(s)$; constitute a transform pair. The path of integration in the complex plane is called the Bromwich path where Bromwich path is a part of integration in the complex plane running from $c - i\infty$ to $c + i\infty$, where c is a real positive number chosen so that the path lies to the right of all singularities of the analytic. Equation (1.1) determines $f(x)$ uniquely if the Mellin transform is an analytic function of the complex variable s for $c_1 \le \Re(s) = c \le c_2$ where c_1 and c_2 are real numbers and $\Re(s)$ denotes the real part of s. In the case of a continuous nonnegative random variable whose density function is $f(x)$, the Mellin transform is its moment of order $(s-1)$ and the inverse Mellin transform yields $f(x)$. Letting

$$M_f(s) = \frac{\left\{\prod_{j=1}^{m} \Gamma\left(b_j + B_j s\right)\right\}\left\{\prod_{i=1}^{n} \Gamma\left(1 - a_i - A_i s\right)\right\}}{\left\{\prod_{j=m+1}^{q} \Gamma\left(1 - b_j - B_j s\right)\right\}\left\{\prod_{i=n+1}^{p} \Gamma\left(a_i + A_i s\right)\right\}}, \tag{1.2}$$

where m, n, p, q are nonnegative integers such that $0 \le n \le p,\ 1 \le m \le q,\ A_i,\ i = 1, ..., p$, $B_j,\ j = 1, ..., q,$ are positive number and $a_i, i = 1, ..., p,\ b_j,\ j = 1, ..., q$, are complex number such that $-A_i\left(b_j + v\right) \neq B_j\left(1 - a_i + \lambda\right)$ and $v, \lambda = 0, 1, 2, ...,\ j = 1, ..., m$, and $i = 1, ..., n$, the H-function can be defined as follows in terms of the inverse Mellin transform of $M_f(s)$:

$$f(x) = H_{p,q}^{m,n}\left(x \middle| \begin{matrix} (a_1, A_1), ..., (a_p, A_p) \\ (b_1, B_1), ..., (b_p, B_p) \end{matrix}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_f(x)\, x^{-s} ds, \tag{1.3}$$

where $M_f(s)$ is as defined in (1.2) and the Bromwich path $(c - i\infty, c + i\infty)$ separates the points $s = -\left(b_j + \nu\right)/B_j$, $j = 1, ..., m$, $\nu = 0, 1, 2, ...$, which are the poles of $\Gamma\left(b_j + B_j s\right)$, $j = 1, ..., m$, from the points $s = \left(1 - a_i + \lambda\right)/A_i$, $i = 1, ..., n$, $\lambda = 0, 1, 2, ...,$ which are the poles of $\Gamma\left(1 - a_i - A_i s\right)$, $i = 1, ..., n$. Thus, one must have

$$\mathfrak{Max}_{1 \le j \le m} \Re\left\{-b_j / B_j\right\} < c < \mathfrak{Min}_{1 \le i \le n} \Re\left\{1 - a_i / A_i\right\}. \qquad (1.4$$

The inverse Mellin transform approach is believed to be the only one that provides a closed form representation of the moment-generating function. If, for certain parameter values, an H-function remains positive on the entire domain, then whenever the existence conditions are satisfied, a probability density function can be generated by normalizing it. For example, the Weibull, chi-square, half–normal and F distributions can all be expressed in terms of H-functions. For the main properties of the H-function as well as applications to various disciplines, the reader is referred to Mathai and Saxena (1978) and Mathai (1993). When $A_i = B_j = 1$ for $i = 1, ..., p$ and $j = 1, ..., q$, the H-function reduces to Meijer's G-function, that is,

$$G_{p,q}^{m,n}\left(x \middle| \begin{matrix} a_1, ..., a_p \\ b_1, ..., b_p \end{matrix}\right) = H_{p,q}^{m,n}\left(x \middle| \begin{matrix} (a_1, 1), ..., (a_p, 1) \\ (b_1, 1), ..., (b_p, 1) \end{matrix}\right) \tag{1.5}$$

Moreover,

$$G_{p,q}^{m,n}\left(x \middle| \begin{matrix} a_1, ..., a_p \\ b_1, ..., b_p \end{matrix}\right) = G_{q,p}^{n,m}\left(\frac{1}{x} \middle| \begin{matrix} 1 - b_1, ..., 1 - b_q \\ 1 - a_1, ..., 1 - a_p \end{matrix}\right). \tag{1.6}$$

2. THE GAMMA-WEIBULL DISTRIBUTION

2.1 Introduction

The two-parameter Weibull density function is usually expressed as follows:

$$f_2\left(x;k,\lambda\right)=\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-\left(x/\lambda\right)^k}\ I_{\mathbb{R}^+}\left(x\right), \tag{2.1}$$

where $\lambda>0$ is a scale parameter and $k>0$ is a shape parameter. This distribution can be shifted by subtracting the location parameter η from x in the density function, which yields,

$$f_2^*\left(x;k,\lambda,\eta\right)\equiv f_2\left(x-\eta;k,\lambda\right)\ I_{(\eta,\infty)}\left(x\right). \tag{2.2}$$

This distribution will be called the shifted Weibull distribution.

We consider the reparameterized three-parameter extension,

$$f\left(x;\xi,k,\theta\right)=\frac{k\theta^{\frac{\xi}{k}+1}x^{\xi+k-1}e^{-\theta x^k}}{\Gamma\left(1+\xi/k\right)}I_{\mathbb{R}^+}\left(x\right), \tag{2.3}$$

with $\xi+k>0$ where ξ is the additional shape parameter and, referring to (2.1), $\theta=\lambda^{-k}$. Since the gamma and Weibull pdf's can both readily be obtained as particular cases, the distribution whose pdf is specified in (2.3) will be referred to as the gamma-Weibull distribution. The gamma-Weibull is a reparameterization of the generalized gamma model proposed by Stacy (1962).

The cdf of the gamma-Weibull distribution with parameters ξ,k and θ is given by

$$F\left(t;\xi,k,\theta\right)=1-\Gamma\left(1+\xi/k,t^k\theta\right)\Big/\Gamma\left(1+\xi/k\right),\ \Re\left(t^k\theta\right)>0, \tag{2.4}$$

where $\Gamma\left(a,x\right)=\int_x^\infty\ y^{a-1}e^{-y}dy$ denotes an incomplete gamma function.

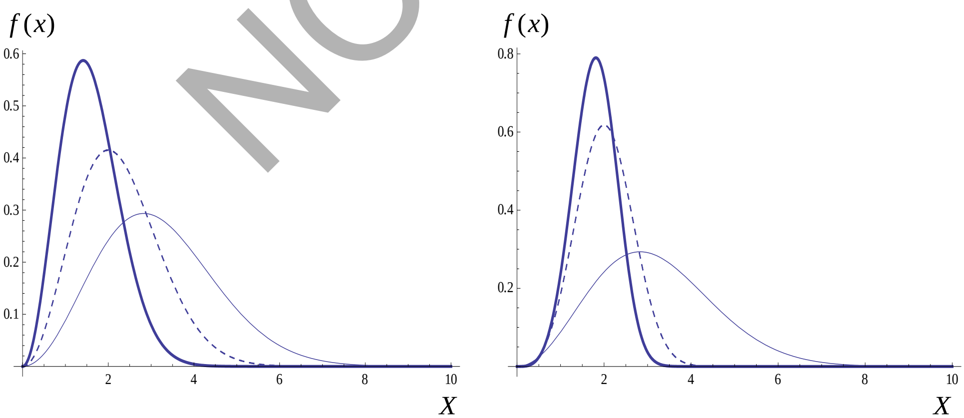


Fig. 2.1: The gamma-Weibull pdf. Left panel: $\xi=1,k=2$ and $\theta=1/2$ (thick line), $\theta=1/4$ (dashed line), $\theta=1/8$ (solid line). Right panel: $\theta=1/8,\xi=1$ and $k=3.5$ (thick line), $k=3$ (dashed line), $k=2$ (solid line).

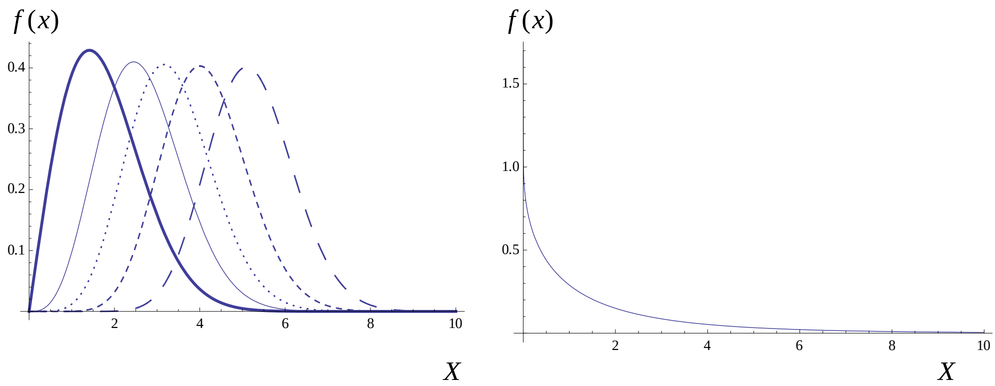


Fig. 2.2: The gamma-Weibull pdf. Left panel: $\theta = 1/4$, $k = 2$ **and** $\xi = 0$ **(thick line),** $\xi = 2$ **(solid line),** $\xi = 4$ **(dotted line),** $\xi = 7$ **(short dashes),** $\xi = 12$ **(long dashes). Right panel:** $\theta = 1/8$, $\xi = .25$ **and** $k = 0.7$**.**

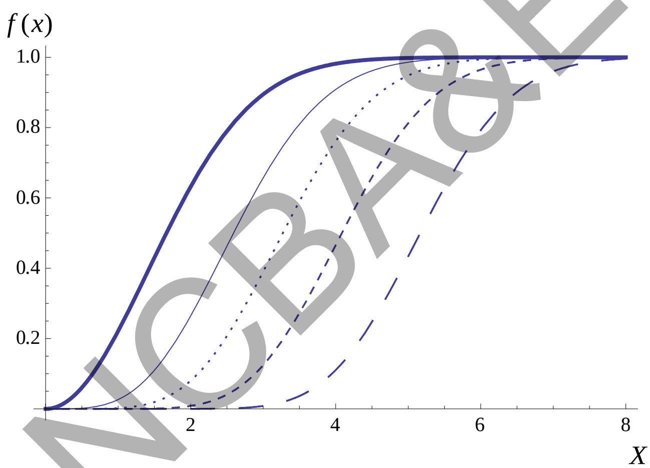


Fig. 2.3: The cdf for $\theta = 1/4$, $k = 2$ **and** $\xi = 0$ **(thick line),** $\xi = 2$ **(solid line),** $\xi = 4$ **(dotted line),** $\xi = 7$ **(short dashes),** $\xi = 12$ **(long dashes).**

The left and right panels of Figure 2.1 illustrate how the parameters θ and k effect the gamma-Weibull distribution while the left panel of Figure 2.2 as well as Figure 2.3 show the effect of the additional parameter ξ on the distribution. Interestingly, the additional shape parameter acts more or less like a location parameter while the support of the distribution still remains the positive half-line. As seen from the right panel of Figure 2.2, the pdf decreases exponentially when $0 < \xi + k < 1$.

2.2 Derivation of the Moment Generating Function

The inverse Mellin transform technique will be used to derive the moment generating function of the gamma-Weibull distribution. Let X be a random variable whose pdf is specified by (2.3). Since

$$\int_0^\infty \theta k\, x^{j+\xi+k-1} e^{-\theta x^k}\, dx = \theta^{-(j+\xi)/k}\Gamma\left(1+(j+\xi)/k\right) \equiv m(j),$$

the j^{th} raw moment of X is given by

$$\mu'_j = \frac{m(j)}{m(0)} = \frac{\Gamma\left(1+(j+\xi)/k\right)}{\theta^{j/k}\Gamma\left(1+\xi/k\right)}. \tag{2.5}$$

By definition, the moment generating function of X is

$$M_X(s) = \frac{k\theta^{\frac{\xi}{k}+1}}{\Gamma\left(1+\xi/k\right)}\int_0^\infty e^{sx} x^{\xi+k-1} e^{-\theta x^k}\, dx. \tag{2.6}$$

We now show that the integral

$$\int_0^\infty e^{sx} x^{\xi+k-1} e^{-\theta x^k}\, dx \tag{2.7}$$

is proportional to the pdf of the ratio of the random variables X_1 and X_2 whose pdf's are

$$g_1(x_1) = c_1 e^{-x_1^k}$$

and

$$g_2(x_2) = c_2 e^{s x_2} x_2^{\xi+k-2},$$

respectively, c_1 and c_2 being normalizing constants. Let $u = x_1/x_2$ and $v = x_2$ so that $x_1 = u\,v$ and $x_2 = v$, the absolute value of the Jacobian of the inverse transformation being v. Thus, the joint pdf of the random variables U and V is $v\, g_1(uv) g_2(v)$ and the marginal pdf of $U = X_1/X_2$ is

$$h_1(u) = \int_0^\infty v\, g_1(uv)\, g_2(v)\, dv,$$

that this

$$h_1(u) = c_1 c_2 \int_0^\infty e^{-(uv)^k} v\, v^{\xi+k-2} e^{sv} dv,$$

which, on letting $u = \theta^{1/k}$ and $v = x$, becomes

$$h_1\left(\theta^{1/k}\right) = c_1 c_2 \int_0^\infty e^{sx} x^{\xi+k-1} e^{-\theta x^k} dx. \tag{2.8}$$

Alternatively, the pdf of X_1/X_2 can be obtained by means of the inverse Mellin transform technique. The required moments of X_1 and X_2 are respectively

$$E\left(X_1^{t-1}\right) = c_1 \int_0^\infty x_1^{t-1} e^{-x_1^k} dx_1 = \frac{c_1}{k} \Gamma\left(t/k\right)$$

and

$$E\left(X_2^{1-t}\right) = c_2 \int_0^\infty x_2^{\xi+k-t-1} e^{s x_2} dx_2 = c_2 \left(-\frac{1}{s}\right)^{\xi+k-t} \Gamma\left(\xi+k-t\right),$$

Provided $\Re(s) < 0$ and $\Re(\xi+k-t) > 0$. Then, the inverse Mellin transform of $U = X_1 / X_2$ is

$$h_1(u) = \frac{c_1 c_2}{k(-s)^{\xi+k}} \frac{1}{2\pi i} \int_c \left(-u/s\right)^{-t} \Gamma\left(t/k\right) \Gamma\left(\xi+k-t\right) dt, \qquad (2.9)$$

where C denotes the Bromwich path described in the Introduction. Thus in terms of an H-function as defined in (1.3), one has

$$h_1(u) = \frac{c_1 c_2}{k(-s)^{\xi+k}} H_{1,1}^{1,1}\left(-\frac{u}{s}\middle| \begin{matrix}(1-\xi-k,1)\\(0,1/k)\end{matrix}\right), s < 0. \qquad (2.10)$$

Since (2.10) is equal to (2.8) when $u = \theta^{1/k}$ and the integrals in (2.8) and (2.6) are identical, it follows that the moment generating function of X is

$$M_X(s) = \frac{\theta^{\frac{\xi}{k}+1}}{\Gamma(1+\xi/k)(-s)^{\xi+k}} H_{1,1}^{1,1}\left(-\frac{\theta^{1/k}}{s}\middle| \begin{matrix}(1-\xi-k,1)\\(0,1/k)\end{matrix}\right), s < 0. \qquad (2.11)$$

Accordingly, it will be assumed that $s < 0$ in the remainder of this dissertation.

When k is rational number such that $k = p/q$, where p and $q \neq 0$ are integers, one can express the integral in (2.9) as a Meijer's G-function by letting $z = t/p$ and making use of the Gauss-Legendre multiplication formula,

$$\Gamma\left(r+qz\right) = \left(2\pi\right)^{\frac{1-q}{2}} q^{r+qz-\frac{1}{2}} \prod_{k=0}^{q-1} \Gamma\left(\frac{k+r}{q}+z\right). \qquad (2.12)$$

Then,

$$h_1(u) = \frac{c_1 c_2 q}{(-s)^{\xi+p/q}} \frac{1}{2\pi i} \int_c \left(-u/s\right)^{-pz} \Gamma\left(qz\right) \Gamma\left(\xi+p/q-pz\right) dz$$

$$= \frac{c_1 c_2 q}{(-s)^{\xi+p/q}} \frac{i}{2\pi i} \int_c \left(-u/s\right)^{-pz} \left(2\pi\right)^{\frac{1-q}{2}+\frac{1-p}{2}}$$

$$\times q^{qz-1/2} p^{\xi+p/q-pz-1/2} \left\{\prod_{j=0}^{q-1} \Gamma\left(\frac{j}{q}+z\right)\right\}$$

$$\times \left\{\prod_{i=0}^{p-1} \Gamma\left(\frac{i+\xi+p/q}{p}-z\right)\right\} dz,$$

that is,

$$h_1(u) = \frac{c_1 c_2 (2\pi)^{1-(p+q)/2} q^{1/2} p^{\xi+p/q-1/2}}{(-s)^{\xi+p/q}} \times G_{p,q}^{q,p}\left(\left(-\frac{up}{s}\right)^p q^{-q} \middle| \begin{matrix} 1-\frac{i+\xi+p/q}{p}, i=0,1,\ldots,p-1 \\ j/q, \quad j=0,1,\ldots,q-1 \end{matrix} \right), \tag{2.13}$$

which on letting $u = \theta^{q/p} \left(= \theta^{1/k}\right)$, yields

$$h_1\left(\theta^{1/k}\right) = \frac{c_1 c_2 (2\pi)^{1-(p+q)/2} q^{1/2} p^{\xi+p/q-1/2}}{(-s)^{\xi+p/q}} \times G_{p,q}^{q,p}\left(\left(-\frac{p}{s}\right)^p \left(\frac{\theta}{q}\right)^q \middle| \begin{matrix} 1-\frac{i+\xi+p/q}{p}, i=0,1,\ldots,p-1 \\ j/q, \quad j=0,1,\ldots,q-1 \end{matrix} \right). \tag{2.14}$$

Since the expression in (2.14) and (2.8) are equal when $k = p/q$, the moment generating function of the gamma-Weibull distribution as given in (2.6) can be expressed as

$$M_X(s) = \frac{(2\pi)^{1-(q+p)/2} \theta^{q\xi/p+1}}{\Gamma(1+q\xi/p)} \frac{q^{-1/2} p^{\xi+p/q+1/2}}{(-s)^{\xi+p/q}} \times G_{p,q}^{q,p}\left(\left(-\frac{p}{s}\right)^p \left(\frac{\theta}{q}\right)^q \middle| \begin{matrix} 1-\frac{i+\xi+p/q}{p}, i=0,1,\ldots,p-1 \\ j/q, \quad j=0,1,\ldots,q-1 \end{matrix} \right). \tag{2.15}$$

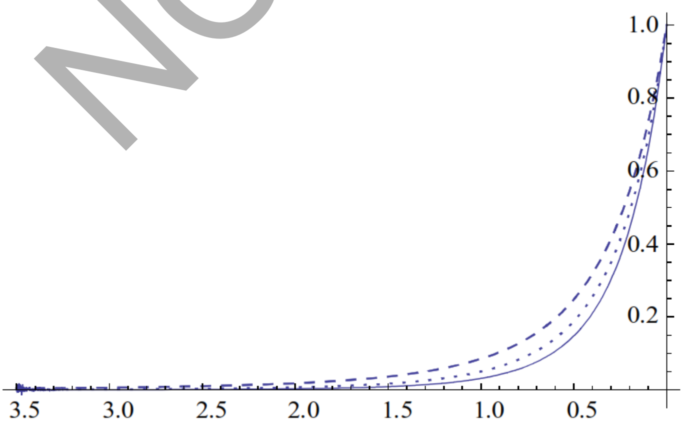


Fig. 2.4: Moment generating function for p = 2; q = 1; θ = 1/8 and ξ = 1 (short dashes), ξ = 2 (dotted line), ξ = 3 (solid line).

Equivalently, in light of (1.6), one has

$$M_X(s) = \frac{(2\pi)^{1-(q+p)/2}\,\theta^{q\xi/p+1}}{\Gamma(1+q\xi/p)}\;\frac{q^{-1/2}\,p^{\xi+p/q+1/2}}{(-s)^{\xi+p/q}}$$
$$\times G^{q,p}_{p,q}\left(\left(-\frac{p}{s}\right)^{-p}\left(\frac{\theta}{q}\right)^{-q}\middle|\begin{matrix}1-j/q, & j=0,1,\ldots,q-1\\ \frac{i+\xi+p/q}{p} & i=0,1,\ldots,p-1\end{matrix}\right). \qquad (2.16)$$

This representation, unlike that given in (2.11), can readily be evaluated by means of computational packages such as Maple or Mathematica. Letting X denote a gamma-Weibull random variable, the mgf of its shifted counterpart $Y = X - \eta$ whose pdf, referring to (2.3), is $f(x-\eta;\xi,k,\theta)$ is simply $M_Y(t) = e^{-\eta t}E(X^j)$, and its h^{th} moment is given by

$$E(X-\eta)^h = \sum_{j=0}^{\infty}\binom{h}{j}(-\eta)^{h-j}\,E(X^j).$$

We note that, in most instances, the location parameter η may not be required due to the presence of the parameter ξ .

2.3 Moments, Hazard Rate, Entropy and Mean Residue Life Function

Closed form representations of the moments of a three-parameter gamma-Weibull random variable which is denoted by X, as well as the associated hazard rate, entropy and mean residue life function are provided in this section.

i) The j^{th} raw moment X is available from (2.5). Accordingly, the mean and variance of X are

$$E(X) = \frac{\Gamma(1+(\xi+1)/k)}{\theta^{1/k}\,\Gamma(1+\xi/k)} \qquad (2.17)$$

and

$$Var(X) = \frac{\Gamma(1+\xi/k)\Gamma(1+(\xi+2)/k) - \Gamma(1+(\xi+1)/k)^2}{\theta^{2/k}\Gamma(1+\xi/k)^2}. \qquad (2.18)$$

ii) The factorial moment of X are

$$E\big(X(X-1)(X-2)\ldots(X-\gamma+1)\big) \equiv \sum_{j=0}^{\gamma-1}\phi_j\,(-1)^j\,E\left(X^{\gamma-j}\right)$$
$$= \sum_{j=0}^{\gamma-1}\phi_j\,(-1)^j\,\theta^{(\gamma-j)/k}\times\frac{\Gamma(1+(\gamma-j+\xi)/k)}{\Gamma(1+\xi/k)}. \qquad (2.19)$$

iii) The i^{th} negative moments is

$$E\left(X^{-i}\right)=\frac{\theta^{i/k}\Gamma\left(1+\left(\xi-i\right)/k\right)}{\Gamma\left(1+\xi/k\right)}, \tag{2.20}$$

provided $\Re\left(k-i+\xi\right)>0$.

iv) The hazard rate function defined as

$$Z\left(x;\xi,k,\theta\right)=\frac{f\left(x;\xi,k,\theta\right)}{\bar{F}\left(x;\xi,k,\theta\right)},$$

where $\bar{F}\left(x;\xi,k,\theta\right)=1-F\left(x;\xi,k,\theta\right)>0$ and $F\left(x;\xi,k,\theta\right)$ is the cdf given in (2.4), is

$$Z\left(x;\xi,k,\theta\right)=\frac{k\,e^{-\theta x^k}\,x^{k+\xi-1}\,\theta^{1+\xi/k}}{\Gamma\left(1+\xi/k,\theta x^k\right)}. \tag{2.21}$$

v) The mean residual life function defined as

$$K\left(x;\xi,k,\theta\right)=\frac{1}{\bar{F}\left(x;\xi,k,\theta\right)}\int_0^\infty\left(y-x\right)f\left(y\right)dy,$$

$$=\frac{\theta^{-1/k}\left(1+\left(\xi+1\right)/k,\ \theta x^k\right)}{\Gamma\left(1+\xi/k,\theta x^k\right)}-x. \tag{2.22}$$

vi) An extension of the Shannon entropy for the continuous case defined as

$$H\left(f\right)=-\int_0^\infty f\left(x\right)\log\left(f\left(x\right)\right)dx,$$

is given by

$$H\left(f\right)=-\log\left(k\right)+\left(\xi-\log\left(\theta\right)\right)/k+\log\left(\Gamma\left(1+\xi/k\right)\right)$$
$$+\left(1+\left(\xi-1\right)/k\right)\psi^{(0)}\left(1+\xi/k\right)+1, \tag{2.23}$$

where $\psi^{(0)}\left(z\right)=\Gamma'\left(z\right)/\Gamma\left(z\right)$ is the polygamma function.

3. PARTICULAR CASES OF INTEREST

Some special cases of the gamma -Weibull distribution are enumerated below. The associated moment generating functions are also provided in terms of G-functions for the two-parameter Weibull, Maxwell, Rayleigh and half-normal distributions.

i) The mgf of the two-parameter Weibull density function which can be expressed as

$$f\left(x\right)=\theta k\,x^{k-1}e^{-\theta x^k}\quad I_{\mathbb{R}^+}\left(x\right)$$

with $\theta=\lambda^{-k}$ in its original representation is

$$M_X(s)=\frac{(2\pi)^{1-(q+p)/2}\,\theta q^{-1/2}p^{p/q+1/2}}{(-s)^{p/q}}$$
$$\times G_{p,q}^{q,p}\left(\left(-\frac{p}{s}\right)^p\left(\frac{\theta}{q}\right)^q\middle|\begin{matrix}1-\frac{i+p/q}{p},\, i=0,1,\ldots,p-1\\ j/q,\quad j=0,1,\ldots,q-1\end{matrix}\right) \tag{3.1}$$

which is a special case of (2.3) and (2.15) with $\xi=0$, assuming that $k=p/q,\ q\neq 0a$ in the latter. An alternative form of the moment generating function was recently derived by Nadarajah and Kotz (2007).

ii) The Maxwell density function

$$f(x)=\frac{\sqrt{2}}{\sqrt{\pi}\phi^3}x^2e^{-x^2/(2\phi^2)}\quad I_{\mathbb{R}^+}(x)$$

and its associated moment generating function,

$$M_X(s)=\frac{2^{5/2}}{\pi(\phi s)^3}G_{1,2}^{2,1}\left(\frac{(\phi s)^2}{2}\middle|\begin{matrix}1\\3/2,2\end{matrix}\right)$$
$$=\sqrt{\frac{2}{\pi}}s\phi+e^{s^2\phi^2/2}\left(s^2\phi^2+1\right)\left(1+erf\left(\frac{s\phi}{\sqrt{2}}\right)\right) \tag{3.2}$$

where

$$erf(z)=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt\,,$$

are particular case of (2.3) and (2.16) with $\xi=1,\theta=1/\left(2\phi^2\right),k=2\quad p=2$ and $q=1$.

iii) The half-normal density function

$$f(x)=\frac{2\phi}{\pi}e^{-x^2\phi^2/\pi}\quad I_{\mathbb{R}^+}(x),\ \phi>0$$

and its associated moment generating function,

$$M_X(s)=\frac{2\phi}{\pi^{3/2}s}G_{1,2}^{2,1}\left(\frac{\pi s^2}{4\phi^2}\middle|\begin{matrix}1\\1/2,1\end{matrix}\right)\quad=\frac{e^{\pi s^2/(4\phi^2)}}{\sqrt{\pi}}\Gamma\left(\frac{1}{2},\frac{\pi s^2}{4\phi^2}\right)$$
$$=e^{4\pi s^2}\left(erf\left(2\sqrt{\pi}\,s\right)+1\right) \tag{3.3}$$

are special cases of (2.3) and (2.16) with $\xi=-1$, $\theta=\phi^2/\pi$, $k=2$, $p=2$ and $q=1$.

iv) The Rayleigh density function

$$f(x) = \frac{x e^{-x^2/\left(2a^2\right)}}{a^2} \; I_{\mathbb{R}^+}(x)$$

and its associated moment generating function,

$$M_X(s) = \frac{2}{\sqrt{\pi}\,(a,s)^2} G_{1,2}^{2,1}\left(\frac{(a\,s)^2}{2} \middle| \begin{matrix} 1 \\ 1, 3/2 \end{matrix} \right) \tag{3.4}$$

$$= \frac{a\,s\,e^{a^2 s^2/2}}{2\sqrt{2}} \Gamma\left(-\frac{1}{2}, \frac{a^2 s^2}{2} \right), \tag{3.5}$$

also turn out to be particular cases of (2.3) and (2.16) with $\xi = 0$, $\theta = 1/\left(2a^2\right)$, $k = 2$, $p = 2$ and $q = 1$. It is known that the moment generating function of the Rayleigh distribution can be expressed as

$$M_X(s) = 1 + a\,s\,e^{a^2 s^2/2} \sqrt{\frac{\pi}{2}} \left(erf\left(\frac{a\,s}{\sqrt{2}} \right) + 1 \right) \tag{3.6}$$

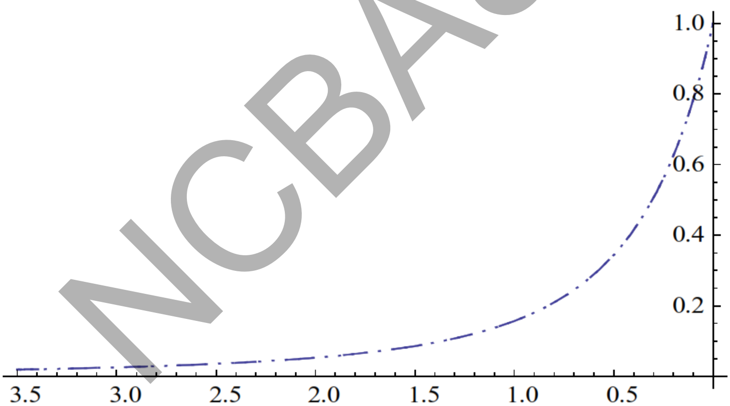


Fig. 3.1: The mgf of the Rayleigh distribution evaluated for a = 2 from (3.4) and (3.5) (dotted line) and (3.6) (long dashes)

v) The gamma distribution with density function

$$f(x) = \frac{x^{\nu-1} e^{-x/\phi}}{\phi^{\nu}\Gamma(\nu)} \; I_{\mathbb{R}^+}(x),\ \nu,\ \phi > 0,$$

is a special case of (2.3) with $\xi = \nu - 1$, $\theta = 1/\phi$ and $k = 1$, whose moment generating function is well known. It follows that the chi-square distribution with N

degrees of freedom, which is a gamma distribution with parameters $v = N/2$ and $\phi = 2$, and the exponential distribution, which is a gamma distribution with parameters $\phi > 0$ and $v = 1$ are also special cases.

It should be noted that on letting X denote a gamma-Weibull random variable with parameters ξ, θ and k, one has that θX^k is distributed as a standard gamma random variable with parameter $1 + \xi/k$, whose density function is $z^{\xi/k} e^{-z} / \Gamma(1+\xi/k) \; I_{\mathbb{R}^+}(z)$.

vi) The Erlang distribution with density function

$$f(x) = \frac{\theta^{(\xi+1)} x^{(\xi+1)-1} e^{-\theta x}}{\Gamma(\xi+1)} I_{\mathbb{R}^+}(x)$$

is also a particular case of (2.3) wherein $\xi = -1, 0, 1, 2, \ldots$ and $k = 1$.

One could also consider the symmetrized versions of the above distributions whose density functions are given by

$$f_s(x) = \frac{f(|x|)}{2} I_{\mathbb{R}}(x)$$

For instance the normal distribution whose density function is

$$f(x) = \frac{e^{-x^2/(2\phi^2)}}{\phi\sqrt{2\pi}} \; I_{\mathbb{R}}(x), \; \phi > 0,$$

is the symmetrized form of the half-normal distribution. Similarly, the double-exponential distribution can be obtained from the exponential distribution.

4. PARAMETER ESTIMATION

The maximum likelihood approach is used to estimate the parameters of the shifted Weibull and the gamma-Weibull distributions. Two goodness-of-fit measures are also defined.

4.1 Maximum Likelihood Estimation

Johnson and Kotz (1976) provided the following three equations for estimating the parameters of the shifted Weibull distribution whose pdf is specified by (2.2):

$$\hat{k} = \left(\sum_{i=1}^{n} \left(x_i - \hat{\eta}\right)^{\hat{k}} \log\left(x_i - \hat{\eta}\right) \left(\sum_{i=1}^{n} \left(x_i - \hat{\eta}\right)^{\hat{k}} \right)^{-1} - n^{-1} \sum_{i=1} \log\left(x_i - \hat{\eta}\right) \right)^{-1} \qquad (4.1)$$

$$\hat{\lambda} = \left(n^{-1} \sum_{i=1}^{n} \left(x_i - \hat{\eta}\right)^{\hat{k}} \right)^{\frac{1}{\hat{k}}} \qquad (4.2)$$

and

$$\left(\hat{k}-1\right)\sum_{i=1}^{n}\frac{1}{x_i-\hat{\eta}}=\hat{k}\,\hat{\lambda}^{-\hat{k}}\sum_{i=1}^{n}\left(x_i-\hat{\eta}\right)^{\hat{k}-1}, \tag{4.3}$$

where the x_i 's denote the observations from a random sample of size n.

When $\eta=0$, k is estimated from (4.1) and then $\hat{\lambda}$ is determined from (4.2). When $\eta\neq 0$, (4.1), (4.2) and (4.3) are solved simultaneously by making use of the symbolic computational package Mathematica with its command FindRoot. The estimates of k and λ obtained for the case $\eta=0$ can be used as initial values, either for determining the parameters of the shifted Weibull distribution or those of the gamma-Weibull distribution discussed below.

Given the independent observations $x_1,\ldots,x_n$, the loglikelihood function of the gamma-Weibull distribution is

$$\begin{aligned}\ell(\xi,k,\theta)&=\sum_{i=1}^{n}\log\left(f\left(x_i;\xi,k,\theta\right)\right)\\&=n\log(k)+n(1+\xi/k)\log(\theta)-n\log\left(\Gamma\left(1+\xi/k\right)\right)\\&\quad+(k+\xi-1)\sum_{i=1}^{n}\log\left(x_i\right)-\theta\sum_{i=1}^{n}x_i^{k},\end{aligned} \tag{4.4}$$

where $f(x;\xi,k,\theta)$ is as given in (2.3). On equating the partial derivatives of (4.4) with respect to ξ, k and θ to zero, one obtains the following equations:

$$\frac{n}{\hat{k}}\left(\log\left(\hat{\theta}\right)\right)-\psi^{(0)}\left(1+\hat{\xi}/\hat{k}\right)+\sum_{i=1}^{n}\log\left(x_i\right)=0, \tag{4.5}$$

where $\psi^{(0)}(z)=\Gamma'(z)/\Gamma(z)$,

$$\frac{n\left(\hat{k}-\hat{\xi}\log\left(\hat{\theta}\right)+\hat{\xi}\psi^{(0)}\left(1+\hat{\xi}/\hat{k}\right)\right)}{\hat{k}^2}+\sum_{i=1}^{n}\log\left(x_i\right)-\hat{\theta}\sum_{i=1}^{n}\log\left(x_i\right)x_i^{\hat{k}}=0 \tag{4.6}$$

and

$$\frac{n\left(1+\hat{\xi}/\hat{k}\right)}{\hat{\theta}}-\sum_{i=1}^{n}x_i^{\hat{k}}=0. \tag{4.7}$$

which can be solved simultaneously for $\hat{k}$, $\hat{\theta}$ and $\hat{\xi}$.

4.2 Goodness-of-Fit Statistics

The following statistics are widely utilized to determine how closely a distribution whose associated cumulative distribution function is denoted by cdf (.) fits that of a given data set:

1. The Anderson-Darling statistic given by

$$A_0^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(z_i\left(1-z_{n-i+1}\right)\right), \tag{4.8}$$

where $z_i = cdf\left(y_i\right)$, the y_i's being the ordered observations.

2. The Cramer-von Mises statistic given by

$$W_0^2 = \sum_{i=1}^{n}\left(z_i - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}. \tag{4.9}$$

The smaller these statistics are, the better is the fit. Upper tail percentiles of the asymptotic distributions of A_0^2 and W_0^2 are tabulated in Stephens (1976).

4.3 The Asymptotic Result

Even though both the gamma-Weibull and shifted Weibull have an additional parameter controlling location, the gamma-Weibull belongs to the exponential family model, while the shifted Weibull does not. Since the Cramer-Rao regularity conditions hold for the gamma-Weibull model (which is not the case for the shifted Weibull), large sample properties of the MLEs can be developed for statistical inference. For example, the MLEs have large sample normal distributions (which are useful for computing confidence regions or calibrating tests). Accordingly, asymptotically, one has

$$\sqrt{n}\begin{pmatrix}\hat{\theta}_n - \theta \\ \hat{k}_n - k \\ \hat{\xi}_n - \xi\end{pmatrix} \xrightarrow{d} \text{Normal}\left(O_3, \left[I\left(\theta, k, \xi\right)\right]^{-1}\right)$$

Letting

$$\log\left(f\left(X;\xi,k,\theta\right)\right) = -\theta x^k + \log(k) + (k+\xi-1)\log(x) + \left(\frac{\xi}{k}+1\right)\log(\theta) - \log\left(\Gamma\left(\frac{\xi}{k}+1\right)\right) \tag{4.10}$$

and

$$H(X) = \begin{pmatrix} \frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{(\partial\xi)^2} & \frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\partial\xi\partial k} & \frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\partial\xi\partial\theta} \\ \frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\partial k\partial\xi} & \frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{(\partial k)^2} & \frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\partial k\partial\theta} \\ \frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\partial\theta\partial\xi} & \frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\partial\theta\partial k} & \frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{(\partial\theta)^2} \end{pmatrix}, \tag{4.11}$$

the inverse of the asymptotic covariance matrix is $I(\xi,k,\theta) = -E\left(H\left(X\right)\right)$ with

$$\frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\left(\partial\xi\right)^2}=\frac{\psi^{(1)}\left(\frac{\xi}{k}+1\right)}{k^2},$$

$$\frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\left(\partial k\right)^2}=-\theta\log^2\left(x\right)x^k+\frac{2\xi\log\left(\theta\right)}{k_3}$$
$$-\frac{2\xi\psi^{(0)}\left(\frac{\xi}{k}+1\right)}{k^3}-\frac{\xi^2\psi^{(1)}\left(\frac{\xi}{k}+1\right)}{k^4}-\frac{1}{k^2},$$

$$\frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\left(\partial\theta\right)^2}=\frac{\frac{\xi}{k}+1}{\theta^2},$$

$$\frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\partial\xi\partial k}=\frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\partial k\partial\xi}$$
$$=\frac{\log\left(\theta\right)}{k^2}+\frac{\psi^{(0)}\left(\frac{\xi}{k}+1\right)}{k^2}+\frac{\xi\psi^{(1)}\left(\frac{\xi}{k}+1\right)}{k^3}$$

$$\frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\partial\xi\partial\theta}=\frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\partial\theta\partial\xi}=\frac{1}{k\theta}$$

$$\frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\partial k\partial\theta}=\frac{\partial^2\left(\log\left(f\left(X;\xi,k,\theta\right)\right)\right)}{\partial\theta\partial k}$$
$$=-\log\left(x\right)x^k-\frac{\xi}{k^2\theta} \qquad (4.12)$$

In practice, $-E\left(H\left(X\right)\right)$ is often estimated by differentiating the log likelihood function and substituting the MLEs in the resulting expression. This is referred to as the observed Fisher information.

5. NUMERICAL EXAMPLES

In this section we give two practical examples using well known data sets easily available in literature, i.e. the ball bearing data set and the Carcinoma data set.

5.1 The Ball Bearings Data Set

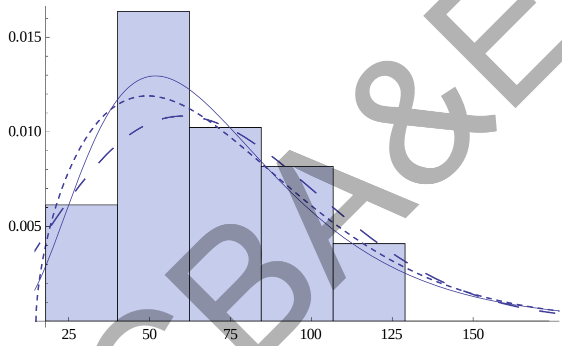
The gamma-Weibull model is applied to a data set published in Lawless (1982, p. 228) and given in Table 5.1, which consists of the number of million revolutions before failure for each of 23 ball bearings in a life testing experiment. Meeker and Escobar (1998) fitted an extended generalized gamma to the ball bearing data and determined that the best fit corresponds to a generalized gamma model.

Table 5.1
Ball Bearings Data

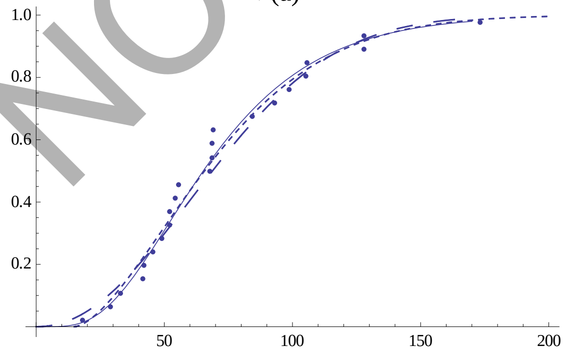
17.88	28.92	33.	41.52	42.12
45.6	48.8	51.84	51.96	54.12
55.56	67.8	68.44	68.88	84.12
93.12	98.64	105.12	105.84	105.84
127.92	128.04	173.4		

Table 5.2
Estimates of Parameters and Goodness-of-Fit Statistics for the Ball Bearings Data

Distribution	$\hat{k}$	$\hat{\theta}$	$\hat{\eta}$	$\hat{\xi}$	A_0^2	W_0^2
Two-Parameter Weibull	2.102	0.00009	-	-	0.329	0.058
Shifted Weibull	1.595	0.001	14.87	-	0.222	0.035
Gamma-Weibull	0.604	0.815	-	5.759	0.190	0.033



(a)



(b)

Fig. 5.1: (a) Two-parameter Weibull (dashed line), shifted Weibull (dotted line) and gamma-Weibull (solid line) density estimates superimposed on the histogram for the ball bearings data.
(b) Right panel: Two-parameter Weibull (short dashes), shifted Weibull (long dashes) and gamma-Weibull (solid line) cdf estimates and empirical cdf.

The pdf and cdf estimates are plotted in Figure 5.1 for the two-parameter Weibull, shifted Weibull and gamma-Weibull distributions. The estimates of the parameters are given in Table 5.2 along with the values of the goodness-of-fit statistics. Clearly, the best fit is obtained with the gamma-Weibull model.

5.2 The Carcinoma Data Set

We also apply the proposed model to a data set published in Lee and Wang (2003, Example 6.2), which is given in Table 5.3. The observations consist of the number of days elapsed until the appearance of a carcinoma in 19 rats that were painted with the carcinogen DMBA (dimethylbenz [a] anthracene).

Table 5.3
Carcinoma Data

143	164	188	188	190
192	206	209	213	216
216	220	227	230	234
244	246	265	304	

Table 5.4
Estimates of Parameters and Goodness-of-Fit Statistics for the Carcinoma Data

Distribution	$\hat{k}$	$\hat{\theta}$	$\hat{\eta}$	$\hat{\xi}$	A_0^2	W_0^2
Two-Parameter Weibull	6.260	1.61×10^{-15}	-	-	0.457	0.069
Shifted Weibull	2.849	1.725×10^{-6}	121.426	-	0.299	0.044
Gamma-Weibull	1.308	0.019	-	27.462	0.243	0.035

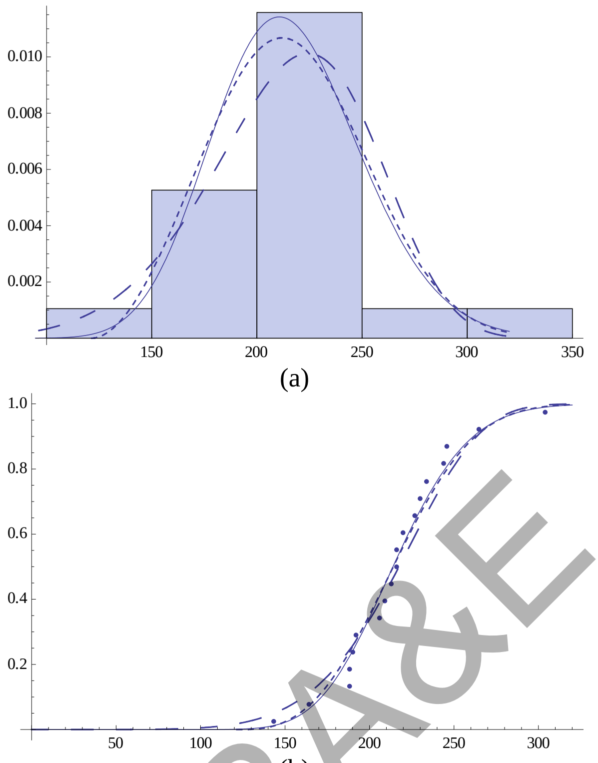


Fig. 5.2: (a) Left panel: Two-parameter Weibull (dashed line), shifted Weibull (dotted line) and gamma-Weibull (solid line) density estimates superimposed on the histogram for the carcinoma data.
(b) Right panel: Two-parameter Weibull (short dashes), shifted Weibull (long dashes) and gamma-Weibull (solid line) cdf estimates and empirical cdf.

The pdf and cdf estimates are plotted in Figure 5.2 for the two-parameter Weibull, the shifted Weibull and the gamma-Weibull distributions. The estimates of the parameters are given in Table 5.4 along with the values of the Anderson-Darling and Cramer-von Mises goodness-of-fit statistics. Once again, the gamma-Weibull model provides the best fit.

ACKNOWLEDGMENTS

The second author wishes to thank the Higher Education Commission of Pakistan and the University of Western Ontario, Canada, for providing the financial support and the facilities needed to carry out this research. The financial assistance of the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged by the first author.

REFERENCES

1. Berrettoni, J. (1964). Practical applications of the Weibull distribution. *Industrial Quality Control*, 21, 71-79.
2. Cheng, R. and Traylor, L. (1995). Non-regular maximum likelihood problems. *J. Roy. Statist. Soc.*, Series B, 57, 3-44.
3. Evans, M., Hastings, N. and Peacock, B. (1993). *Statistical Distributions*. Second Edition. Wiley, New York.
4. Farewell, V. and Prentice, R. (1977). A study of distributional shape in life testing. *Technometrics*, 19, 69-75.
5. Hirose, H. and Lai, T. (1997). Inference from grouped data in three-parameter Weibull models with applications to breakdown voltage experiments. *Technometrics*, 39, 199-210.
6. Johnson, N. and Kotz, S. (1976). *Distributions in Statistics, Continuous Univariate Distributions-1*, Wiley, New York.
7. Lawless, J. (1982). *Statistical Models and Methods for Lifetime Data*. John Wiley & Sons, New York.
8. Lee, E. and Wang, J. (2003). *Statistical Methods for Survival Data Analysis*. 3rd Edition, Wiley & Sons, New York.
9. Mathai, A. (1993). *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*. Oxford University Press, Oxford.
10. Mathai, A. and Saxena, R. (1978). *The H-Function with Applications in Statistics and Other Disciplines*. Wiley Halsted, New York.
11. Meeker, W. and Escobar, L. (1988). *Statistical Methods for Reliability Data*. Wiley, New York.
12. Mudholkar, G., Srivastava, D. and Kollia, G. (1996). A generalization of the Weibull distribution with application to the analysis of survival data. *J. Amer. Statist. Assoc.*, 91, 1575-1583.
13. Nadarajah, S. and Kotz, S. (2006). On the product and ratio of gamma and Weibull random variables. *Econometric Theory*, 22, 338-344.
14. Nadarajah, S. and Kotz, S. (2007). On the Weibull MGF. *IEEE Trans. on Commun.*, 55, 1287-1287.
15. Nordman D. and Meeker, W. (2002). Weibull prediction intervals for a future number of failures. *Technometrics*, 44, 15-23.
16. Prentice, R. (1974). A log gamma model and its maximum likelihood estimation. *Biometrika*, 61, 539-544.
17. Smith, R. and Naylor, R. (1987). A comparison of maximum likelihood and Bayesian estimators for the three parameter Weibull distribution. *Applied Statistics*, 36, 358-369.
18. Stacy, E. (1962). A generalization of the gamma distribution. *Ann. Math. Statist.*, 33, 1187-1192.
19. Stephens, M. (1976). Asymptotic results for goodness-of-fit statistics with unknown parameters. *The Ann. of Statist.*, 4, 357-359.
20. Weibull, W. (1939). A statistical theory of the strength of material. *Ingenirs Vetenskaps Akademiens Handligar Stockholm*, No. 151.

DISTRIBUTION OF MEAN OF CORRELATION COEFFICIENTS[*]

Syed Anwer Hasnain[1] and **Munir Ahmad**[2]
National College of Business Administration & Economics, Lahore, Pakistan
Email: [1]anwerleo@yahoo.com; [2]drmunir@brain.net.pk

ABSTRACT

In this paper, we derive a distribution of the mean of k-independent sample correlation coefficients each of which is based on "n" pairs of observations. This distribution has been developed by expanding the r^{th} power of the modified Bessel function by using the Taylor series. The mean distribution of sample correlation coefficients (r) has many applications in the field of medicine, accounting, stock exchange market, economics and finance. Particularly, if one is interested to find the distribution of average rate of return correlated with risk free, one may have high and/or low risk, in a multiple investment portfolios.

KEY WORDS

Bessel Function, Correlation coefficient, Modified Bessel Function, Taylor series, Characteristic function.

1. INTRODUCTION

Sample correlation is a widely discussed issue among the researchers. Correlation was explored much before the 20th century. Galton (1877) originally conceived the modern concept of correlation and regression on the basis of a problem of heredity. Galton (1877, 1888) introduced the concepts of regression and correlation and first referred to the term "correlation" and developed the product-moment correlation. Pearson (1896) published his work on correlation and regression and credited Bravais (1846) for his initial mathematical formulae of correlation. [See Weldon (1892) and Yule (1897, 1907)]. Some important features of correlation coefficient (ρ) were studied by Student (1908) who discovered that the sample correlation coefficient is symmetrically distributed about zero. The exact distribution of sample correlation was derived by Fisher (1915), who showed that the sampling distribution of Z-transformation of correlation tends to normality. Any inference about the value of ρ is equivalent to an inference about the independence between two variables when the assumption of a bivariate normal distribution holds. Hotelling (1953) derived some mathematical properties of the distribution function of sample correlation coefficient r, and improved the Z-transformation to find better approximation to the distribution of r. He gave the moments of r and Z, that are closer to the normal moments, even for the small samples.

The problem of estimating a common correlation coefficient ρ from $k \geq 2$ independent random samples drawn from normal populations were investigated by Anderson (1958), Rao (1965) and Sendecor and Cochran (1967). Donner and Rosner

[*] Published in Pak. J. Statist. (2011), Vol. 27(2).

(1980) made a comparative study of four different methods of estimating a common correlation ρ. It is usually assumed that each of the corresponding k population has a bivariate normal distribution with means (μ_{1i}, μ_{2i}) and standard deviations $(\sigma_{1i}, \sigma_{2i})$ and common correlation ρ. The usual correlation coefficient might not hold true over all pairs of observations because the means and variances in these populations are not necessarily homogeneous. Fisher's method computed the correlation coefficient r_i for each sample and converted each r_i to Z_i (Fisher's Z-transformation) which was approximately normally distributed.

Donner and Rosner (1980) developed the standard score method to estimate ρ. This method is a weighted average of correlation coefficients. This showed that the standard-score estimator, r_s, is approximately a minimum variance estimator and is also more "natural" estimator of ρ. The standard-score and Hotelling's estimators provide better estimators than Fisher's and the maximum likelihood estimators in small and moderate sized samples in terms of their relative efficiency. The standard-score estimator can also be used for cases of unequal sample sizes.

Samiuddin (1970) and Kraemer (1973) put forward function of r. Both used a test statistic which is a function of both r and ρ. It provided an appropriate test for a specified value of ρ. It offered a simple method for estimating an interval for ρ. Kreamer (1973) derived and compared various approximations to the non-null distribution of correlation coefficient.

Paul (1988) discussed the estimation and testing the significance of a common correlation coefficient. It is also called equi-familial and / or sometimes related to intra class correlation. To test the hypothesis $H_0 : \rho = 0$ *vs.* $H_1 : \rho \neq 0$, Paul (1988) used the test statistic $Z = \sum_{i=1}^{k} n_i r_i \Big/ \sqrt{N}$, where $N = \sum_{i=1}^{k} n_i$ and the statistic $Z = (z_w - \zeta_0)\sqrt{\sum_{i=1}^{k}(n_i - 3)}$, where $z_w = \sum_{i=1}^{k}(n_i - 3)Z_i \Big/ \sum_{i=1}^{k}(n_i - 3)$ and $\zeta_0 = \frac{1}{2}\ln\left[(1+\rho_0)/(1-\rho_0)\right]$ to test $H_0 : \rho \geq 0$. Paul (1988) showed that if $|\rho|$ is less than 0.5 then a better estimate of the population correlation coefficient is $r_w = \sum_{i=1}^{k}(n_i - 1)Z_i' \Big/ \sum_{i=1}^{k}(n_i - 1)$, where $Z_i' = Z_i - \left(3Z_i + r_i\right)/4(n_i - 1)$. Similarly to compare ρ at a specified value ρ_0, ($n_i - 3$) would be used instead of ($n_i - 1$). Paul (1989) also derived likelihood-ratio statistic for testing the equality of several correlation coefficients for $k \geq 2$ independent random samples from bivariate normal populations. The hypothesis of interest is $H_\circ : \rho_i = \rho,\ i = 1, 2, \ldots\ldots, k$ vs $H_1 : \rho_i \neq \rho_j$ for some $i \neq j$. The likelihood-ratio statistic has an asymptotic distribution as chi-square with (k-1) degrees of freedom and is given by $\chi_l^2 = 2\left(l_l - l_0\right) = \sum_{i=1}^{k} n_i \ln\left[\left(1-\hat{\rho}r_i\right)^2 \Big/ \left(1-r_i^2\right)\left(1-\hat{\rho}^2\right)\right]$, where $\hat{\rho}$ is the maximum-likelihood estimate of ρ, which is obtained by solving $\sum_{i=1}^{k} n_i\left(r_i-\hat{\rho}\right)\big/\left(1-r_i\hat{\rho}\right)=0$ iteratively [See Cox and Hinkley (1974)].

Bhatti (1990) developed the moment generating function of the mean distribution of correlation coefficients and found the upper tail area. Later Bhatti (1994) applied a linear regression procedure for diabetic patients to compute optimum quantity of insulin in testing the small sample mean correlation coefficient ($\bar{r}$). He used the probability density function of ($\bar{r}$) to compute the critical values of the null distribution of average sample correlation coefficient to make an efficient decision. Finally he used regression bands to assess the quantity of insulin which minimizes the risk of damage of diabetic patients. Aboukalam (1997) improved Bhatti's procedure by using robustness technique. He (1997)computed more improved quantity of insulin with minimal risk to diabetic patient. He succeeded to narrow Bhatti's confidence bands to make his estimates more efficient. However, both of them could not find the distribution of the average sample correlation coefficient $\left(\bar{r}\right)$.

2. DERIVATION OF THE DISTRIBUTION OF THE MEAN OF CORRELATION COEFFICIENTS

We extend the results to k -independent values of the sample correlation coefficients. As mentioned earlier that Bhatti's and Abukalam's works use the classical and linear regression models. Unfortunately, the results obtained by the classical and linear regression procedure are unreliable if some outlying observations are present in the data. To overcome these problems we derive a distribution of mean of correlation coefficients. This method results in the exact distribution of the correlation coefficients based on k^{th} power of modified Bessel function and provides $1-\alpha$ confidence interval as compared to the critical values of Bhatti (1994).

Bhatti (1990) derived the characteristic function of the distribution of the mean correlation coefficients ($\bar{r}$) for k independent values of r and is

$$\phi_{\bar{r}}\left(t\right)=\left[\Gamma\left[\left(n-1\right)/2\right]\quad 2^{\frac{n-3}{2}}\left(t/k\right)^{-\left(\frac{n-3}{2}\right)}\quad J_{\frac{n-3}{2}}\left(t/k\right)\right]^{k},$$

where $J_n(x)=\sum_{k=0}^{n}\frac{(-1)^k\left(x/2\right)^{n+2k}}{k!\ \Gamma\left(n+k+1\right)}$. Expressing $\phi_{\bar{r}}\left(t\right)$ in terms of modified Bessel function, by using the relation $I_\nu\left(t\right)=i^{-\nu}J_\nu(it)$ (See Gradshteyn and Ryzhik (1963)), we have the characteristic function in modified Bessel function form as;

$$\begin{aligned}\phi_{\bar{r}}\left(t\right)&=\left[2^{\nu}\Gamma\left(\nu+1\right)\right]^{k}\left(it/k\right)^{-\nu k}\left[I_\nu\left(it/k\right)\right]^{k}\\&=\left(\Gamma\left(\nu+1\right)\right)^{k}2^{\nu k}\left(it/k\right)^{-\nu k}\left[I_\nu\left(it/k\right)\right]^{k}.\end{aligned}$$

The distribution of $\bar{r}$ is symmetric as, $\varphi_{\bar{r}}(-t)=\varphi_{\bar{r}}(t)$. By using the inversion formula for characteristic function, Bhatti (1990) found the pdf of the distribution of $\bar{r}$;

$$f(\bar{r})=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\bar{r}}\varphi_{\bar{r}}(t)\,dt$$

$$=\frac{1}{\pi}\int_{0}^{\pi/2} Cos(t\bar{r})\varphi_{\bar{r}}(t)\,dt.$$

By substituting the value of $\varphi_{\bar{r}}(t)$ in the above expression we obtain

$$f(\bar{r})=\frac{\left(\Gamma(\nu+1)\right)^{k}2^{\nu k}}{\pi}\int_{0}^{\pi/2} Cos(t\bar{r})(it/k)^{-\nu k}\left[I_{\nu}(it/k)\right]^{k}dt\,. \tag{2.1}$$

We know from Bender et al. (2003) that the k[th] power of the modified Bessel function derived for an arbitrary value k is

$$\left[I_{\nu}(t)\right]^{k}=\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+\nu+1)\left[\Gamma(\nu+1)\right]^{k-1}}B_{m}^{\nu}(k)\left(t/2\right)^{2m+k\nu}. \tag{2.2}$$

The result is expressed in terms of a recursive formula for a class of polynomial where the polynomials $B_{m}^{\nu}(k)$ have the following relationship for different values of m=0, 1, 2, 3,……

$$B_{0}^{\nu}(k)=1,\ B_{1}^{\nu}(k)=k\,,\ B_{2}^{\nu}(k)=\frac{\nu+2}{\nu+1}k^{2}-\frac{1}{\nu+1}k \text{ and}$$

$$B_{3}^{\nu}(k)=\frac{(\nu+2)(\nu+3)}{(\nu+1)^{2}}k^{3}-\frac{3(\nu+3)}{(\nu+1)^{2}}k^{2}+4.\frac{1}{(\nu+1)^{2}}k\text{, and so on.}$$

By putting the value of (2.2) in (2.1), we have

$$f(\bar{r})=\frac{2^{\nu k}\left(\Gamma(\nu+1)\right)^{k}}{\pi}\int_{0}^{\pi/2}Cos(t\bar{r})(it/k)^{-\nu k}\sum_{m=0}^{\infty}\frac{B_{m}^{\nu}(k)\left((it/2k)\right)^{2m+\nu k}}{m!\Gamma(m+\nu+1)\left(\Gamma(\nu+1)\right)^{k-1}}\,\mathrm{dt}$$

$$=\frac{\Gamma(\nu+1)}{\pi}\sum_{m=0}^{\infty}\frac{(i)^{2m}B_{m}^{\nu}(k)}{m!(2k)^{2m}\Gamma(\nu+1+m)}\int_{0}^{\pi/2}t^{2m}Cos(t\bar{r})\,dt\,.$$

Let $y=t\bar{r}\,,\quad dy=\bar{r}dt$

$$f(\bar{r})=\frac{\Gamma(\nu+1)}{\bar{r}\,\pi}\sum_{m=0}^{\infty}\frac{(i)^{2m}B_{m}^{\nu}(k)}{m!(2k)^{2m}(\bar{r})^{2m}\Gamma(m+\nu+1)}\int_{0}^{\frac{\pi\bar{r}}{2}}y^{2m}Cos\,y\,dy\,. \tag{2.3}$$

Let

$$I_m = \sum_{m=0}^{\infty} \frac{(i)^{2m} B_m^{\nu}(k)}{m!(2k)^{2m}(\bar{r})^{2m}\Gamma(m+\nu+1)} \int_0^{\frac{\pi\bar{r}}{2}} y^{2m} Cos\, y\, dy\,.$$

By putting $m=0$ in the above equation, we get

$$I_0 = \frac{B_0^{\nu}(k)}{\Gamma(\nu+1)} \int_0^{\frac{\pi\bar{r}}{2}} Cos\;\; y\;\; dy$$

$$= \frac{1}{\Gamma(\nu+1)} \frac{Sin(\pi\bar{r}/2)}{\pi\bar{r}/2}, \because \;\; B_0^{\nu}(k) = 1. \tag{2.4}$$

If m = 1, we have

$$I_1 = \frac{-B_1^{\nu}(k)}{(2k\bar{r})^2\,\Gamma(\nu+2)} \int_0^{\frac{\pi\bar{r}}{2}} y^2 \;\; Cos\;\; y\;\; dy$$

$$= \frac{-k}{(2k\bar{r})^2\,\Gamma(\nu+2)} \Big[(\pi\bar{r}/2)^2 \;\; S\,\text{in}(\pi\bar{r}/2)$$

$$+\pi\bar{r}\;\; Cos\;(\pi\bar{r}/2) - 2\,Sin(\pi\bar{r}/2) \Big]. \tag{2.5}$$

If m = 2, we have

$$I_2 = \frac{B_2^{\nu}(k)}{2!(2k\bar{r})^4\,\Gamma(\nu+3)} \int_0^{\frac{\pi\bar{r}}{2}} y^4 \;\; Cos\;\; y\;\; dy\,, \tag{2.6}$$

where $B_2^{\nu}(k) = \frac{\nu+2}{\nu+1}k^2 - \frac{1}{\nu+1}k$ and

$$\int_0^{\frac{\pi\bar{r}}{2}} y^4 Cos\;\; y\;\; dy = \left(y^4\; Sin\, y + 4\,y^3\; Cos\; y - 12y^2\; Sin\, y - 24y\,Cos\; y + 24\,Sin\; y \Big|_0^{\frac{\pi\bar{r}}{2}} \right)$$

$$= (\pi\bar{r}/2)^4\; Sin(\pi\bar{r}/2) + (\pi\bar{r}/2)^3\; Cos(\pi\bar{r}/2)$$

$$-3(\pi\bar{r})^2\; Sin(\pi\bar{r}/2) - 12(\pi\bar{r})Cos(\pi\bar{r}/2) + 24\,Sin(\pi\bar{r}/2).$$

So by substituting these two values in (2.6), we have

$$I_2 = \frac{(\nu+2)k^2 - k}{2!(2k\bar{r})^4(\nu+1)\Gamma(\nu+3)} \times$$

$$\Big[(\pi\bar{r}/2)^4 \, Sin(\pi\bar{r}/2) + (\pi\bar{r}/2)^3 \, Cos(\pi\bar{r}/2)$$

$$-3(\pi\bar{r})^2 \, Sin(\pi\bar{r}/2) - 12(\pi\bar{r}) Cos(\pi\bar{r}/2) + 24 \, Sin(\pi\bar{r}/2)\Big]. \qquad (2.7)$$

Now if we take m = 3, we obtain I_3 as

$$I_3 = \frac{-B_3^{\nu}(k)}{3!(2k\bar{r})^6 \, \Gamma(\nu+4)} \int_0^{\frac{\pi\bar{r}}{2}} y^4 \; Cos \; y \; dy \, .$$

By integrating it and substituting the value of the polynomial, we have

$$I_3 = -\frac{(\nu+2)(\nu+3)k^3 - 3(\nu+3)k^2 + 4k}{3!(2k\bar{r})^6(\nu+1)^2 \, \Gamma(\nu+4)} \qquad (2.8)$$

$$\times\Big[(\pi\bar{r}/2)^6 \, Sin(\pi\bar{r}/2) + 6(\pi\bar{r}/2)^5 \, Cos(\pi\bar{r}/2)$$

$$-30(\pi\bar{r}/2)^4 \, Sin(\pi\bar{r}/2) - 120(\pi\bar{r}/2)^3 \, Cos(\pi\bar{r}/2)$$

$$+360(\pi\bar{r}/2)^2 \, Sin(\pi\bar{r}/2) + 720(\pi\bar{r}/2) Cos(\pi\bar{r}/2)$$

$$-720 \, Sin(\pi\bar{r}/2)\Big].$$

By substituting (2.4), (2.5), (2.6), (2.7) and (2.8) values in equation (2.3), we have

$$f(\bar{r}) = \frac{\Gamma(\nu+1)}{\bar{r}\,\pi}\Bigg[\frac{1}{\Gamma(\nu+1)} \frac{Sin(\pi\bar{r}/2)}{\pi\bar{r}/2}$$

$$-\frac{k}{(2k\bar{r})^2 \, \Gamma(\nu+2)}\Big\{(\pi\bar{r}/2)^2 \; S\,\text{in}(\pi\bar{r}/2)$$

$$+\pi\bar{r} \; Cos \; (\pi\bar{r}/2) - 2 \, Sin(\pi\bar{r}/2)\Big\}$$

$$+\frac{\left[(\nu+2)k^2 - k\right]}{2!(2k\bar{r})^4(\nu+1)\Gamma(\nu+3)}\Big\{(\pi\bar{r}/2)^4 \, Sin(\pi\bar{r}/2)$$

$$+(\pi\bar{r}/2)^3 \, Cos(\pi\bar{r}/2) - 3(\pi\bar{r})^2 \, Sin(\pi\bar{r}/2)$$

$$-12(\pi\bar{r}) Cos(\pi\bar{r}/2) + 24 \, Sin(\pi\bar{r}/2)\Big\} + \ldots\ldots\Bigg]$$

$$f(\bar{r}) = \frac{1}{\bar{r}\,\pi}\left[\frac{Sin(\pi\bar{r}/2)}{\pi\bar{r}/2} - \frac{k}{(2k\bar{r})^2(\nu+2)}\left\{(\pi\bar{r}/2)^2\ S\text{in}(\pi\bar{r}/2)\right.\right.$$
$$\left. + \pi\bar{r}\ Cos\ (\pi\bar{r}/2) - 2\,Sin(\pi\bar{r}/2)\right\}$$
$$+ \frac{\left[(\nu+2)k^2 - k\right]}{2!(2k\bar{r})^4(\nu+1)(\nu+2)(\nu+3)}\left\{(\pi\bar{r}/2)^4\ Sin(\pi\bar{r}/2)\right.$$
$$+ (\pi\bar{r}/2)^3\ Cos(\pi\bar{r}/2) - 3(\pi\bar{r})^2\ Sin(\pi\bar{r}/2)$$
$$\left.\left. -12(\pi\bar{r})Cos(\pi\bar{r}/2) + 24\,Sin(\pi\bar{r}/2)\right\} + \ldots\ldots \right].$$

(2.9)

Now by replacing $\upsilon = (n-3)/2$ in the above equation, we have

$$f(\bar{r}) = \frac{1}{\bar{r}\,\pi}\left[\frac{Sin(\pi\bar{r}/2)}{\pi\bar{r}/2} - \frac{k}{(2k\bar{r})^2(\nu+2)}\left\{(\pi\bar{r}/2)^2\ S\text{in}(\pi\bar{r}/2)\right.\right.$$
$$\left. + \pi\bar{r}\ Cos\ (\pi\bar{r}/2) - 2\,Sin(\pi\bar{r}/2)\right\}$$
$$+ \frac{2\left[(n+1)k^2 - 2k\right]}{(2k\bar{r})^4(n-1)(n+1)(n+3)}\left\{(\pi\bar{r}/2)^4\ Sin(\pi\bar{r}/2)\right.$$
$$+ (\pi\bar{r}/2)^3\ Cos(\pi\bar{r}/2) - 3(\pi\bar{r})^2\ Sin(\pi\bar{r}/2)$$
$$\left.\left. -12(\pi\bar{r})Cos(\pi\bar{r}/2) + 24\,Sin(\pi\bar{r}/2)\right\} + \ldots\ldots \right].$$

It is the required distribution of mean of correlation coefficients for k independent sample values.

3. CENTRAL MOMENTS OF THE DISTRIBUTION OF MEAN CORRELATION COEFFICIENTS FOR K INDEPENDENT VALUES

In the light of (2.2) taken from Bender et al. (2003), we express the characteristic function of $(\bar{r})$ as follows

$$\phi_{\bar{r}}(t) = \Gamma(\nu+1)\sum_{m=0}^{\infty}\frac{1}{m!\,\Gamma(m+\nu+1)}B_m^{\nu}(k)(it/2k)^{2m}. \qquad (3.1)$$

By expanding the above expression (3.1) over different values of m, we obtain

$$\phi_{\bar{r}}(t) = \Gamma(\nu+1)\left[\frac{1}{\Gamma(\nu+1)}B_0^\nu(k)(it/2k)^0 + \frac{1}{\Gamma(\nu+2)}B_1^\nu(k)(it/2k)^2 + \frac{1}{2!\Gamma(\nu+3)}B_2^\nu(k)(it/2k)^4 + \ldots\ldots\right]$$

$$= B_0^\nu(k) + \frac{1}{(\nu+1)}B_1^\nu(k)(it/2k)^2 + \frac{1}{2!(\nu+2)(\nu+1)}B_2^\nu(k)(it/2k)^4 + \ldots \quad (3.2)$$

By substituting the different values of m = 0, 1, …. in the polynomial $B_m^\nu(k)$, the equation (3.2) written as

$$\phi_{\bar{r}}(t) = \left[1 + \frac{1}{(\nu+1)}.k(it/2k)^2 + \frac{1}{2!(\nu+2)(\nu+1)} \times \frac{(\nu+2)k^2 - k}{\nu+1}.(it/2k)^4 + \ldots\right]$$

$$= 1 + \frac{1}{2k(\nu+1)}.\frac{(it)^2}{2!} + \frac{3[(\nu+2)k-1]}{4k^3(\nu+2)(\nu+1)^2}.\frac{(it)^4}{4!} + \ldots.. \quad (3.3)$$

The coefficient of $(it)^m / m!$ in the (3.3) expression of characteristic function will provide the mth moment about zero. Here all odd order moments are zero i.e. $\mu_1' = \mu_3' \ldots\ldots = 0$ and even orders give

$$\mu_2' = \frac{1}{2k(\nu+1)} \quad \text{and} \quad \mu_4' = \frac{3[(\nu+2)k-1]}{4k^3(\nu+2)(\nu+1)}.$$

Now the first four moments about mean are

$$\mu_2 = \frac{1}{2k(\nu+1)}.$$

When $\nu = (n-3)/2$, we obtain

$$\mu_2 = \frac{1}{2k\left(\frac{n-3}{2}+1\right)} = \frac{1}{k(n-1)}. \quad (3.4)$$

and

$$\mu_4 = \mu_4' = \frac{3[(n+2)k-2]}{k^3(n+1)(n-1)^2}.$$

The moment ratios are

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0 \text{ and } \beta_2 = 3 - \frac{6}{(n+1)k}.$$

As $\beta_1 = 0$ and for large n $\beta_2 = 3$, so the asymptotic distribution of mean correlation coefficients will have similar properties of a normal population.

REFERENCES

1. Aboukalam, M.A.F. (1997). On Robust Linear Regression of Hepatic extraction of Insulin. *Biometrical J.*, 39, 171-181.
2. Anderson, T.W. (1958). *An Introduction to Multivariate Statistical Analysis.* New York, Wiley.
3. Bender, Carl M., Brody, D.C. and Meister, B.K. (2003). On powers of Bessel functions. *J. Math. Phy.*, 44(1), 309-314.
4. Bhatti, M.I. (1994). Assessment of Hepatic Extraction of Insulin by Testing the Small Sample Mean Correlation coefficient. *Applied Math. and Compu.*, 66, 77-102.
5. Bhatti, M.I. (1990). Null distribution of the Small Sample mean Correlation Coefficient. *Biometrical J.* 32, 435-440.
6. Bhatti, M.I. (1995), Optimal Testing for Equicorrelated Linear Regression Models. *Statistical Papers*, 36(4), 299-312.
7. Bravais, A. (1846). Mathematical analysis on probabilities of errors in separation of a point. *Memories of various scholars French royal Academy and institute of Sciences*, 9, 255-332.
8. Cox, D.R., and Hinkley, D.V. (1974). *Theoretical Statistics*. Chapman and Hall, London.
9. Donner, A. and Rosner, B. (1980). On Inferences concerning a common correlation coefficient. *Applied Statistics*, 29, 69-76.
10. Fisher, R.A. (1915). Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population. *Biometrika*, 10, 507-521.
11. Galton, F. (1877). Typical Laws of Heredity in Man. *Proceeding of Royal Institute of Great Britain*, 8, 282-301. In *Encyclopedia of Statistics* Volume 1, edited by Kotz, S. and Johnson, N.L.
12. Galton, F. (1888). Correlations and their Measurement Chiefly from Anthropolotric data. *Proceeding of Royal Society of London*, 45,135-145. In *International Encyclopedia of Statistics*, Volume 1, edited by Kruskal, William H. and Tanur, Judith M.
13. Gradshteyn I.S. and Ryzhik M.I. (1963). *Tables of integrals*, Series and Products. Brox, N.Y.
14. Hotelling, Harold. (1953). New light on correlation coefficient and its transformation. *J. Roy. Statist. Soc.*, Series B, 15, No. 2, 193-232.
15. Kraemer, Helena Chmura. (1973). Improved Approximation to the Non-Null distribution of the correlation coefficient. *J. Amer. Statist. Assoc.,* 68(344), 1004-1008.
16. Paul, S.R. (1988). *Estimation and testing significance for a common correlation coefficient. Computational Statistics Theory and Methods*, A 17, 39-53. *In*

Biostatistical Analysis, 3rd Edition by Jerrold H. Zar (1996). Prentice Hall, New Jersey.

17. Paul, S.R. (1989). Test for the equality of several correlation coefficients. *The Canadian J. Statist.*, 17(2), 217-227.
18. Pearson, K. (1896). Mathematical Contributions to the Theory of Evolution. III. Regerssion, *Heredity and Panmixia. Philosophical Transactions of the Royal Society A,* 187, 253-318.
19. Rao, C.R. (1965). *Linear Statistical Inference and its Applications*, 2nd edition. New York: Wiley.
20. Samiuddin, M. (1970). On a test for an assigned value of correlation in a bivariate normal distribution. *Biometrika*, 57, 461-464.
21. Sendecor, G.W. and Cochran, W.G. (1967). *Statistical Methods*, 6th edition. Ames, Iowa: The Iowa State University Press.
22. Student (1908). Probable Error of a Correlation Coefficient. *Biometrika*, 6, 302-310.
23. Weldon, W.F.R. (1892). Certain correlated variations in crangon vulgaris *Proceedings of the Royal Society of London*. 51, 2-21. In *Encyclopedia of Statistics*, Vol. 1, Edited by Kotz, S. and Johnson, N.L.
24. Yule, G.U. (1907). On the Theory of Correlation for any number of variables, Treated by a New System of Notation. *Proceedings of the Royal Society A*, 79, 182-193.
25. Yule, G.U. (1897). On the significance of Bravais' Formulae for Regression, and Correlation, in the case of Skew Correlation. *Proceedings of the Royal Society of London*. 51, 2-21.

CHARACTERIZATION OF EXTENDED KATZ'S (EK) FAMILY VIA NEGATIVE MOMENTS*

Masood Anwar[1] and **Munir Ahmad**[2]

[1] Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan.
Email: masoodanwar@comsats.edu.pk

[2] National College of Business Administration and Economics, Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

A characterization theorem based on the recursive relation of negative moments is given for the Extended Katz's (EK) family of discrete distributions. It is found that numerous discrete probability distributions belong to EK family. The theorem is then applied to numerous discrete probability distributions, providing specific characterizations for each of them.

KEYWORDS

Characterization; Discrete distributions; Negative moments; Recursive relation.

1. INTRODUCTION

The irreversible damage to manufacturing materials is generally caused by a damage process such as fatigue, creep, fracture, corrosion, wear and aging (Jiang and Xiao 2003). Virtuoso and Vieira (2004) have discussed the creep, shrinkage, cracking and deformation of concrete flange on the basis of negative moments. The probability models corresponding to the reciprocal transformation arise in these different types of stresses. A number of authors have studied the moments of reciprocals of random variables and negative moments of positive random variables, see Chao and Straderman (1972), Kabe (1976), Kumar and Consul (1979), Jones (1987), Jones and Zhigljavsky (2004), Ahmad and Roohi (2004a, 2004b) and Anwar and Ahmed (2009). Less work is done on characterizations through negative moments. Ahmad and Z_i Roohi (2004a, 2004b, 2007) obtain negative moments of some discrete distributions in terms of hypergeometric series functions. Using the properties of hypergeometric series functions the recurrence relations between first order negative moments are obtained and used for characterization.

This paper proves a characterization theorem for the EK family of discrete probability distributions, based on recursive relations of first order negative moments. It is found that most well-known discrete distributions (see Table 1), such as binomial, negative binomial, geometric, Engset, Poisson, hyper-Poisson, Logarithmic series, Waring, Yule, geometric compound, discrete rectangular, hyper-logarithmic, hyper-negative binomial Naor's, Sundt and Jewell Family, Factorial, Poisson-Lindley distributions belong to EK family. The theorem is then applied to numerous discrete probability distributions providing specific characterizations for each of the above distributions.

* Published in Pak. J. Statist. (2012), Vol. 28(3).

Table 1
Values of α, β, γ for Discrete Distributions

Distribution	α	β	γ
Binomial	np/q	$-p/q$	1
Negative Binomial	kq	q	1
Geometric	q	q	1
Engset	np	$-p$	1
Logarithmic	θ	θ	2
Hyper-Poisson	θ	0	λ
Poisson	θ	0	1
Waring	a	1	$c+1$
Yule	1	1	$c+1$
Geometric Compound	a	1	$a+b+1$
Discrete rectangular	1	1	1
Hyper-Logarithmic	θ	θ	$\lambda+1$
Hyper-negative Binomial	kq	q	θ
Sundt and Jewell Family	$a+b$	a	1
Naor's	$-n(n-1)$	n	$-n$
Factorial	$n-\lambda+1$	1	$\lambda+1$
Poisson-Lindley	$(\theta+3)/(\theta+1)$	$1/(\theta+1)$	$\theta+2$

The article proceeds as follows. Section 2 introduces briefly the EK family that is going to be characterized. Section 3 contains a general characterization theorem based on recursive relations of first order negative moments for EK family of discrete distributions. The theorem is then applied to numerous discrete probability distributions, providing specific characterization theorems for each of them.

2. EXTENDED KATZ'S (EK) FAMILY OF DISCRETE DISTRIBUTIONS

A three-parameter family of distributions which belongs to Kemp's wide class, and which extends a two-parameter family of Katz, is investigated by Tripathi and Gurland (1977). They designated it as EK family and the recurrence relation between probabilities can be written as

$$f_{x+1}/f_x = (\alpha+\beta x)/(\gamma+x), \ \alpha > 0, \ \beta < 1, \gamma > 0, x = 0,1,2,.... \qquad (1)$$

The solution of Eq. (1) is:

$$f_x = f_0 \frac{(\alpha/\beta)_x \beta^x}{(\gamma)_x}, \ x = 0,1,2,.... \qquad (2)$$

where

$$f_o^{\,-1} = \sum_{j=0}^{\infty} \frac{(\alpha/\beta)_j (1)_j}{(\gamma)_j} \frac{\beta^j}{j!} = {}_2F_1(\alpha/\beta, 1; \gamma; \beta)$$

and

$$(a)_j = a(a+1)(a+2)\cdots(a+j-1), \quad j = 1, 2, \ldots.$$
$$(a)_0 = 1.$$

The probability generating function (*pgf*) of EK family has the form

$$G(z) = \frac{{}_2F_1(\alpha/\beta, 1; \gamma; \beta z)}{{}_2F_1(\alpha/\beta, 1; \gamma; \beta)}.$$

3. CHARACTERIZATION OF EK FAMILY

Theorem 1

A non-negative integer-valued random variable X defined over a given domain, belongs to the general family with probability mass function (pmf) f_x *(2) if and only if*

$$(1-\beta) + (\gamma - A)E(X + A - 1)^{-1} - (\gamma - 1)(A-1)^{-1} f_0 = (\alpha - \beta A)E(X + A)^{-1}, \qquad (3)$$

holds for all $A > 1$ *and* $\gamma \neq A \neq (\alpha/\beta)$,

where $f_0 = P(X = 0)$, $E(X + A)^{-1}$ *denotes the negative moment of first order,* α, β, γ *are real numbers that provide a pmf.*

Proof:

Suppose *X* follows a *pmf* (2), we have

$$E(X + A)^{-1} = f_0 \sum_{x=0}^{\infty} \frac{(\alpha/\beta)_x \beta^x}{(x + A)(\gamma)_x} = \frac{f_0}{A}\, {}_3F_2(A, \alpha/\beta, 1; A+1, \gamma; \beta), \text{ for all } A > 1, \qquad (4)$$

after replacing *A* by (*A*-1) we get

$$E(X + A - 1)^{-1} = \frac{f_0}{(A-1)}\, {}_3F_2((A-1), \alpha/\beta, 1; A, \gamma; \beta), \text{ for all } A > 1. \qquad (5)$$

Using the recurrence relation (Rainville 1960)

$$\begin{aligned}(1-z)\,{}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) = {}&{}_3F_2(\alpha_1 - 1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) \\ &+ z\left\{\frac{(\alpha_2 - \beta_1)(\alpha_3 - \beta_1)}{\beta_1(\beta_2 - \beta_1)}\, {}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1 + 1, \beta_2; z)\right. \\ &\left. - \frac{(\alpha_2 - \beta_2)(\alpha_3 - \beta_2)}{\beta_2(\beta_2 - \beta_1)}\, {}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2 + 1; z)\right\}. \qquad (6)\end{aligned}$$

On putting $\alpha_1 = A, \alpha_2 = \alpha/\beta, \alpha_3 = 1, \beta_1 = A, \beta_2 = \gamma, z = \beta$, in (6) we get;

$$
\begin{aligned}
(1-\beta)\,{}_2F_1\left(\alpha/\beta,1;\gamma;\beta\right) = {}_3F_2\left(A-1,\alpha/\beta,1;A,\gamma;z\right) \\
-\frac{(\alpha-\beta A)(A-1)}{A(\gamma-A)}\,{}_3F_2\left(A,\alpha/\beta,1;A+1,\gamma;\beta\right) \\
+\frac{(\alpha-\beta\gamma)(\gamma-1)}{\gamma(\gamma-A)}\,{}_2F_1\left(\alpha/\beta,1;\gamma+1;\beta\right).
\end{aligned} \tag{7}
$$

Again using the recurrence relation (Rainville 1960)

$$
(1-z)\,{}_2F_1(\alpha_1,\alpha_2;\beta_1;z) = {}_2F_1(\alpha_1,\alpha_2-1;\beta_1;z) - \frac{z(\beta_1-\alpha_1)}{\beta_1}\,{}_2F_1(\alpha_1,\alpha_2;\beta_1+1;z)\,. \tag{8}
$$

On putting $\alpha_1=\alpha/\beta, \alpha_2=1, \beta_1=\gamma, z=\beta$, in (8) we get;

$$
\frac{(\alpha-\beta\gamma)}{\gamma}\,{}_2F_1\left(\alpha/\beta,1;\gamma+1;\beta\right) = \left[1-(1-\beta)\,{}_2F_1\left(\alpha/\beta,1;\gamma;\beta\right)\right]. \tag{9}
$$

Putting (9) in (7) and after applying (4), (5) we get (3).

Suppose (3) holds, then we have

$$
(1-\beta)\sum_{x=0}^{\infty} f_x + (\gamma-A)\sum_{x=0}^{\infty}(x+A-1)^{-1} f_x - (\gamma-1)(A-1)^{-1} f_0 = (\alpha-\beta A)\sum_{x=0}^{\infty}(x+A)^{-1} f_x,
$$

$$
\sum_{x=1}^{\infty} f_x + (\gamma-A)\sum_{x=1}^{\infty}(x+A-1)^{-1} f_x = \alpha\sum_{x=0}^{\infty}(x+A)^{-1} f_x - \beta A\sum_{x=0}^{\infty}(x+A)^{-1} f_x + \beta\sum_{x=0}^{\infty} f_x,
$$

$$
\sum_{x=1}^{\infty} f_x + (\gamma-A)\sum_{x=1}^{\infty}(x+A-1)^{-1} f_x = \alpha\sum_{x=0}^{\infty}(x+A)^{-1} f_x + \beta\sum_{x=0}^{\infty}(x+A-A)(x+A)^{-1} f_x,
$$

$$
\sum_{x=0}^{\infty} f_{x+1} + (\gamma-A)\sum_{x=0}^{\infty}(x+A)^{-1} f_{x+1} = \sum_{x=0}^{\infty}(\alpha+\beta x)(x+A)^{-1} f_x\,,
$$

$$
\sum_{x=0}^{\infty}(x+A)^{-1}(\gamma+x) f_{x+1} = \sum_{x=0}^{\infty}(x+A)^{-1}(\alpha+\beta x) f_x\,,
$$

hence, we can get the following equivalent set of equations (for similar results, see also Osaki and Li 1988, Ahmed 1991, Kemp and Kemp 2004):

$$
(\gamma+x) f_{x+1} = (\alpha+\beta x) f_x\,,\ \ x=0,1,2,\ldots. \tag{10}
$$

its solution gives f_x.

4. SPECIAL CASES

In this section the Theorem 1 is applied to numerous discrete probability distributions by taking different values of α,β,γ and provided specific characterizations as corollaries.

1. Putting $\alpha = \frac{np}{q}, \beta = \frac{-p}{q}, \gamma = 1$, in theorem 1; we have

Corollary 3.1
A non-negative integer-valued random variable X has binomial distribution with pmf

$$f_x = \binom{n}{x} p^x q^{n-x}, x = 0,1,....,n \text{ , } 0 < p < 1, q \equiv 1-p \text{ ,}$$

if and only if

$$1 - q(A-1)E(X+A-1)^{-1} = p(n+A)E(X+A)^{-1}, \quad (3.1)$$

holds for all $A > 1$.

When $n = 1$, it reduces to *Bernoulli distribution*.

Corollary 3.1 was proved by Ahmad and Roohi (2007).

2. Putting $\alpha = kq, \beta = q, \gamma = 1$, in theorem 1; we have

Corollary 3.2
A non-negative integer-valued random variable X has negative binomial distribution with pmf

$$f_x = \binom{x+k-1}{x} p^k q^x, \quad x = 0,1,2,\cdots, 0 < p < 1, q \equiv 1-p \text{ , } k > 0 \text{ ,}$$

if and only if

$$p - (A-1)E(X+A-1)^{-1} = q(k-A)E(X+A)^{-1}, \quad (3.2)$$

holds for all $A > 1$ *and* $A \neq k$.

Corollary 3.2 was proved by Ahmad and Roohi (2007).

Put $k = 1$ in corollary 3.2; corollary 3.3 results.

Corollary 3.3
A non-negative integer-valued random variable X has geometric distribution with pmf

$$f_x = pq^x, \quad x = 0,1,2,\cdots, 0 < p < 1, q \equiv 1-p \text{ ,}$$

if and only if

$$p(1-A)^{-1} + E(X+A-1)^{-1} = qE(X+A)^{-1}, \quad (3.3)$$

holds for all $A > 1$.

Corollary 3.3 was proved by Ahmad and Roohi (2007).

3. Putting $\alpha = np, \beta = -p, \gamma = 1$, in theorem 1; we have

Corollary 3.4
A non-negative integer-valued random variable X has Engset distribution with pmf

$$f_x = \left[\sum_{x=0}^{k} \binom{n}{x} p^x \right]^{-1} \binom{n}{x} p^x, \; x = 0,1,2,.....,k \,, 0 < p < 1 \,,$$

if and only if

$$1-(A-1)E(X+A-1)^{-1} = p(n+A)E(X+A)^{-1} - p \,, \tag{3.4}$$

holds for all $A > 1$.

4. Putting $\alpha = \beta = \theta, \gamma = 2,$ in theorem 1; we have

Corollary 3.5
A non-negative integer-valued random variable X has Logarithmic series distribution with pmf

$$f_x = \frac{\theta^x}{(x+1)\,{}_2F_1\left(1,1;2;\theta\right)}, \; x = 0,1,2,....,0 < \theta < 1 \,,$$

if and only if

$$(A-2)E(X+A-1)^{-1} - (1-\theta) + (A-1)^{-1} f_0 = \theta(A-1)E(X+A)^{-1}, \tag{3.5}$$

holds for all $A > 1$ and $A \neq 2$.

Corollary 3.5 was proved by Ahmad and Roohi (2007).

5. Putting $\alpha = \theta, \beta = 0, \gamma = \lambda,$ in theorem 1; we have

Corollary 3.6
A non-negative integer-valued random variable X has hyper-Poisson distribution with pmf

$$f_x = \frac{\theta^x}{{}_1F_1(1;\lambda;\theta)(\lambda)_x}, \; (\lambda)_x = \lambda(\lambda+1).....(\lambda+x-1), \; (\lambda)_0 = 1, \lambda > 0, \theta > 0 \,,$$

if and only if

$$1+(\lambda-A)E(X+A-1)^{-1} - (\lambda-1)(A-1)^{-1} f_0 = \theta E(X+A)^{-1} \,,$$
(3.6)

holds for all $A > 1$ *and* $A \neq \lambda$.

Corollary 3.6 was proved by Ahmad and Roohi (2004b).

Put $\lambda = 1$ in corollary 3.6; corollary 3.7 results.

Corollary 3.7

A non-negative integer-valued random variable X has Poisson distribution with pmf

$$f_x = \frac{e^{-\theta}\theta^x}{x!},\ \theta > 0, x = 0,1,....,$$

if and only if

$$1-(A-1)E(X+A-1)^{-1} = \theta E(X+A)^{-1}, \tag{3.7}$$

holds for all $A > 1$.

Corollary 3.7 was proved by Ahmad and Roohi (2004b).

6. Putting $\alpha = a, \beta = 1, \gamma = c+1$ in theorem 1; we have

Corollary 3.8
A non-negative integer-valued random variable X has Waring distribution with pmf

$$f_x = \frac{(c-a)(a+x-1)!(c)!}{c(a-1)!(c+x)!},\quad c > a \geq 2,\ x = 0,1,2,....,$$

if and only if

$$(c-A+1)E(X+A-1)^{-1} - c(A-1)^{-1} f_0 = (a-A)E(X+A)^{-1}, \tag{3.8}$$

holds for all $A > 1$ *and* $(c+1) \neq A \neq a$.

Put $a = 1$ in corollary 3.8; corollary 3.9 results.

Corollary 3.9
A non-negative integer-valued random variable X has Yule distribution with pmf

$$f_x = \frac{(c-1)(x)!(c)!}{c(c+x)!},\quad c > 1,\ x = 0,1,2,....,$$

if and only if

$$(c-A+1)E(X+A-1)^{-1} - c(A-1)^{-1} f_0 = (1-A)E(X+A)^{-1}, \tag{3.9}$$

holds for all $A > 1$ *and* $A \neq (c+1)$.

7. Putting $\alpha = a, \beta = 1, \gamma = a+b+1$ in theorem 1; we have

Corollary 3.10
A non-negative integer-valued random variable X has geometric compound distribution with pmf

$$f_x = \frac{\Gamma(a+b)\ \ \Gamma(a+x)\ \ \Gamma(b+1)}{\Gamma a\ \ \Gamma b\ \ \Gamma(a+b+x+1)},\ a > 0, b > 0, x = 0,1,2,....,$$

if and only if

$$(a+b+1-A)E(X+A-1)^{-1}-(a+b)(A-1)^{-1}f_0=(a-A)E(X+A)^{-1}, \quad (3.10)$$

holds for all $A>1$ *and* $(a+b+1)\neq A\neq a$.

8. Putting $\alpha=\beta=\gamma=1$, in theorem 1; we have

Corollary 3.11
A non-negative integer-valued random variable X has discrete rectangular distribution with pmf

$$f_x=\frac{1}{(n+1)}, \quad x=0,1,2,\cdots,n,$$

if and only if

$$E(X+A-1)^{-1}=E(X+A)^{-1}, \quad (3.11)$$

holds for all $A>1$.

9. Putting $\alpha=\beta=\theta, \gamma=\lambda+1$, in theorem 1; we have

Corollary 3.12
A non-negative integer-valued random variable X has hyper-logarithmic distribution with pmf

$$f_x=\frac{\lambda!x!\theta^x}{(\lambda+x)!\,{}_2F_1(1,1;\lambda+1;\theta)}, 0<\theta<1, x=0,1,2,....,$$

if and only if

$$(1-\theta)+(\lambda+1-A)E(X+A-1)^{-1}-\lambda(A-1)^{-1}f_0=\theta(1-A)E(X+A)^{-1}, \quad (3.12)$$

holds for all $A>1$ *and* $A\neq\lambda$.

10. Putting $\alpha=kq, \beta=q, \gamma=\theta$ in theorem 1; we have

Corollary 3.13
A non-negative integer-valued random variable X has hyper-negative binomial distribution with pmf

$$f_x=\frac{(k+x+1)!(\theta-1)!q^x}{(k-1)!(\theta+x-1)!}\Bigg/\sum_{x=0}^{\infty}\frac{(k+x+1)!(\theta-1)!q^x}{(k-1)!(\theta+x-1)!}, 0<q<1, k>0, \theta>0, x=0,1,2,\cdots,$$

if and only if

$$(\theta-A)E(X+A-1)^{-1}-(\theta-1)(A-1)^{-1}f_0+p=q(k-A)E(X+A)^{-1}, \quad (3.13)$$

holds for all $A>1$ *and* $\theta\neq A\neq k$.

11. Putting $\alpha = -n(n-1), \beta = n, \gamma = -n$ in theorem 1; we have

Corollary 3.14
A non-negative integer-valued random variable X has Naor's distribution with pmf

$$f_x = \frac{(n-1)!(n-x)}{x!n^{n-x}}, \; x = 0,1,2,\ldots,n-1, \; n > 1,$$

if and only if

$$(n+A)E(X+A-1)^{-1} - (n+1)(A-1)^{-1} f_0 + (n-1) = n(n-1+A)E(X+A)^{-1}, \quad (3.14)$$

holds for all $A > 1$ *and* $-n \neq A \neq -(n-1)$.

12. Putting $\alpha = a+b, \beta = a, \gamma = 1$ in theorem 1; we have

Corollary 3.15
A non-negative integer-valued random variable X has Sundt and Jewell Family with pmf

$$f_x = \frac{\left((a+b)/a\right)_x a^x}{{}_1F_0\left((a+b)/a;-;a\right) x!}, \; x = 0,1,2,\ldots,$$

if and only if

$$(1-a) - (A-1)E(X+A-1)^{-1} = (a+b-aA)E(X+A)^{-1}, \quad (3.15)$$

holds for all $A > 1$ *and* $A \neq (a+b)/a$.

13. Putting $\alpha = n-\lambda+1, \beta = 1, \gamma = \lambda+1$ in theorem 1; we have

Corollary 3.16
A non-negative integer-valued random variable X has Factorial distribution with pmf f_x

$$f_x = \frac{\left(n-\lambda+1\right)_x}{\left(\lambda+1\right)_x \, {}_2F_1\left(n-\lambda+1, 1; \lambda+1; 1\right)}, \; x = 0,1,2,\ldots,$$

if and only if

$$(\lambda+1-A)E(X+A-1)^{-1} - \lambda(A-1)^{-1} f_0 = (n-\lambda+1-A)E(X+A)^{-1}, \quad (3.16)$$

holds for all $A > 1$ *and* $(n+1-\lambda) \neq A \neq (\lambda+1)$.

14. Putting $\alpha = (\theta+3)/(\theta+1)$, $\beta = 1/(\theta+1)$, $\gamma = (\theta+2)$ in theorem 1; we have

Corollary 3.17
A non-negative integer-valued random variable X has Poisson-Lindley distribution with pmf

$$f_x = \frac{\theta^2 (x+\theta+2)}{(\theta+1)^{x+3}},\ \theta > 0, x = 0,1,2,....,$$

if and only if

$$\theta + (\theta+1)(\theta+2-A)E(X+A-1)^{-1} - (\theta+1)^2 (A-1)^{-1} f_0 = (\theta+3-A)E(X+A)^{-1}, \tag{3.17}$$

holds for all $A > 1$ *and* $\theta + 2 \neq A \neq \theta + 3$.

REFERENCES

1. Anwar, M. and Ahmad, M. (2009). On negative moments of certain discrete distributions. *Pak. J. Statist.*, 25(2), 135-140.
2. Ahmed, A.N. (1991). Characterization of the beta, binomial and Poisson distributions. *IEEE Transactions on Reliability*, 40, 403-407.
3. Ahmad, M., and Roohi, A. (2004a). Characterization of the Poisson probability distribution. *Pak. J. Statist.*, 20(2), 301-304.
4. Ahmad, M. and Roohi, A. (2004b). Characterization of the hyper-Poisson probability function using first order negative moments. *Proceedings of International Conference on Distributional Theory, Order Statistics and Inference* in honour of Barry C. Arnold, held at University of Cantabria, Santander Spain, June 16-18.
5. Ahmad, M. and Roohi, A. (2007). Characterizations of Discrete Distributions by Negative Moments. Proceedings of *Third National Conference on Statistical Sciences and its Application to Engineering, Biostatistics, Industrial, Computer and Telecom Technology* held at FAST Lahore, Vol. 14, 283-288.
6. Chao, M.T. and Strawderman, W.E. (1972). Negative moments of positive random variables. *J. Amer. Stat. Assoc.*, 67, 429-431.
7. Jiang R., Ji, P. and Xiao, X. (2003). Aging property of unimodal failure rate models. *Reliability Engineering and System Safety*, 79, 113-116.
8. Jones, M.C. (1987). Inverse factorial moments. *Stat. Prob. Lett.*, 6, 37-42.
9. Jones, M.C. and Zhigljavsky, A.A. (2004). Approximating the negative moments of the Poisson distribution. *Stat. Prob. Lett.*, 66, 171-181.
10. Kabe, D.G. (1976). Inverse moments of discrete distributions. *Can. J. Stat.*. 4, 133-141.
11. Kemp, A.W. and Kemp, C.D. (2004). Factorial moment characterizations for certain binomial-type distributions. *Commun. Stat. Theory Meth.*, 33, 3059-3068.
12. Kumar A, and Consul, P.C. (1979). Negative moments of a modified power series distribution and bias of the maximum likelihood estimator. *Commun. Stat. Theory Meth.*, 8, 151-166.
13. Osaki, S. and Li, X. (1988). Characterization of gamma and negative binomial distributions. *IEEE Transactions on Reliability,* 37, 379-382.
14. Rainville, E.D. (1960). *Special functions*. Chelsea publication company, Bornmx, New York.
15. Tripathi, R.C. and Gurland, J. (1977). A general family of discrete distributions with Hypergeometric probabilities. *J. Roy Statist. Soc.*, B, 39(3), 349-356.
16. Virtuoso, F. and Vieira, R. (2004). Time dependent behavior of continuous composite beams with flexible connection. *J. Constructional Steel Research*, 60, 451-463.

FURTHER PROPERTIES OF A BIVARIATE GAMMA-TYPE FUNCTION AND ITS PROBABILITY DISTRIBUTION*

Abdus Saboor[1] and **Munir Ahmad**[2]

[1] Department of Mathematics, Kohat University of Science and Technology, Kohat, Pakistan. Email: dr.abdussaboor@kust.edu.pk

[2] National College of Business Administration and Economics, Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

Saboor and Ahmad [4] defined a bivariate gamma-type function involving a confluent hypergeometric function of two variables and discussed some of its mathematical properties. They also gave its probability density function and derived some of its statistical properties. Saboor *et al.* [5] generalized their probability density function by introducing one more parameter and obtained its generalized moment generating function in terms of a generalized hypergeometric function using inverse Mellin transform technique and discussed its application. In this paper, contiguous function relations and pure recurrence relationships of a bivariate gamma-type function defined by Saboor and Ahmad [4] are derived. Explicit expression for reliability for probability density function defined by Saboor *et al.* [5] is also discussed.

KEYWORDS

Hypergeometric function of two variables; Bivariate gamma-type function; Contiguous functions; Recurrence relations; Reliability.

1. INTRODUCTION

Recently, Saboor and Ahmad [4] introduced a bivariate gamma-type function involving a confluent hypergeometric function of two variables [1]:

$$\int_0^\infty\int_0^\infty x^{\lambda_1-1}y^{\lambda_2-1}\exp\left[-p\left(x^{\delta_1}+y^{\delta_2}\right)\right]\Psi_2\left(a;b,c;\alpha x^{\delta_1},\beta y^{\delta_2}\right)dxdy$$
$$=B\left(a,b,c,\alpha,\beta,\delta_1,\delta_2;\lambda_1,\lambda_2,p\right), \tag{1.1}$$

where $\text{Re}(\lambda_2)>0$, $\text{Re}(p)>0$, $\text{Re}(\delta_2)>0$, with

$$B\left(a,b,c,\alpha,\beta,\delta_1,\delta_2;\lambda_1,\lambda_2,p\right)$$
$$=\frac{\Gamma\left(\lambda_1/\delta_1\right)\Gamma\left(\lambda_2/\delta_2\right)}{\delta_1\delta_2 p^{\left(\lambda_1/\delta_1\right)+\left(\lambda_2/\delta_2\right)}}F_2\left(a;\lambda_1/\delta_1,\lambda_2/\delta_2;b,c;\alpha/p,\beta/p\right), \tag{1.2}$$

where Ψ_2 and F_2 are hypergeometric functions of two variables [1] defined as

* Published in Pak. J. Statist. (2013), Vol. 29(1).

$$\Psi_2\left(\xi,\varsigma,\rho;x,y\right)=\sum_{j,m=0}^{\infty}\frac{(\xi)_{j+m}\,x^j\,(y)^m}{(\varsigma)_j\,(\rho)_m\,j!m!}$$

and

$$F_2\left(\xi,\kappa,\tau;\varsigma,\rho;x,y\right)=\sum_{j,m=0}^{\infty}\frac{(\xi)_{j+m}(\kappa)_j(\tau)_m\,x^j\,(y)^m}{(\varsigma)_j\,(\rho)_m\,j!m!} \tag{1.3}$$

$$=(1-x)^{-\xi}\;{}_2F_1\left(\xi,\tau;\rho,\frac{y}{1-x}\right). \tag{1.4}$$

Moreover,

$$B\left(a,b,c,0,\beta,\delta_1,\delta_2;\lambda_1,\lambda_2,p\right)$$

$$=\int_0^{\infty}\int_0^{\infty}x^{\lambda_1-1}y^{\lambda_2-1}\exp\left[-p\left(x^{\delta_1}+y^{\delta_2}\right)\right]{}_1F_1\left(a;\,c;\,\beta y^{\delta_2}\right)dxdy \tag{1.5}$$

$$=\frac{\Gamma(\lambda_1/\delta_1)\Gamma(\lambda_2/\delta_2)}{\delta_1\delta_2\,p^{(\lambda_1/\delta_1)+(\lambda_2/\delta_2)}}\;{}_2F_1\left(a,\lambda_2/\delta_2\,;\,c;\,\beta/p\right), \tag{1.6}$$

and

$$B\left(a,b,c,\alpha,0,\delta_1,\delta_2;\lambda_1,\lambda_2,p\right)$$

$$=\int_0^{\infty}\int_0^{\infty}x^{\lambda_1-1}y^{\lambda_2-1}\exp\left[-p\left(x^{\delta_1}+y^{\delta_2}\right)\right]{}_1F_1\left(a;\,b;\,\alpha x^{\delta_1}\right)dxdy \tag{1.7}$$

$$=\frac{\Gamma(\lambda_1/\delta_1)\Gamma(\lambda_2/\delta_2)}{\delta_1\delta_2\,p^{(\lambda_1/\delta_1)+(\lambda_2/\delta_2)}}\;{}_2F_1\left(a,\,\lambda_1/\delta_1\,;\,b;\;\alpha/p\right), \tag{1.8}$$

where ${}_1F_1$ is confluent hypergeometric function and ${}_2F_1$ is Gauss hypergeometric function [3] defined as

$${}_1F_1\left(\xi;\,\tau;\;z\right)=\sum_{k=0}^{\infty}\frac{(\xi)_k\,z^k}{(\tau)_k\,k!},$$

and

$${}_2F_1\left(\xi,\,\tau;\rho;\;z\right)=\sum_{k=0}^{\infty}\frac{(\xi)_k\,(\tau)_k\,z^k}{(\rho)_k\,k!},\;\;|z|<1.$$

A confluent hypergeometric function ${}_1F_1\left(\xi-1;\,\tau;\,z\right)$ in which one parameter is increased or decreased by unity is called contiguous to ${}_1F_1\left(\xi;\,\tau;\,z\right)$. There are four functions contiguous to ${}_1F_1\left(\xi;\,\tau;\,z\right)$. A homogenous linear relation exists between ${}_1F_1$

and two of its contiguous functions. Here, the following contiguous function relations [6] of ${}_1F_1$ are given which will be used in next Section:

$$\tau\,{}_1F_1(\xi;\,\tau;\,z)=\tau\,{}_1F_1(\xi-1;\,\tau;\,z)+z\,{}_1F_1(\xi;\,\tau+1;\,z), \tag{1.9}$$

$$(1+\xi-\tau)\,{}_1F_1(\xi;\,\tau;\,z)=\xi\,{}_1F_1(\xi+1;\,\tau;\,z)-(\tau-1)\,{}_1F_1(\xi;\,\tau-1;\,z), \tag{1.10}$$

Similarly, a function ${}_2F_1(\xi-1,\,\tau;\rho;\,z)$ in which one parameter is increased or decreased by unity is called contiguous to ${}_2F_1(\xi,\,\tau;\rho;\,z)$. There are six functions contiguous to ${}_2F_1(\xi,\,\tau;\rho;\,z)$. Gauss proved that between ${}_2F_1$ and two of its contiguous functions, there exists a linear relation with coefficients at most linear in z. For Gauss contiguous functions, a common notation shall be used illustrated by

$$F={}_2F_1(\xi,\,\tau;\rho;\,z),$$

$$F_{\xi-1}={}_2F_1(\xi-1,\,\tau;\rho;\,z),$$

$$F_{\tau+1}={}_2F_1(\xi,\,\tau+1;\rho;\,z).$$

The few relations by Gauss which will be used in next section are:

$$(\xi-\tau)F=\xi F_{\xi+1}-\tau F_{\tau+1},$$

$$\left(2\xi-\rho+(\tau-\xi)z\right)F=\xi(1-z)F_{\xi+1}-(\rho-\xi)F_{\xi-1},$$

$$(\xi-\rho+1)F=\xi F_{\xi+1}-(\rho-1)F_{\rho-1},$$

$$(\xi+\tau-\rho)F=\xi(1-z)F_{\xi+1}-(\rho-\tau)F_{\tau-1}$$

$$(1-z)F=F_{\xi-1}-\rho^{-1}(\rho-\tau)zF_{\rho+1},$$

$$(1-z)F=F_{\tau-1}-\rho^{-1}(\rho-\xi)zF_{\rho+1},$$

$$(\tau-\xi)(1-z)F=(\rho-\xi)F_{\xi-1}-(\rho-\tau)F_{\tau-1},$$

$$\left(1-\xi+(\rho-\tau-1)z\right)F=(\rho-\xi)F_{\xi-1}-(\rho-1)(1-z)F_{\rho-1}.$$

2. CONTIGUOUS FUNCTIONS

We obtain contiguous functions relationships for the bivariate gamma-type function defined in (1.2) and (1.6) using Gauss hypergeometric function ${}_2F_1$ given in Section 1 to facilitate computation. For them we shall use a common notation illustrated by

$$B=B(a,b,c,\alpha,\beta,\delta_1,\delta_2;\lambda_1,\lambda_2,p),$$

$$B_{a-1}=B(a-1,b,c,\alpha,\beta,\delta_1,\delta_2;\lambda_1,\lambda_2,p),$$

$$B_{b-1} = B(a, b-1, c, \alpha, \beta, \delta_1, \delta_2; \lambda_1, \lambda_2, p),$$

$$B_{a-1,c-1} = B(a-1, b, c-1, \alpha, \beta, \delta_1, \delta_2; \lambda_1, \lambda_2, p),$$

$$\mathrm{B} = B(a, b, c, 0, \beta, \delta_1, \delta_2; \lambda_1, \lambda_2, p),$$

$$\mathrm{B}_{a\text{-}1} = B(a-1, b, c, 0, \beta, \delta_1, \delta_2; \lambda_1, \lambda_2, p),$$

$$\mathrm{B}_{b\text{-}1} = B(a, b-1, c, 0, \beta, \delta_1, \delta_2; \lambda_1, \lambda_2, p),$$

$$\mathrm{B}_{a\text{-}1,c\text{-}1} = B(a-1, b, c-1, 0, \beta, \delta_1, \delta_2; \lambda_1, \lambda_2, p).$$

The following relations are obtained by using (1.2), (1.4) and contiguous functions of Gauss hypergeometric function given in Section 1. They are derived by replacing ξ by a, τ by λ_2/δ_2, ρ by c, x by α/p and y by β/p.

$$(a-c+1)B = a\,B_{a+1} - (c-1)B_{c-1}, \tag{2.1}$$

$$\left(2a - c + (\lambda_2/\delta_2 - a)\frac{\beta}{p-\alpha}\right)B = a\left(1 - \frac{\beta}{p-\alpha}\right)B_{a+1} - (c-a)B_{a-1}, \tag{2.2}$$

$$(a - \lambda_2/\delta_2)B = a\,B_{a+1} - (\lambda_2/\delta_2)B_{\lambda_2/\delta_2+1},$$

$$(a + \lambda_2/\delta_2 - c)B = a\left(1 - \frac{\beta}{p-\alpha}\right)B_{a+1} - (c - \lambda_2/\delta_2)B_{\lambda_2/\delta_2-1},$$

$$\left(1 - \frac{\beta}{p-\alpha}\right)B = B_{a-1} - c^{-1}(c - \lambda_2/\delta_2)\frac{\beta}{p-\alpha}B_{c+1},$$

$$\left(1 - \frac{\beta}{p-\alpha}\right)B = B_{\lambda_2/\delta_2-1} - c^{-1}(c-a)\frac{\beta}{p-\alpha}B_{c+1},$$

$$(\lambda_2/\delta_2 - a)\left(1 - \frac{\beta}{p-\alpha}\right)B = (c-a)B_{a-1} - (c - \lambda_2/\delta_2)B_{\lambda_2/\delta_2-1},$$

$$\left(1 - a + (c - \lambda_2/\delta_2 - 1)\frac{\beta}{p-\alpha}\right)B = (c-a)B_{a-1} - (c-1)\left(1 - \frac{\beta}{p-\alpha}\right)B_{c-1}.$$

The following relations are obtained by using (1.6) and contiguous functions of Gauss hypergeometric function given in Section 1. They are derived by replacing ξ by a, τ by λ_2/δ_2 and ρ by c, z by βy^{δ_2}.

$$(a-c+1)\mathbb{B} = a\,\mathbb{B}_{a+1} - (c-1)\mathbb{B}_{c-1},$$

$$\left(2a-c+\left(\lambda_2/\delta_2-a\right)\frac{\beta}{p-\alpha}\right)\mathrm{B}=a\left(1-\frac{\beta}{p-\alpha}\right)\mathrm{B}_{a+1}-\left(c-a\right)\mathrm{B}_{a-1},$$

$$\left(a-\lambda_2/\delta_2\right)\mathrm{B}=a\mathrm{B}_{a+1}-\left(\lambda_2/\delta_2\right)\mathrm{B}_{\lambda_2/\delta_2+1},$$

$$\left(a+\lambda_2/\delta_2-c\right)\mathrm{B}=a\left(1-\frac{\beta}{p-\alpha}\right)\mathrm{B}_{a+1}-\left(c-\lambda_2/\delta_2\right)\mathrm{B}_{\lambda_2/\delta_2-1},$$

$$\left(1-\frac{\beta}{p}\right)\mathrm{B}=\mathrm{B}_{a-1}-c^{-1}\left(c-\lambda_2/\delta_2\right)\frac{\beta}{p}\mathrm{B}_{c+1},$$

$$\left(1-\frac{\beta}{p}\right)\mathrm{B}=\mathrm{B}_{\lambda_2/\delta_2-1}-c^{-1}\left(c-a\right)\frac{\beta}{p}\mathrm{B}_{c+1},$$

$$\left(\lambda_2/\delta_2-a\right)\left(1-\frac{\beta}{p}\right)\mathrm{B}=\left(c-a\right)\mathrm{B}_{a-1}-\left(c-\lambda_2/\delta_2\right)\mathrm{B}_{\lambda_2/\delta_2-1},$$

$$\left(1-a+\left(c-\lambda_2/\delta_2-1\right)\frac{\beta}{p}\right)\mathrm{B}=\left(c-a\right)\mathrm{B}_{a-1}-\left(c-1\right)\left(1-\frac{\beta}{p}\right)\mathrm{B}_{c-1}.$$

2. RECURRENCE RELATIONS

Pure recurrence relationships are derived for the bivariate gamma-type function defined in (1.2) and (1.6), using contiguous function relations of bivariate gamma-type function given in Section 2 and for (1.5), and using contiguous function relations of confluent hypergeometric function given in Section 1, to facilitate computation.

One can derive many recurrence relationships for the bivariate gamma-type function defined in (1.2), using contiguous function relations of the bivariate gamma-type function discussed in Section 2.

Eliminating B_{a+1} from (2.1) and (2.2), then replacing a by $a-1$, one obtains the following pure recurrence relationship

$$\left(1-a-\frac{\beta}{p-\alpha}\left(1+\lambda_2/\delta_2-c\right)\right)B=\left(\frac{\beta}{p-\alpha}\left(c-1\right)+1-c\right)B_{c-1}+\left(c-a\right)B_{a-1}.$$

Similarly, one can derive following recurrence relationships for the bivariate gamma-type function defined in (1.2).

$$\left(a-\lambda_2/\delta_2+\left(\lambda_2/\delta_2-a\right)\frac{\beta}{p-\alpha}\right)B=\left(a-c\right)B_{a-1}-\left(\lambda_2/\delta_2-c\right)B_{\left(\lambda_2/\delta_2\right)-1},$$

$$\left(1-\lambda_2/\delta_2-\frac{\beta}{p-\alpha}(a-c+1)\right)B=\left(1-c+\frac{\beta}{p-\alpha}(c-1)\right)B_{c-1}+(\lambda_2/\delta_2-c)B_{(\lambda_2/\delta_2)-1},$$

$$(\lambda_2/\delta_2-1)B=(c-1)B_{c-1,(\lambda_2/\delta_2)-1}+(\lambda_2/\delta_2-c)B_{(\lambda_2/\delta_2)-1},$$

$$\left(\frac{\beta}{p-\alpha}-1\right)(\lambda_2/\delta_2-1)B=\left(\frac{\beta}{p-\alpha}-a\right)B_{a-1,(\lambda_2/\delta_2)-1}$$
$$-\left(c-\lambda_2/\delta_2-a+1+2(\lambda_2/\delta_2-a)\frac{\beta}{p-\alpha}\right)B_{(\lambda_2/\delta_2)-1},$$

$$\left(\frac{\beta}{p-\alpha}-1\right)(\lambda_2/\delta_2-1)B=(c-\lambda_2/\delta_2+1)B_{(\lambda_2/\delta_2)-2}$$
$$-\left(c-2(\lambda_2/\delta_2)+2-(a-\lambda_2/\delta_2+1)\frac{\beta}{p-\alpha}\right)B_{(\lambda_2/\delta_2)-1},$$

$$(\lambda_2/\delta_2-a)\left(1-\frac{\beta}{p-\alpha}\right)B=(c-a)B_{a-1}+(\lambda_2/\delta_2-c)B_{(\lambda_2/\delta_2)-1},$$

$$\left(\lambda_2/\delta_2-2a-1-(c-a-1)\frac{\beta}{p-\alpha}\right)B=(\lambda_2/\delta_2-c)B_{(\lambda_2/\delta_2)-1}+(1-c)\frac{\beta}{p-\alpha}B_{c-1}.$$

Similarly, using same procedure discussed above, one can get following recurrence relationships for the bivariate gamma-type function defined in (1.6).

$$\left(1-a-\frac{\beta}{p}(1+\lambda_2/\delta_2-c)\right)\mathbb{B}=\left(\frac{\beta}{p}(c-1)+1-c\right)\mathbb{B}_{c-1}+(c-a)\mathbb{B}_{a-1}.$$

$$\left(a-\lambda_2/\delta_2+(\lambda_2/\delta_2-a)\frac{\beta}{p}\right)\mathbb{B}=(a-c)\mathbb{B}_{a-1}-(\lambda_2/\delta_2-c)\mathbb{B}_{(\lambda_2/\delta_2)-1},$$

$$\left(1-\lambda_2/\delta_2-\frac{\beta}{p}(a-c+1)\right)\mathbb{B}=\left(1-c+\frac{\beta}{p}(c-1)\right)\mathbb{B}_{c-1}+(\lambda_2/\delta_2-c)\mathbb{B}_{(\lambda_2/\delta_2)-1},$$

$$(\lambda_2/\delta_2-1)\mathbb{B}=(c-1)\mathbb{B}_{c-1,(\lambda_2/\delta_2)-1}+(\lambda_2/\delta_2-c)\mathbb{B}_{(\lambda_2/\delta_2)-1},$$

$$\left(\frac{\beta}{p}-1\right)(\lambda_2/\delta_2-1)\mathbb{B}=\left(\frac{\beta}{p}-a\right)\mathbb{B}_{a-1,(\lambda_2/\delta_2)-1}$$
$$-\left(c-\lambda_2/\delta_2-a+1+2(\lambda_2/\delta_2-a)\frac{\beta}{p}\right)\mathbb{B}_{(\lambda_2/\delta_2)-1},$$

$$\left(\frac{\beta}{p}-1\right)\left(\lambda_2/\delta_2-1\right)\mathrm{B}=\left(c-\lambda_2/\delta_2+1\right)\mathrm{B}_{(\lambda_2/\delta_2)-2}$$
$$-\left(c-2\left(\lambda_2/\delta_2\right)+2-\left(a-\lambda_2/\delta_2+1\right)\frac{\beta}{p}\right)\mathrm{B}_{(\lambda_2/\delta_2)-1},$$

$$\left(\lambda_2/\delta_2-a\right)\left(1-\frac{\beta}{p}\right)\mathrm{B}=\left(c-a\right)\mathrm{B}_{a-1}+\left(\lambda_2/\delta_2-c\right)\mathrm{B}_{(\lambda_2/\delta_2)-1},$$

$$\left(\lambda_2/\delta_2-2a-1-\left(c-a-1\right)\frac{\beta}{p}\right)\mathrm{B}=\left(\lambda_2/\delta_2-c\right)\mathrm{B}_{(\lambda_2/\delta_2)-1}+\left(1-c\right)\frac{\beta}{p}\mathrm{B}_{c-1}.$$

We also obtain recurrence relationships for the bivariate gamma-type function defined in (1.5). Substituting the value of ${}_1F_1$ from (1.9) in (1.5) and replacing ξ by a, τ by c, z by βy^{δ_2}, one obtains

$$\mathrm{B}=\int_0^\infty\int_0^\infty x^{\lambda_1-1}y^{\lambda_2-1}\exp\left[-p\left(x^{\delta_1}+y^{\delta_2}\right)\right]{}_1F_1\left(a-1;\,c;\,\beta y^{\delta_2}\right)dxdy$$
$$+\frac{\beta}{c}\int_0^\infty\int_0^\infty x^{\lambda_1-1}y^{\lambda_2+\delta_2-1}\exp\left[-p\left(x^{\delta_1}+y^{\delta_2}\right)\right]{}_1F_1\left(a;\,c+1;\,\beta y^{\delta_2}\right)dxdy$$
$$=\mathrm{B}_{a-1}+\frac{\beta}{c}\frac{\Gamma\left(\lambda_1/\delta_1\right)\Gamma\left(1+\lambda_2/\delta_2\right)}{\delta_1\delta_2 p^{1+(\lambda_1/\delta_1)+(\lambda_2/\delta_2)}}\,{}_2F_1\left(a,\,\lambda_2/\delta_2+1;\,c+1;\,\beta/p\right).$$

Simplifying and then replacing λ_2/δ_2 by $\left(\lambda_2/\delta_2\right)-1$, c by $c-1$ in second term of right hand side of (2.1), one gets

$$\left(\lambda_2-\delta_2\right)\beta\mathrm{B}=\left(c-1\right)p\delta_2\left(\mathrm{B}_{(\lambda_2/\delta_2)-1,c-1}-\mathrm{B}_{a-1,(\lambda_2/\delta_2)-1}\right).$$

Similarly, substituting the value of ${}_1F_1$ from (1.10) in (1.5) and replacing ξ by a, τ by c, z by βy^{δ_2}, one can have

$$\left(a-1\right)\mathrm{B}=\left(a-c\right)\mathrm{B}_{a-1}+\left(a-2\right)\mathrm{B}_{a-1,c-1}.$$

Using the methods discussed above, one can also obtain recurrence relationships for the bivariate gamma-type function defined in (1.7) and (1.8).

4. RELIABILITY

Bivariate gamma distributions arise as tractable ’lifetime’ models in many areas, including telecommunications, reliability engineering, extreme value theory, failure analysis, life testing, industrial engineering (manufacturing times and distribution process), risk management (probability of ruin) and queuing systems. *In the context of* reliability, the stress-strength model describes the life of a component which has a random strength Y and is subjected to a random stress X. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $Y > X$. Thus, $R = \Pr(X < Y)$ is a measure of the component reliability. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures, and the aging of concrete pressure vessels. e.g. If X represents the maximum chamber pressure generated by ignition of a solid propellant and Y represents the strength of the rocket chamber, then R is the probability of successful firing of the engine. For further details reader is referred to Nadarajah [2].

Saboor *et al.* [3] defined following probability density function

$$f(x,y) = C\, x^{\lambda_1 - 1} y^{\lambda_2 - 1} \exp\left[-p_1 x^{\delta_1} - p_2 y^{\delta_2}\right] \Psi_2\left(a;b,c;\alpha x^{\delta_1},\beta y^{\delta_2}\right) dxdy\;\; I_{R^+\times R^+}(x,y), \tag{4.1}$$

where,

$$C^{-1} = \frac{\Gamma(\lambda_1/\delta_1)\Gamma(\lambda_2/\delta_2)}{\delta_1\delta_2\, p_1^{\lambda_1/\delta_1}\, p_2^{\lambda_2/\delta_2}} F_2\left(a;\lambda_1/\delta_1,\lambda_2/\delta_2;b,c;\alpha/p_1,\beta/p_2\right) \tag{4.2}$$

and Ψ_2 and F_2 are defined in Section 1.

Reliability for a bivariate distribution can be calculated by following formula

$$R = \int_0^\infty \int_x^\infty f(x,y)\, dy\, dx. \tag{4.3}$$

$$R = C\int_0^\infty \int_x^\infty x^{\lambda_1 - 1} y^{\lambda_2 - 1} \exp\left[-p_1 x^{\delta_1} - p_2 y^{\delta_2}\right] \Psi_2\left(a;b,c;\alpha x^{\delta_1},\beta y^{\delta_2}\right) dydx$$

$$= C\int_0^\infty \int_x^\infty x^{\lambda_1 - 1} y^{\lambda_2 - 1} \exp\left[-p_1 x^{\delta_1} - p_2 y^{\delta_2}\right] \sum_{j,m=0}^{\infty} \frac{(a)_{j+m}\left(\alpha x^{\delta_1}\right)^j\left(\beta y^{\delta_2}\right)^m}{(b)_j\,(c)_m\; j!m!} dydx$$

$$= C\sum_{j,m=0}^{\infty} \frac{(a)_{j+m}\,\alpha^j\beta^m}{(b)_j\,(c)_m\; j!m!} \int_0^\infty x^{\lambda_1+\delta_1 j-1} e^{-p_1 x^{\delta_1}} \int_x^\infty y^{\lambda_2+\delta_2 m-1} e^{-p_2 y^{\delta_2}}\, dy\, dx. \tag{4.4}$$

Since

$$\int_x^\infty y^{\lambda_2+\delta_2 m-1} e^{-p_2 y^{\delta_2}}\, dy = \frac{1}{\delta_2}\, p_2^{-m-\lambda_2/\delta_2}\, \Gamma\left(m+\lambda_2/\delta_2,\, p_2\, x^{\delta_2}\right). \tag{4.5}$$

Substituting (4.5) in (4.4), we obtain,

$$R = \frac{C}{\delta_2\, p_2^{\lambda_2/\delta_2}} \sum_{j,m=0}^{\infty} \frac{(a)_{j+m}\,\alpha^j\left(\beta/p_2\right)^m}{(b)_j\,(c)_m\; j!m!} \int_0^\infty x^{\lambda_1+\delta_1 j-1} e^{-p_1 x^{\delta_1}}\; \Gamma\left(m+\lambda_2/\delta_2,\, p_2\, x^{\delta_2}\right) dx. \tag{4.6}$$

For $\delta_1 = \delta_2 = 1$, (4.6) becomes,

$$R = \frac{p_1^{\lambda_1} p_2^{\lambda_2}}{\Gamma(\lambda_1)\Gamma(\lambda_2) F_2\left(a;\lambda_1,\lambda_2;b,c;\alpha/p_1,\beta/p_2\right)} \sum_{j,m=0}^{\infty} \frac{(a)_{j+m}\,\alpha^j (\beta/p_2)^m}{(b)_j (c)_m\, j!m!}$$

$$\times \int_0^{\infty} x^{\lambda_1+j-1} e^{-p_1 x}\, \Gamma(m+\lambda_2, p_2 x)\, dx.$$

$$= \frac{p_1^{\lambda_1} p_2^{\lambda_2}}{\Gamma(\lambda_1)\Gamma(\lambda_2) F_2\left(a;\lambda_1,\lambda_2;b,c;\alpha/p_1,\beta/p_2\right)} \sum_{j,m=0}^{\infty} \frac{(a)_{j+m}\,\alpha^j (\beta/p_2)^m}{(b)_j (c)_m\, j!m!}$$

$$\times \frac{\Gamma(j+m+\lambda_1+\lambda_2)}{(j+\lambda_1)}\, {}_2F_1\left(j+\lambda_1, j+m+\lambda_1+\lambda_2; j+\lambda_1+1; -p_1/p_2\right). \quad (4.7)$$

REFERENCES

1. Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. (1953).*Higher Transcendental Functions Vol. I*. McGraw-Hill Book Co, New York.
2. Nadarajah, S. (2005). Reliability for some bivariate gamma distributions. *Mathematical Problems in Engineering*, 2, 151-163.
3. Rainville, E.D. (1960). *Special Functions*, the Macmillan Company, New York.
4. Saboor, A. and Ahmad, M. (2012). Bivariate gamma-type probability function using a hypergeometric function of two variables. *Pak. J. Statist.*, 28(1), 81-91.
5. Saboor, A. Provost, S.B. and Ahmad, M. (2012). The moment generating function of a bivariate gamma-type distribution. *Appl. Math. Comput.*, 218(24), 11911-11921.
6. Slater, L.J. (1960). *Confluent Hypergeometric Functions*, Cambridge Univ. Press, Cambridge.

DISCRETE INVERSE RAYLEIGH DISTRIBUTION*

Tassaddaq Hussain[1,2] and **Munir Ahmad**[2]

[1] Department of Statistics, Government Post Graduate College, Rawalakot (AJK), Pakistan. Email: taskho2000@yahoo.com

[2] National College of Business Administration and Economics, Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

Rayleigh distribution is one of the well-known continuous distribution developed by Lord Rayleigh and J.W. Strutt (1880, 1919) used in modeling lifetime data. A reciprocal transformation of Rayleigh variable generates inverse Rayleigh distribution derived by Voda (1972) which is also being used in lifetime experiments. While keeping in mind the famous of Rayleigh and inverse Rayleigh distribution, we hereby proposed the discrete version of continuous inverse Rayleigh distribution by adopting the simple approach and presented as an appropriate lifetime model for discrete data. Non-monotonicity of the hazard function of discrete Inverse Rayleigh distribution is studied, suitability of the model in over dispersed data is highlighted with real lifetime data examples, basic mathematical properties, order statistics and characterization issues of the model are also presented.

KEY WORDS

Inverse Rayleigh distribution; reliability parameters; negative moments; discretized version; generating functions.

1. INTRODUCTION

Generally one associates the lifetime of the product with continuous non-negative lifetime distributions however, in some situations lifetime can be best described through non-negative integer valued random variables e.g. life of a switch is measured by the number of strokes, life of equipment is measured by the number of cycles it completes or the number of times it is operated prior to failure, life of a weapon is measured by the number of rounds fired until failure, number of years of a married couple successfully completed. Although continuous lifetime distributions are playing their roles in reliability analysis very well, yet in certain scenarios, when measured data is discrete and realized from continuous set up, researchers are trying to search out a proper alternate. For this purpose they developed discretized version of continuous lifetime distributions. This development is generally based on discrete lifetime phenomena which are expressed through grouping or finite precision measurement of continuous time phenomena. Such discretized versions are too much functional in small set of discrete type data and have applications in reliability theory in situation where clock time is not an appropriate scale for measuring the reliability of the product. The discretization approaches have expanded the scope of reliability modeling and provides methods for approximating integrals coming out of the continuous phenomena. The reliability of the discrete or counted data

* Published in Pak. J. Statist. (2014), Vol. 30(2).

is measured on the bases of success or failure and number of successes or failures are modeled through binomial, negative binomial, geometric and Poisson distributions. Such models yield imprecise results for small samples. However with the initiation of discretization approach, the discrete success-failure data based on small and to some extent large samples can efficiently be modeled through discretized version of continuous lifetime distributions.

The discrete success-failure data is realized from continuous set up in two common situations (i) a product is scrutinized only once a time period i.e. a day, an hour and a month etc. and observation is made on the number of time period successfully completed prior to failure of the product (ii) an equipment operates in cycles and researcher observes the number of cycles successfully completed prior to failure of the device. If the observed data values are very large e.g. thousands of revolutions, cycles, blows etc. then for modeling such a data it is better to use a continuous counterpart. In order to get an appropriate model for the success-failure data various discretizing approaches exist in the literature which are (i) Moment equalization approach (ii) Discrete concentration approach (iii) Failure rate approach (see Roy and Ghosh, 2009) (iv) Discrete differential equation approach (see Sreehari, 2008) (v) Time discretization approach. Due to these approaches discretized distributions are finding their way into survival analysis. In this regard, an initial attempt was made by Nakagawa and Osaki (1975) who discretized the Weibull distribution. Later on, a number of researchers like Stein and Dattero (1984), Khan et al. (1989), Szablowski (2001), Bracquemond and Gaudoin (2003), Roy (2003, 2004), Kemp (1997, 2004, 2006), Inusah and Kozubowski (2006), Kozubowski and Inusah (2006), Krishna and Pundir (2007, 2009), Sreehari (2008), Roy and Ghosh (2009) and Jazi et al. (2010), Gómez-Déniz and Calderín-Ojeda (2011), Chakaraborty and Chakravarty (2012), Hussain and Ahmad (2012) and Al-Huniti and Al-Dayian (2012) and Nekoukhou et al. (2012) developed discretized version of continuous lifetime distributions and applied them on discrete sets of data in various disciplines of life like engineering, social sciences, medical sciences, and forestry etc.. Classifications of discrete distribution have been made by number of researchers like Khalique (1989) and Kemp (2004). In order to develop reliability theory in discrete discipline various attempts have been initiated in multiple directions. We hereby made an attempt to develop suitable discrete lifetime model in terms of discrete inverse Rayleigh distribution which is defined and discussed in section two along with failure rate function and related conditions, mathematical properties, order statistics and the link between discrete inverse Rayleigh and continuous distributions like Rayleigh and inverse Rayleigh and in section three the parameter's estimation and goodness of fit with real data examples are studied.

2. DISCRETE INVERSE RAYLEIGH DISTRIBUTION

As discretization of continuous lifetime distribution is an emerging issue of discrete reliability theory, so various discretization approaches exist in the literature. However, these approaches are used by various researchers under different circumstances. For example while using the discrete concentration approach researchers considered the discrete time space either as $N=\{0,1,2,3,...\}$ or as $N=\{1,2,3,...\}$ which was usually based on the support of continuous random variable i.e. if support of continuous random variable is $[0,\infty)$ or $x \geq 0$ then the support for discretized random variable will be as

$N=\{0,1,2,3,...\}$ with survival function $S(x)=P_r(X\geq x)$ (see Nakagawa and Osaki (1975), Roy (2003, 2004), Krishna and Pundir (2007, 2009), Gómez-Déniz and Calderín-Ojeda (2011), Chakaraborty and Chakravarty (2012), Hussain and Ahmad (2012) and Al-Huniti and Al-Dayian (2012)) and if support of continuous random variable is $(0,\infty)$ or $x>0$, then discretized random variable will based on $N=\{1,2,3,...\}$ with survival function $S(x)=P_r(X > x)$ (see Khan et al. (1989), Bracquemond and Gaudoin (2003) and Jazi et al. (2010)). The simplest discretized model of continuous exponential distribution is the geometric distribution which is obtained after preserving the survival function of exponential distribution as

$$S_x=\sum_{j\geq x} p_j=q^x,\, p_x=S_x-S_{x+1}=q^x-q^{x+1}, 0<\exp(-\lambda)=q<1,\ x=0,1,2,3,...$$

Since there is one to one correspondence between survival function of geometric distribution and exponential distribution, so a number of researchers considered the geometric distribution as discrete exponential distribution with lack of memory property. Moreover, if the survival functions of discretized distribution retain the same functional forms as their continuous counterparts then many reliability measures and class properties under series, parallel and coherent structures will remain unchanged (see Roy, 2004). In view of the above characteristics we have adopted this approach for discretizing the inverse Rayleigh distribution. The Inverse Rayleigh distribution is a special case of inverse Weibull distribution i.e. if $Y\sim W(\theta,\beta)$ then $X\left(=\frac{1}{Y}\right)\sim IW(\theta,\beta)$ and for $\beta=2$ we have $X\sim IR(\theta)$ with survival function as $S(x)=P_r(X\geq x)=1-\exp\left(-\theta/x^2\right),\ \theta>0,\ x\geq 0.$ Although most of the continuous distribution exhibit monotonic failure rate yet the inverse Rayleigh distribution is among the rare distributions which is being effectively used in the area of reliability studies where failure rate exhibits non-monotonic behavior. It is used in lifetime experiments (see Voda, 1972), record values from Inverse Rayleigh distribution are being used for prediction purposes in real life problems like weather, economic and support data (see Soliman et al., 2010) and used in acceptance sampling plans (see Rosaiah and Kantam (2005) and Aslam and Jun, 2009) etc. The important feature of this distribution is that its variance and higher order moments do not exist. However its r^{th} moment, mean, mode and failure rate function are expressed as

$$\mu_r{}'=\theta^{r/2}\Gamma\left(1-\frac{r}{2}\right), \text{mean}=\sqrt{\pi\theta}\ ,\ \text{mode}=\sqrt{\frac{\theta}{3/2}}$$

and

$$h(x)=\frac{f(x)}{S(x)}=2\theta x^{-3}\left(\exp\left(\theta/x^2\right)-1\right)^{-1}.$$

It was first considered by Voda (1972), Mukherjee and Saran (1984) stated that a single parameter inverse Rayleigh distribution possessed increasing failure rate (IFR) for $x < 1.0694543\sqrt{\theta}$ and decreasing failure rate (DFR) for $x > 1.0694543\sqrt{\theta}$.

2.1 Definition

A random variable Y is said to have discrete inverse Rayleigh distribution with parameter $0 < q < 1$, denoted by dIR(q), if

$$P_r\left(Y = x\right) = p_x = S_x - S_{x+1} = q^{1/(x+1)^2} - q^{1/x^2},\ 0 < \exp\left(-\theta\right) = q < 1,\ x = 0,1,2,3,... \quad (1)$$

where S_x is the preserved survival function of inverse Rayleigh distribution at integers expressed as

$$S_x = P_r\left(Y \geq x\right) = \sum_{j \geq x} p_j = 1 - q^{1/x^2},\ 0 < q < 1,\ x = 0,1,2,3,....,\ \text{where } q^{\infty} = 0, S_0 = 1,$$

$Y = [X]$ denote the observed discrete random variable i.e. Y is equal to the greatest integer less than or equal to X . If Y is a random variable denoting the number of times a product fail in any given week/month/year and q denotes the probability of failure of a product in any given week/month/year than $P\left(Y = 0\right)$ gives the probability of no failure in any given week/month/year.

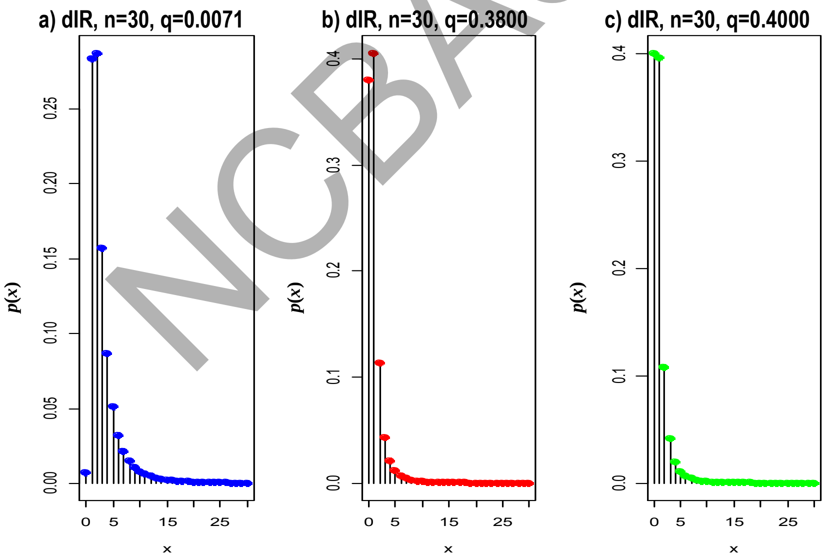


Fig. (2.2.1): Discrete Inverse Rayleigh Distribution

Fig. (2.2.1) shows the probability plots for discrete Inverse Rayleigh distribution for different values of the parameter q , which portrays that as $q \to 0$ the mode of the distribution shifted towards the right and as q increases the mode of the distribution shifted towards the left and distribution shows a reverse J-shaped.

2.2 Hazard Function

Let Y be a discrete random variable with probability mass function $p_x = P(Y = x)$ and reliability function $S_x = P_r(Y \geq x)$ then the failure rate function of Y is defined as the conditional probability that failure is observed at x given that the product has not failed before x and expressed as

$$h_x = \frac{p_x}{S_x} = \frac{S_x - S_{x+1}}{S_x}, \quad x = 0,1,2,3,... \tag{2}$$

The hazard function defined in equation (2) has some misconception i.e.

i) It is bounded i.e. $h_x \leq 1.$ this may add some confusion in industry that failure rate and failure probability are sometimes mixed up (see Xie et al., 2002).
ii) Suppose that if we have m discrete component connected independently in series then their failure rate is not additive i.e. $h_x = 1 - \prod_{i=1}^{m}(1 - h_{ix}) \neq \sum_{i=1}^{m} h_{ix}$. (see Xie et al., 2002).
iii) The cumulative hazard function $H_x = \sum_{x=1}^{k} h_x \neq -l\,nS_x$. (see Xie et al., 2002).

Due to these problems an alternative hazard function for the discrete random variable is defined as $h_x^* = l\,n\left(\frac{S_x}{S_{x+1}}\right)$, which is based on the fact that the hazard function defined in the continuous case and expressed as $h(x) = \frac{p(x)}{S(x)} = -\frac{d(l\,nS(x))}{dx}$ can be defined into discrete case by replacing $h(x)$ by h_x^* and $-\frac{d(l\,nS(x))}{dx}$ by -$(l\,nS_{x+1} - l\,nS_x)$ so $h_x^* = l\,n\left(\frac{S_x}{S_{x+1}}\right), x = 0,1,2,3,...$ which is not bounded like $h(x)$, and shows the same monotonicity as shown by h_x, $H_x^* = -l\,nS_x$ and additive in series system (see Xie et al., 2002). Roy and Gupta (1999) named this failure rate function as second failure rate function. Now by using the above definitions the failure rate and second failure rate functions for discrete inverse Rayleigh distribution are defined as

$$h_x = \frac{p_x}{S_x} = \left(q^{(x+1)^{-2}} - q^{(x)^{-2}}\right)\left(1 - q^{(x)^{-2}}\right)^{-1}, \quad 0 < q < 1,\ x = 0,1,2,....$$

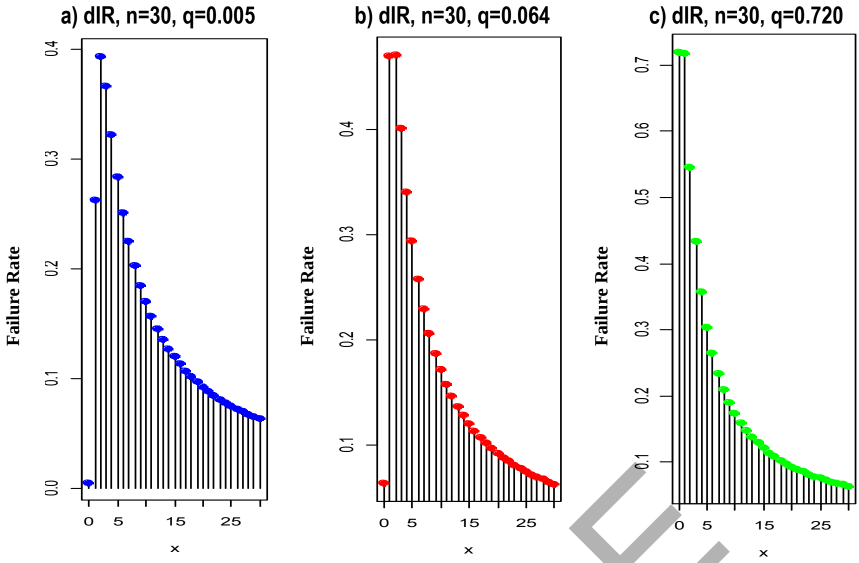


Fig. (2.2.2): Failure Rate of Discrete Inverse Rayleigh Distribution

and $h_x^* = ln\left(\dfrac{S_x}{S_{x+1}}\right) = ln\left(\dfrac{1-q^{x^{-2}}}{1-q^{(x+1)^{-2}}}\right)$, $0 < q < 1$, $x = 0,1,2,....$

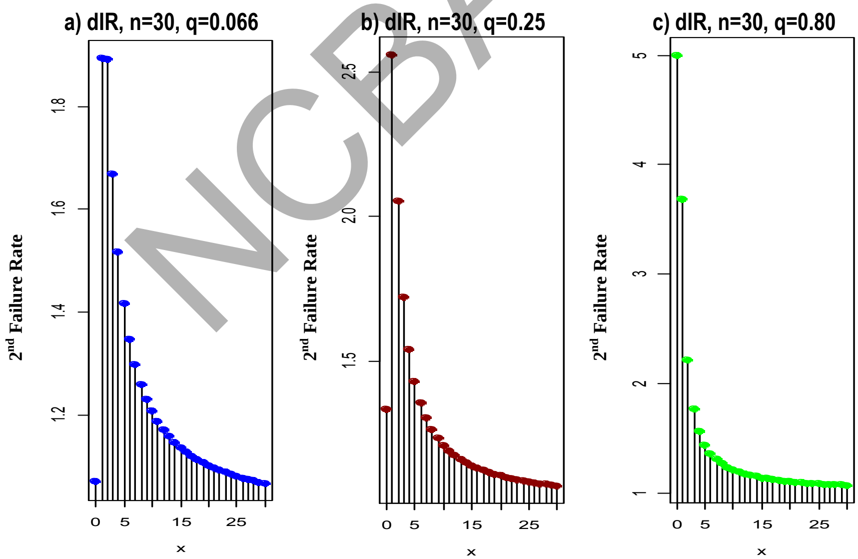


Fig. (2.2.3): Second Failure rate of discrete inverse Rayleigh distribution

Mukherjee and Saran (1984) stated that a single parameter inverse Rayleigh distribution possessed increasing failure rate (IFR) for $x < 1.0694543\sqrt{\theta}$ and decreasing failure rate (DFR) for $x > 1.0694543\sqrt{\theta}$. The same is true with the discrete inverse Rayleigh distribution which has discrete increasing failure rate (dIFR) when $x < 1.0694543\sqrt{-\ln q}$ and discrete decreasing failure rate when (dDFR) $x > 1.0694543\sqrt{-\ln q}$ at integers. However certain features of this non-monotonic behavior of hazard function are also explored like

i) The hazard function of discrete Inverse Rayleigh distribution is an upside down bath tub function of x with change points either at $x=1$ or at $x=2$ for $0<q<0.75$. However as $q \to 0$ the change point appears at $x=2$ see Fig. (2.2.2 and 2.2.3).
ii) The hazard function of discrete Inverse Rayleigh distribution is an upside down bath tub function of x with constant failure rate at $x=1$ and $x=2$ for $0.0636 < q < 0.0673$ see Fig. (2.2.2 and 2.2.3).
iii) The hazard function of discrete Inverse Rayleigh distribution is a decreasing function of x for $0.71 < q < 1.00$ see Fig. (2.2.2 and 2.2.3).

Theorem 2.1.1:

Let $Y = [X]$ be an integer valued random variable which follows the discrete Inverse Rayleigh distribution with parameter q i.e. $Y \sim dIR(q)$. Then expectation for $Y = \varphi(x)$ is expressed as

$$E(\varphi(x)) = \sum_{x=1}^{\infty}\{\varphi(x)-\varphi(x-1)\}\left(1-q^{x^{-2}}\right)+\varphi(0),$$

where $P_r(Y \ge x) = 1-q^{x^{-2}}$ and $\exp(-\theta)=q,\ \ 0<q<1,\ x=0,1,2,3,....$

Proof:

We have by definition $E(\varphi(x)) = \sum_{x=0}^{\infty}\varphi(x)P(Y=x)$

Consider $\{\varphi(x)-\varphi(x-1)\}P_r(Y \ge x) = \varphi(x)P_r(Y \ge x) - \varphi(x-1)P_r(Y \ge x),$

Taking summation over x from 1 to ∞, we get

$$\sum_{x=1}^{\infty}\{\varphi(x)P_r(Y \ge x)-\varphi(x-1)P_r(Y \ge x)\} = \varphi(1)P_r(Y \ge 1)-\varphi(0)P_r(Y \ge 1)$$
$$+\varphi(2)P_r(Y \ge 2)-\varphi(1)P_r(Y \ge 2)+\varphi(3)P_r(Y \ge 3)-\varphi(2)P_r(Y \ge 3)+.....,$$
$$\sum_{x=0}^{\infty}\varphi(x)P(Y=x) = \sum_{x=1}^{\infty}\{\varphi(x)P_r(Y \ge x)-\varphi(x-1)P_r(Y \ge x)\}+\varphi(0)$$

where $P_r(Y \ge 0)=1$.

$$E(\varphi(x)) = \sum_{x=1}^{\infty}\{\varphi(x)-\varphi(x-1)\}P_r(Y \ge x)+1 \Rightarrow E(\varphi(x))$$

$$=\sum_{x=1}^{\infty}\left\{\varphi(x)-\varphi(x-1)\right\}\left(1-q^{x^{-2}}\right)+\varphi(0).$$

where $P_r(Y\geq x)=1-q^{x^{-2}}$, $x=0,1,2,3,.....$

The series converges i.e. as $x\to\infty$ the tail probabilities approaches zero. This completes the proof.

Corollary:

If $\varphi(x)=t^x$ then resulting expression is probability generating function of discrete inverse Rayleigh distribution.

Corollary:

If $\varphi(x)=e^{tx}$ then resulting expression is the moment generating function of discrete inverse Rayleigh distribution.

Corollary:

If $\varphi(x)=x^r$ then resulting expression is the r^{th} moment about origin of discrete inverse Rayleigh distribution.

Its mean and variance are

$$\mu_1'=\sum_{x=1}^{\infty}\left(1-q^{x^{-2}}\right) \text{ and } Var(Y)=\frac{1}{4}+2\sum_{x=1}^{\infty}x\left(1-q^{x^{-2}}\right)-\frac{\left(1+2\mu'_1\right)^2}{4},$$

Corollary:

If $\varphi(x)=(x+a)^{-1}$ then resulting expression is the first order negative moment of discrete inverse Rayleigh distribution.

Corollary:

If $\varphi(x)=(x+a)^{-s}$ then resulting expression is the s^{th} order negative moment of discrete inverse Rayleigh distribution.

Corollary:

If $\varphi(x)=\dfrac{1}{(x+a)_s}$ then resulting expression is the s^{th} order negative factorial moment of discrete inverse Rayleigh distribution, where

$$(x+a)_s=(x+a)(x+a+1)..(x+a+s-1), a>0, 0<q<1 \text{ and } \exp(-\theta)=q.$$

In order to check the suitability of distribution for specific type of data we define immediately the index of dispersion and showed mean and variance of the distribution in Table-1.

2.3 Index of Dispersion

Index of dispersion (ID) for any distribution is defined as the ratio between variance to mean which indicate whether the distribution is suitable for over or under dispersed data. If $ID > 1(<1)$ the distribution is over-dispersed (under-dispersed) (see Chakraborty and Chakravarty, 2012). Table-1 portrays the dispersion pattern of discrete Inverse Rayleigh distribution in which upper values indicate mean and lower values indicate variance of the distribution against particular value of the parameter q. It is observed that discrete Inverse Rayleigh distribution is over dispersed for all values of the parameter q. Moreover it can also be seen that as mean and variance decreases the value of the parameter increases and vice versa.

Table 1
Mean (above) and Variance below of dIR(q)

q	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	--	3.3053 68.5072	3.0118 105.7505	2.8272 94.7891	2.6891 87.0152	2.5774 80.9897	2.4827 76.0670	2.3999 71.9061	2.3259 68.3033	2.2589 65.1258
0.1	2.1970 62.2833	2.1396 59.7178	2.0861 57.3719	2.0356 55.2120	1.9877 53.2180	1.9424 51.3622	1.8990 49.6250	1.8574 47.9912	1.8173 46.4497	1.7788 44.9966
0.2	1.7418 43.6153	1.7057 42.2999	1.6708 41.0486	1.6368 39.8529	1.6038 38.7041	1.5716 37.6082	1.5401 36.5485	1.5093 35.5351	1.4791 34.5546	1.4497 33.6094
0.3	1.42092 32.6959	1.3926 31.8118	1.3649 30.9572	1.3376 30.1281	1.3108 29.3222	1.2843 28.5397	1.2584 27.7801	1.2328 27.0391	1.2075 26.3203	1.1826 25.6177
0.4	1.1579 12.3691	1.1337 12.0406	1.1098 11.7188	1.0861 11.4053	1.0627 11.0983	1.0395 10.7982	1.0167 10.5048	0.9940 10.2170	0.9716 9.9328	0.9494 9.6569
0.5	0.9277 9.3871	0.9059 9.1205	0.8844 8.8585	0.8630 8.6025	0.8419 8.3500	0.8209 8.1025	0.8001 7.8586	0.7795 7.6203	0.7590 7.3838	0.7387 7.1526
0.6	0.7188 9.7928	0.6988 9.4756	0.6789 9.1634	0.6592 8.8557	0.6396 8.5529	0.6202 8.2542	0.6008 7.9612	0.5815 7.6715	0.5624 7.3839	0.5435 7.1029
0.7	0.5246 6.8243	0.5058 6.5500	0.4871 6.2799	0.4686 6.0126	0.4502 5.7499	0.4319 5.4887	0.4137 5.2327	0.3956 4.9792	0.3775 4.7285	0.3595 4.4819
0.8	0.3416 3.1187	0.3238 2.9402	0.3061 2.7631	0.2884 2.5888	0.2709 2.4168	0.2534 2.2467	0.2360 2.0784	0.2187 1.9127	0.2014 1.7491	0.1842 1.5875
0.9	0.1670 1.2030	0.1499 1.0697	0.1331 0.9383	0.1162 0.80874	0.0993 0.6818	0.0826 0.5575	0.0659 0.4357	0.0493 0.3173	0.0328 0.2028	0.0163 0.0942

Theorem 2.1.2:

Let $Y_1 \le Y_2 \le Y_3 \le ... \le Y_i \le ... \le Y_n$ denote an order sample of size n drawn identically independently from the discrete inverse Rayleigh distribution whose distribution function can also be written as $F_{x\text{-}1} = \sum_{j=0}^{x\text{-}1} p_j = q^{(x\text{-}1)^{-2}}$, $S_x = 1 - F_{x\text{-}1}$, $0 < q < 1$, then the probability function of i^{th} order statistics is

$$P_r\left(Y_{(i)}=x\right)=K_i\left\{q^{ix^{-2}}\ {}_2F_1\left(-n+i,i;i+1;q^{x^{-2}}\right)-q^{i(x-1)^{-2}}\ {}_2F_1\left(-n+i,i;i+1;q^{(x-1)^{-2}}\right)\right\},\tag{3}$$

the recurrence relation between i^{th} order statistics's probabilities are

$$(i+1)P_r\left(X_{(i+1)}=x\right)=iP_r\left(X_{(i)}=x\right)-\binom{n}{i}\left\{q^{ix^{-2}}\left(1-q^{x^{-2}}\right)^{n-i}-q^{i(x-1)^{-2}}\left(1-q^{(x-1)^{-2}}\right)^{n-i}\right\},\tag{4}$$

and

$$(i+1)P_r\left(X_{(i+1)}=x\right)=iP_r\left(X_{(i)}=x\right)-\binom{n}{i}\left\{F_x\left(S_{x+1}\right)^{n-i}-F_{x-1}\left(S_x\right)^{n-i}\right\},\tag{5}$$

where

$$K_i=\frac{1}{i}\binom{n}{i},K_{i+1}=\frac{1}{i+1}\binom{n}{i+1}\,and\ {}_2F_1\left(\alpha_1,\alpha_2;\beta_1;z\right)=\sum_{n=0}^{\infty}\frac{\left(\alpha_1\right)_n\left(\alpha_2\right)_n}{\left(\beta_1\right)_n}\frac{z^n}{n!}.$$

Proof:

By definition the probability function of i^{th} order statistics is

$$P_r\left(Y_{(i)}=x\right)=P_r\left(Y_{(i)}\le x\right)-P_r\left(Y_{(i)}\le x\text{-}1\right),$$

$$P_r\left(Y_{(i)}\le x\right)=P_r\left(\text{at least } i \text{ of } Y's \text{ are } \le x\right),$$

$$P_r\left(X_{(i)}\le x\right)=\sum_{j=i}^{n}\binom{n}{j}P_r\left(X_1\le x\right)^j\left(1\text{-}P_r\left(X_1\le x\right)\right)^{n\text{-}j},$$

$$\left(\text{since } X_1,\ X_2,....,X_n \text{ are } i.i.d\right)$$

where

$$\sum_{j=i}^{n}\binom{n}{j}\left(F_x\right)^j\left(1\text{-}F_x\right)^{n\text{-}j}=\int_0^{F_x}\frac{1}{B\left(i,n\text{-}i+1\right)}u^{i\text{-}1}\left(1-u\right)^{n\text{-}i}du=I_{\left(F_x\right)}\left(i,n\text{-}i+1\right),$$

$I_{\left(F_x\right)}\left(i,n\text{-}i+1\right)$ is the incomplete beta function.

Therefore

$$P_r\left(X_{(i)}=x\right)=\int_0^{F_x}\frac{1}{B\left(i,n\text{-}i+1\right)}u^{i\text{-}1}\left(1-u\right)^{n\text{-}i}du-\int_0^{F_{x\text{-}1}}\frac{1}{B\left(i,n\text{-}i+1\right)}u^{i\text{-}1}\left(1-u\right)^{n\text{-}i}du,$$

$$P_r\left(X_{(i)}=x\right)=I_{\left(q^{(x\text{-}1)^{-2}},q^{x^{-2}}\right)}\left(i,n\text{-}i+1\right),\qquad\left(\text{where } F_x=q^{x^{-2}}\right)$$

$$P_r\left(X_{(i)}=x\right)=\frac{n!}{i!\left(n\text{-}i\right)!}\sum_{j=0}^{n-i}\binom{n\text{-}i}{j}\frac{(-1)^j}{(i+j)}\left(q^{(i+j)(x)^{-2}}-q^{(i+j)(x\text{-}1)^{-2}}\right),\tag{6}$$

since

$$\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{(i+j)}\left(q^{x^{-2}}\right)^{(i+j)} = \frac{q^{ix^{-2}}}{i}\,{}_2F_1\left(-n+i,i;i+1;q^{x^{-2}}\right),$$

and

$$\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{(i+j)}\left(q^{(x-1)^{-2}}\right)^{(i+j)} = \frac{q^{i(x-1)^{-2}}}{i}\,{}_2F_1\left(-n+i,i;i+1;q^{(x-1)^{-2}}\right),$$

on substituting the values of above expression in (6) we get (3). From equation (3) we have the probability function of i^{th} *and* $(i+1)^{th}$ order statistics as

$$P_r\left(Y_{(i)}=x\right)=K_i\left\{q^{ix^{-2}}\,{}_2F_1\left(-n+i,i;i+1;q^{x^{-2}}\right)-q^{i(x-1)^{-2}}\,{}_2F_1\left(-n+i,i;i+1;q^{(x-1)^{-2}}\right)\right\},$$

$$P_r\left(X_{(i+1)}=x\right)=K_{i+1}q^{ix^{-2}}q^{x^{-2}}\,{}_2F_1\left(-n+i+1,i+1;i+2;q^{x^{-2}}\right)$$
$$-K_{i+1}q^{i(x-1)^{-2}}q^{(x-1)^{-2}}\,{}_2F_1\left(-n+i+1,i+1;i+2;q^{(x-1)^{-2}}\right), \tag{7}$$

$$P_r\left(X_{(i+1)}=x\right)=K_{i+1}\left(A_2-B_2\right),$$

where

$$A_2=q^{x^{-2}}\,{}_2F_1\left(-n+i+1,i+1;i+2;q^{x^{-2}}\right),\ B_2=q^{(x-1)^{-2}}\,{}_2F_1\left(-n+i+1,i+1;i+2;q^{(x-1)^{-2}}\right),$$

Using the Gauss' recurrence relation for A_2 and B_2 (see Gradshteyn and Ryzhik, 1965).

$$c\,{}_2F_1\left(a,b;c;z\right)-c\,{}_2F_1\left(a,b+1;c;z\right)+az\,{}_2F_1\left(a+1,b+1;c+1;z\right)=0,$$

Let $a=-n+i,\ b=i,\ c=i+1$ and $z=q^{(x-1)^{-2}}$ then

$$(i+1)\,{}_2F_1\left(-n+i,i;i+1;q^{(x-1)^{-2}}\right)-(i+1)\,{}_2F_1\left(-n+i,i+1;i+1;q^{(x-1)^{-2}}\right)$$
$$+(-n+i)q^{(x-1)^{-2}}\,{}_2F_1\left(-n+i+1,i+1;i+2;q^{(x-1)^{-2}}\right)=0,$$

for $A_2\ q^{x^{-2}}\,{}_2F_1\left(-n+i+1,i+1;i+2;q^{x^{-2}}\right)$

$$=\frac{(i+1)}{n-i}\,{}_2F_1\left(-n+i,i;i+1;q^{x^{-2}}\right)-\frac{(i+1)}{n-i}\left(1-q^{x^{-2}}\right)^{n-i},$$

for B_2 $q^{(x\text{-}1)^{-2}}\ {}_2F_1\left(-n+i+1, i+1; i+2; q^{(x\text{-}1)^{-2}}\right)$

$$=\frac{(i+1)}{n-i}\ {}_2F_1\left(-n+i, i; i+1; q^{(x\text{-}1)^{-2}}\right)-\frac{(i+1)}{n-i}\left(1-q^{(x\text{-}1)^{-2}}\right)^{n-i},$$

On substituting above expression for A_2 and B_2 into equation (7) we get

$$P_{i+1}\left(X_{(i+1)}=x\right)=\frac{1}{i+1}\binom{n}{i+1}\left[\frac{(i+1)}{(n\text{-}i)}\left\{q^{ix^{-2}}\ {}_2F_1\left(-n+i, i; i+1; q^{x^{-2}}\right)\right.\right.$$

$$\left.-q^{i(x\text{-}1)^{-2}}\ {}_2F_1\left(-n+i, i; i+1; q^{(x\text{-}1)^{-2}}\right)\right\}$$

$$\left.-\frac{(i+1)}{(n\text{-}i)}\left\{q^{ix^{-2}}\left(1-q^{x^{-2}}\right)^{n-i}-q^{i(x\text{-}1)^{-2}}\left(1-q^{(x\text{-}1)^{-2}}\right)^{n-i}\right\}\right],$$

$$P_{i+1}\left(X_{(i+1)}=x\right)=\frac{i}{i+1}\left[\frac{1}{i}\binom{n}{i}\left\{q^{ix^{-2}}\ {}_2F_1\left(-n+i, i; i+1; q^{x^{-2}}\right)\right.\right.$$

$$\left.-q^{i(x\text{-}1)^{-2}}\ {}_2F_1\left(-n+i, i; i+1; q^{(x\text{-}1)^{-2}}\right)\right\}$$

$$\left.-q^{i(x\text{-}1)^{-2}}\ {}_2F_1\left(-n+i, i; i+1; q^{(x\text{-}1)^{-2}}\right)\right\}$$

and then simplifying it we get (4).

The general recurrence relation is

$$(i+1)P_r\left(X_{(i+1)}=x\right)=iP_r\left(X_{(i)}=x\right)-\binom{n}{i}\left\{F_x\left(S_{x+1}\right)^{n-i}-F_{x\text{-}1}\left(S_x\right)^{n-i}\right\}.$$

This completes the proof.

Theorem 2.1.3:

Let X be non-negative continuous Inverse Rayleigh random variable and $Z=[X]$ be an integer valued random variable. Then $Z \sim dIR(q)$ if $X \sim IR(\theta)$.

Proof:

Let $X \sim IR(\theta)$ with $S_X(x)=P_r(X \geq x)=1-e^{-\theta/x^2}$, $\theta>0,\ x\geq 0,$ then we observe that $\vee x=0,1,2,3,....$

$$S_Z(x)=P_r(Z \geq x),$$

$$=P_r([X] \geq x),$$

$$S_Z(x) = P_r(X \geq x), S_Z(x) = 1 - \exp\left(\frac{-\theta}{x^2}\right), S_Z(x) = 1 - q^{x^{-2}},$$

since $[X] \geq Z \Leftrightarrow X \geq Z$

where $0 < q < 1$ and $q = \exp(-\theta)$.

This completes the proof.

Theorem 2.1.4:

Let X be non-negative continuous Rayleigh random variable and $W = \left[\frac{1}{X}\right]$ be an integer valued random variable. Then $W \sim dIR(q)$ if $X \sim R(\theta)$.

Proof:

Let $X \sim R(\theta)$ with $F_X(x) = P_r(X \leq x) = 1 - e^{-\theta x^2}$, $\theta > 0, x \geq 0,$ then we observe that $\vee x = 0, 1, 2, 3, \ldots$

$$S_W(x) = P_r(W \geq x),$$

$$= P_r\left(\left[\frac{1}{X}\right] \geq x\right),$$

$$S_Z(x) = P_r\left(X \leq \frac{1}{x}\right), S_Z(x) = 1 - \exp\left(\frac{-\theta}{x^2}\right), S_Z(x) = 1 - q^{x^{-2}},$$

since $[X] \geq Z \Leftrightarrow X \geq Z$

where $0 < q < 1$ and $q = \exp(-\theta)$.

This completes the proof.

3. PARAMETERS' ESTIMATION, GOODNESS OF FIT TESTS AND APPLICATIONS

In order to estimate the parameter q of discrete Inverse Rayleigh distribution we have studied three methods like proportions, pseudo-moments (see Khan et al. 1989) and maximum likelihood. Simulation results of these methods are based on 100 replication and presented in Table 2.

3.1 Method of Moments

While using the method of moments for estimating the parameter q, we have to first equate the population moment to the corresponding sample moment than solve the equation for q. Since moments are not in closed form so this equation cannot be solved by ordinary techniques. However Jazi et al. (2010) used the method of pseudo-moments, as proposed by Khan et al. (1989), by minimizing $S(q,\beta)=\left(M_1 - E(X)\right)^2+\left(M_2 - E\left(X^2\right)\right)^2$, with respect to q and β where $M_1=\frac{1}{n}\sum_{i=1}^{n}x_i$ and $M_2=\frac{1}{n}\sum_{i=1}^{n}x_i^2$. We have also used this method to estimate q by minimizing $S(q)=\left(M_1 - E(X)\right)^2$ with respect to q. Unlike Jazi et al. (2010), though this method yields smaller variance yet the deviation from the true value is larger as compared to the other estimators for larger q and n.

3.2 Method of Proportions

The method of proportions, proposed and studied by Khan et al. (1989) and Jazi et al. (2010), are based on the proportions of 1's and 2's. Now we proposed the similar method for discrete inverse Rayleigh distribution but based on proportions of 0's which is outlined below.

Let $x_1, x_2, ..., x_n$ be a random sample of n items from discrete inverse Rayleigh distribution, then the indicator function is defined as

$$I(x_i)=\begin{cases}1, & x_i=0\\ 0, & x_i>0.\end{cases}$$

As $Z=\sum_{i=1}^{n}I(x_i)$ denotes the number of 0's in the sample so the proportions of zeros i.e. $\frac{Z}{n}$ estimates the probability $p_{0,q}=q$. Hence, we have denoted $\hat{q}$ as an estimate of q and z as observed value of Z therefore $\hat{q}=\frac{z}{n}$. It is known that an empirical cumulative distribution function (cdf) is consistent and an unbiased estimator of the actual cdf the same is true for $\hat{q}$ which is an unbiased and consistent estimator of $P(Y\le 0)=q$ (see Jazi et al. 2010).

3.3 Maximum Likelihood:

Let $X_1, X_2, ..., X_n$ be the recorded lifetimes of a random sample of n items. If these recorded lifetimes identically independently follow the dIR i.e. $X_i\text{'s} \sim dIR(q)$ then the likelihood function for dIR can be expressed as

$$L(q)=\prod_{i=1}^{n}p_{x_i}=\prod_{i=1}^{n}\left\{q^{(x_i+1)^{-2}}-q^{(x_i)^{-2}}\right\},\quad \frac{\partial lnL(q)}{\partial q}=\sum_{i=1}^{n}\frac{\dfrac{q^{(x_i+1)^{-2}-1}}{(x_i+1)^2}-\dfrac{q^{(x_i)^{-2}-1}}{(x_i)^2}}{q^{(x_i+1)^{-2}}-q^{(x_i)^{-2}}}=0.$$

A numerical solution of the above equation will yield the MLEs of q.

As MLEs are generally unbiased and consistent estimators so the asymptotic distribution of MLE of q i.e. $\hat{q}$ is normal with mean q and variance of $\hat{q}$ is $Var(\hat{q}) = \left(I(\hat{q})\right)^{-1}$ i.e. $\hat{q} \sim N\left(q, \dfrac{1}{I(q)}\right)$ where $I(q)$ is the Fisher Information and is defined as $I(q) = E\left(-L^{''}(\hat{q})\right)$ an estimate of I(q) is $I(\hat{q})$ by virtue of invariance property of MLE and is expressed as $I(\hat{q}) = -L^{''}(\hat{q})\Big|_{q=\hat{q}}$ so we have variance of $\hat{q}$ in this form $Var(\hat{q}) = \left(I(\hat{q})\right)^{-1}$.

Table 2
Estimation by Method of Proportions, Moments and Maximum Likelihood

		$\tilde{q}$ (PM)	$Var(\tilde{q})$ (PM)	$\widehat{q}$ (MM)	$Var(\widehat{q})$ (MM)	$\hat{q}$ (ML)	$Var(\hat{q})$ (ML)
$q = 0.10$	$n = 60$	0.0858	0.0055	0.1243	0.0110	0.0945	0.0010
	$n = 40$	0.0907	0.0023	0.1375	0.0249	0.0873	0.0014
	$n = 20$	0.0926	0.0045	0.1570	0.0979	0.1019	0.0034
$q = 0.30$	$n = 60$	0.2748	0.0034	0.3167	0.0061	0.2781	0.0030
	$n = 40$	0.2953	0.0053	0.3263	0.0129	0.2940	0.0046
	$n = 20$	0.2886	0.0105	0.3617	0.0516	0.2936	0.0101
$q = 0.60$	$n = 60$	0.5881	0.0040	0.5993	0.0024	0.5871	0.0039
	$n = 40$	0.6021	0.0059	0.6504	0.0052	0.6040	0.0057
	$n = 20$	0.6014	0.0120	0.6335	0.0215	0.5944	0.0112

Table 2 is based on 100 replication, from this table, it is evident that the estimators obtained by the method of moments are asymptotically unbiased and consistent for larger q and n. It is also observed that method of proportion is much better than the method of moments in terms of smaller variances for all n and $q < 0.40$ and smaller deviation about the true value of q for all n and q. Finally, the maximum likelihood method is the most efficient procedure for almost all q and n.

Now in the next section we deals with the goodness of fit tests which uses the above developed MLE to test the suitability of discrete Inverse Rayleigh Distribution in over dispersed data structure.

3.3 Goodness of Fit Tests

Generally the goodness of fit (GOF) tests compute the compatibility of a random sample with a theoretical probability distribution function. In short, these tests measure the suitability of your data to the distribution you have selected. The general procedure consists of defining a test statistic which is some function of the data measuring the distance between the hypothesis and the data. Here, we are using Kolmogorov-Smirnov and Chi-Squared goodness of fit tests for testing the suitability of various data sets.

3.3.1 Kolmogorov-Smirnov Test

This test may be used to decide if a sample comes from a hypothesized continuous/discrete distribution (see Gibbons, 1971, p.85). In it an empirical cumulative distribution function (ECDF) is computed. Suppose that we have a random sample $x_1,\ldots,x_n$ from some continuous/discrete distribution with CDF $F(x)$. The empirical CDF is denoted by

$$F_n(x) = \frac{Number\ of\ observations \leq x}{n}$$

Definition

The Kolmogorov-Smirnov test statistic (D) is based on the largest vertical difference between $F(x)$ and $F_n(x)$. It is defined as

$$D_n = \sup_x \left| F_n(x) - F(x) \right|$$

H_0 : The data follow the specified distribution.

H_A : The data do not follow the specified distribution.

The hypothesis regarding the distributional form is rejected at the chosen significance level (α) if the test statistic, D, is greater than the critical value obtained from a table.

3.3.2 Chi-Squared Test

This test is used to determine whether a sample comes from a population with a specific distribution or not. It is applied to grouped data that is why its test statistic depends on how the data is grouped.

Since there is no optimal choice for the number of classes (k), so there are several formulas which are used to calculate the number of classes which are based on the sample size (N). For this purpose the Sturges' empirical formula is frequently used i.e. $k = 1 + 3.3 \log N$.

Generally the data can be grouped into intervals of equal probability or equal width. Each class should contain at least 5 or more data points, so, in order to satisfy this condition certain adjacent classes sometimes need to be joined together.

Definition

The Chi-Squared test statistic is defined as $\chi^2 = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i}$, where o_i and e_i are the observed and expected frequencies for class i respectively. For testing the compatibility of the data with theoretical probability function we formulate the following null and alternative hypothesis

H_0 : The data follow the specified distribution.

H_A : The data do not follow the specified distribution.

The hypothesis regarding the distributional form is rejected at the chosen significance level (α) if the test statistic is greater than the critical value defined as $\chi^2_{(1-\alpha, k-m-1)}$. Where $k-1-m$ denotes the degree of freedom with m the number of parameters to be estimated. Usually a smaller computed value of chi-square indicates a good fit whereas larger value showed a poor fit.

3.4 Applications

Here, we are now presenting some applications and comparisons of the proposed model with the Poisson under real life scenario.

Example 1

The following data set give the number of times that computer break down in each of the 128 consecutive week of operation (see Chakaraborty and Chakravarty (2012)). The empirical failure function is presented in Fig. 3.4.1.

{4, 0, 0, 0, 3, 2, 0, 0, 6, 7, 6, 2, 1, 11, 6, 1, 2, 1, 1, 2, 0, 2, 2, 1, 0, 12, 8, 4, 5, 0, 5, 4, 1, 0, 8, 2, 5, 2, 1, 12, 8, 9, 10, 17, 2, 3, 4, 8, 1, 2, 5, 2, 2, 2, 3, 1, 2, 0, 2, 1, 6, 3, 3, 6, 11, 10, 4, 3, 0, 2, 4, 2, 1, 5, 3, 3, 2, 5, 3, 4, 1, 3, 6, 4, 4, 5, 2, 10, 4, 1, 5, 6, 9, 7, 3, 1, 3, 0, 2, 2, 1, 4, 2, 13, 0, 2, 1, 1, 0, 3, 16, 22, 5, 1, 2, 4, 7, 8, 6, 11, 3, 0, 4, 7, 8, 4, 4, 5}

By using MLE's, we have fitted the failure functions of discrete inverse Rayleigh and Poisson distributions. Kolmogrov-Smirnov (KS) test for goodness of fit (see Gibbons, 1971, p.85) and AIC are computed to compare their performance. Findings are computed in R computational package.

Table 3
Data on the number of times that computer break down in each of the 128 consecutive week of operation

Model	K.S	AIC	p-value
Discrete Inverse Rayleigh (0.02432)	0.1765	715.718	0.9631
Poisson(2.99993)	0.4706	810.8884	0.04495

From the above table 3, it is evident that discrete inverse Rayleigh distribution provides marginally better fit as compare to Poisson distribution not only in larger p-value but also in least loss of information i.e. smaller AIC.

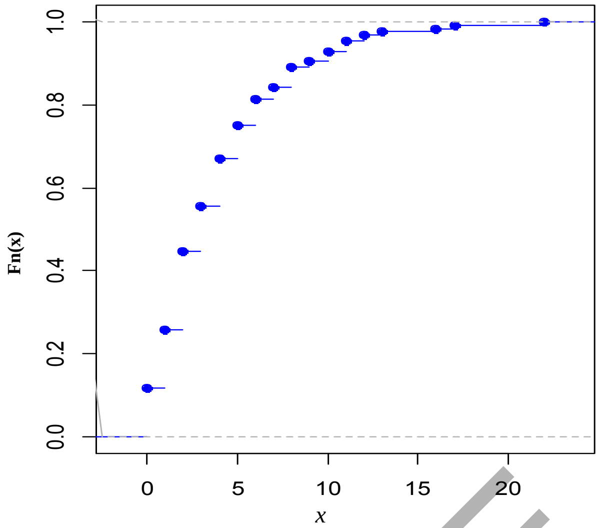


Fig. 3.4.1: Empirical Failure Function

Example 2: Modeling Probability distribution of Count data

We have also investigated that whether the proposed discrete inverse Rayleigh distribution can compete with the Poisson distribution in modeling real life count data, other than reliability. In this example we fitted the proposed and the Poisson distributions to an over-dispersed data set. The following data set is the distribution of yeast cells in 400 squares of haemacytometer observed by "Student" (1907) (see Roy and Gupta 1999).

Table 4
Distribution of yeast cells in 400 squares of haemacytometer observed by "Student" (1907) Data is taken from Roy and Gupta(1999)

No. of Cells	**0**	**1**	**2**	**3**	**4**	**≥5**	**Total**	**Chi-square**	**p-value**
Frequency	213	128	37	18	3	1	400	calculated	right tail
Expected frequencies dIR(0.5335)	213.4	128.5	31.2	11.6	5.5	9.9	400	2.01	0.3660
Expected frequencies Poi.(0.6825)	202.1	138.0	47.1	10.7	1.8	0.3	400	10.09	0.0066

The above is an over dispersed data with mean = 0.6825, variance = 0.8137 and index of dispersion = 1.1922. Based on the value of chi-square and p-value it follows that dIR(q) provide a good fit to the data set. It is also worth mentioning that while using the same data set, our proposed model gives the closest fit among all the alternative models studied by Roy and Gupta (1999).

CONCLUDING REMARKS

In this paper, a discrete Inverse Rayleigh distribution is developed by using the discrete concentration approach, which can be used in over dispersed data structure as an alternate to single parameter Poisson distribution. Moreover, this newly proposed model can be applied to model not only the count data but also seems suitable for modeling number of claims in Actuarial science and in discrete life testing data where the hazard function shows non-monotonic behavior.

ACKNOWLEDGEMENT

The authors are thankful to the referees for their suggestions, which led to the improvement of this paper.

REFERENCES

1. AL-Huniti, A.A. and AL-Dayian, G.R. (2012). Discrete Burr Type-III Distribution. *Amer. J. Math. and Statist.*, 2(5), 145-152.
2. Aslam, M. and Jun, C.-H. (2009). A group acceptance sampling plans for truncated life tests based on the inverse Rayleigh and log-logistic distributions. *Pak. J. Statist.*, 25(2), 107-119.
3. Bracquemond, C. and Gaudoin, O. (2003). A Survey on Discrete Lifetime distributions. *International Journal of Reliability, Quality, Safety Engineering*, 10, 69-98.
4. Chakraborty, S. and Chakravarty, D. (2012). Discrete Gamma Distributions: Properties and Parameter Estimations. *Commun. in Statist. Theo. and Meth.*, 41(18), 3301-3324.
5. Gibbons, J.D. (1971). *Nonparametric Statistical Inference*. Tokyo: McGraw-Hill Kogakusha, Ltd.
6. Gomez-Deniz, E. and Calderin-Ojeda, E. (2011). The Discrete Lindley Distribution: Properties and Application. *J. Statist. Compu. and Simula.*, 81(11), 1405-1416.
7. Gradshteyn, I.S. and Ryzhik, I.M. (1965). *Table of integrals, Series, and Products*. 5th Edition. Academic Press Limited, London.
8. Hussain, T. and Ahmad, M. (2012). Discrete Inverse Gamma Distribution. Proceedings of *9th International Conference on Statistical Sciences: Official Statistics and its Impact on Good Governance and Economy of Pakistan*. Organized by ISOSS, Lahore, Pakistan, Vol. 22, 381-394. ISBN 978-969-8858-10-0.
9. Inusah, S. and Kozubowski, J.T. (2006). A Discrete Analogue of the Laplace Distribution. *J. Statist. Plann. and Infer.*, 136, 1090-1102.
10. Jazi, M.A., Lai, C.D. and Alamatsaz, M.H. (2010).A Discrete Inverse Weibull Distribution and Estimation of its Parameters. *Statistical Methodology*, 7, 121-132.
11. Johnson, N.L., Kotz, S. and Kemp, A.W. (1992). *Univariate Discrete Distributions.* 2nd Edition. New York: John Wiley and Sons.
12. Johnson, N.L., Kotz, S. and Balakrishnan, N. (1994). *Continuous Univariate Distributions Volume 1.* 2nd Edition. New York: John Wiley and Sons.
13. Johnson, N.L. and Kotz, S. (1970). *Continuous Univariate Distributions-2.* New York: Houghton Mifflin Company. Boston.
14. Kemp, A.W. (1997). Characterization of a Discrete Normal Distribution. *J. Statist. Plann. and Infer.*, 63, 223-229.

15. Kemp, A.W. (2004). Classes of Discrete Lifetime Distributions. *Commun. in Statist. Theo. and Meth.*, 33(12), 3069-3093.
16. Kemp, A.W. (2006). The Discrete Half-Normal Distribution. *Advances in Mathematical and Statistical Modeling*, 9, 353-360.
17. Khalique, A. (1989). On Discrete Failure Time Distributions. *Reliability Engineering and System Safety*, 25, 99-107.
18. Khan, M.S.A., Khalique, A. and Abouammoh A.M. (1989). On Estimating Parameters in a Discrete Weibull Distribution. *IEEE Transactions on Reliability*, 38(3), 348-350.
19. Kozubowski, J.T. and Inusah, S. (2006). A Skew Laplace Distribution on Integers. *AISM*, 58, 555-571.
20. Krishna, H. and Pundir, P.S. (2007). *Discrete Maxwell Distribution*. http://interstat.statjournals.net/YEAR/2007/articles/0711003.pdf, 1-15. (accessed in July 2010).
21. Krishna, H. and Pundir, P.S. (2009). Discrete Burr and discrete Pareto Distributions. *Statistical Methodology*, 6, 177-188.
22. Mukherjee, S.P and Saran, L.K. (1984). Bivariate Inverse Rayleigh Distributions in Reliability Studies. *Jour. Ind. Statistical Assoc.*, 2, 23-31.
23. Nakagawa, T. and Osaki, S. (1975).The Discrete Weibull Distribution. *IEEE Transactions on Reliability*, 24(5), 300-301.
24. Nekoukhou, V., Alamatsaz, M.H. and Bidram, H. (2012). A Discrete Analog of the Generalized Exponential Distribution. *Commun. in Statist. – Theo. and Meth.*, 41(11), 2000-2013.
25. Rosaiah, K. and Kantam, R.R.L. (2005). Acceptance sampling based on the inverse Rayleigh distribution. *Eco. Quality Control*, 20(2), 277-286.
26. Roy, D. (2003). Discrete Normal Distribution. *Commun. in Statist. – Theo. and Meth.*, 32(10), 1871-1883.
27. Roy, D. (2004). Discrete Rayleigh Distribution. *IEEE Transactions on Reliability*, 53(2), 255-260.
28. Roy, D. and Ghosh, T. (2009). A New Discretization Approach with Application in Reliability Estimation. *IEEE Transactions on Reliability*, 58(2), 456-461.
29. Roy, D. and Gupta, R.P. (1999). Characterizations and model selections through reliability measures in the discrete case. *Statist. and Probab. Lett.*, 43, 197-206.
30. Soliman, A., Amin, E.A. and Aziz A.A.A. (2010). Estimation and Prediction from Inverse Rayleigh Distribution Based on Lower Record Values. *Applied Mathematical Sciences*, 62(4), 3057-3066.
31. Sreehari, M. (2008). On a Class of Discrete Distributions Analogous to Burr Family. *J. Ind. Statist. Assoc.*, 46(2), 163-162.
32. Stein, W.E. and Dattero, R. (1984). A New Discrete Weibull Distribution. *IEEE Transactions on Reliability*, 33(2), 196-197.
33. Szablowski, P.J. (2001).Discrete Normal Distribution and its relationship with Jacobi Theta Functions. *Statistics & Probability Letters*, 52,289-299.
34. Voda, R. Gh. (1972). On the inverse Rayleigh variable. *Rep. Stat. Apph. Res. Juse*, 19(4), 15-21.
35. Xie, M., Gaudoin, O. and Bracquemond, C. (2002). Redefining failure rate function for discrete distributions. *Int. J. Reliab., Qual. and Saf. Engg.*, 9(3), 275-285.

A NOTE ON RECORD VALUES FROM A TWO-SIDED POWER DISTRIBUTION*

Shakila Bashir§**, Khalil Ahmed** and **Munir Ahmad**
National College of Business Administration & Economics, Lahore
Email: §shakilabashir15@gmail.com

ABSTRACT

In this paper we consider the lower record values from a two-sided power distribution (TSP). Some distributional properties of the record values from the two-sided power distribution are given. The entropy, cumulative distribution function, survival function and hazard function have been derived of lower record values from two-sided power distribution. The possible shapes of pdf, cdf, entropy, survival and hazard functions of TSP from lower record value have also been discussed through graphs.

KEYWORDS AND PHRASES

Cumulative distribution function; moments; entropy; TSP; probability density function; survival function; hazard function.

1. INTRODUCTION

Record values are used in many real life applications such as sports, weather, economics, students grade sheets, purchase order, memos and any other type of documents. Let $X_1, X_2, \cdots$ be a sequence of independently and identically distributed random variables with cumulative distribution function $F(x)$, probability density function $f(x)$. Let $Y_n = \max(\min)\{X_1, X_2, \cdots, X_n\}, n = 1, 2, \cdots$. We say X_j is an upper (lower) record value of this sequence if $Y_j > (<) Y_{j-1}$, $j \geq 2$. X_1 is an upper record value as well as a lower record value.

Chandler (1952) developed the general theory of record values. Ahsanullah (1986) discussed the distributional properties of the record values from a rectangular distribution. Ahsanullah and Houchens (1989) discussed the distributional properties and estimators of record values from Pareto distribution. Ahsanullah and Bhoj (1996) discussed properties of record values from Extreme value distribution and a test statistic based on record values is proposed. Arslan and Ahsanullah (2005) considered two characterizations of the Uniform distribution using record values. Sultan, Dayian and Mohammad (2008) introduced the record values from the gamma distribution and derived the BLUEs for the location and scale parameters of the gamma distribution. Ahsanullah (2009) gave some basic properties of record values of univariate distributions and concomitants of record values. Khan and Zia (2009) gave some recurrence relations satisfied by single and product moments of upper record values from Gompertz distribution. Ahsanullah (2010) gave several properties of upper record values from exponential distribution and some characterizations are

---

* Published in Pak. J. Statist. (2014) Vol. 30(2).

discussed. Shakil and Ahsanullah (2011) investigated the distribution of record values of ratio of two independently distributed Rayleigh random variable. Moments, hazard function and entropy has been derived.

Triangular distribution has been investigated by D. Johnson (1997) as a proxy for the beta distribution, especially in problems of assessment of risk and uncertainty, such as the project evaluation and review technique. The parameters of a triangular distribution have a one-to-one correspondence with an optimistic estimate 'a' , most likely estimate 'm' and pessimistic estimate 'b' of a quantity under consideration, providing to the triangular distribution its intuitive appeal. Similarly to the beta distribution, the triangular distribution can be positively or negatively skewed (or symmetrical) but must remain unimodal. In this paper we consider the three-parameter triangular distribution, to be called the two-sided power (TSP) distribution, as a meaningful alternative to the beta distribution.

Nadarajah (2005) introduced a reformulated two-sided power distribution with the same number of parameters and compare it with the one suggested by Drop and Kotz (2002). They discussed the estimation of two-sided power distribution by the method of moments and method of maximum likelihood. The two-sided power distribution contains as special cases the triangular distribution, the standard power distribution and the uniform distribution. Drop and Kotz derived various properties of two-sided power distribution and discussed its flexibility as compared with beta family. The probability density function of two-sided power distribution is

$$f(x)=\begin{cases} 0 & ,x<a \\ \dfrac{2(x-a)}{(m-a)(b-a)} & ,a\le x\le m \\ \dfrac{2(b-x)}{(b-m)(b-a)} & ,m<x\le b \\ 0 & ,b<x \end{cases} \qquad (1.1)$$

$$a : a\in(-\infty,\infty)$$
$$b : a<b$$
$$m : a\le m\le b$$

where 'm' is most likely or mode value, 'a' is lower limit and 'b' is upper limit. If the distribution is symmetrical then 'm' is also the mean and median of the distribution. Its cumulative distribution function

$$F(x)=\begin{cases} \dfrac{(x-a)^2}{(b-a)(m-a)} & ,a\le x\le m \\ 1-\dfrac{(b-x)^2}{(b-a)(b-m)} & ,m<x\le b \end{cases} \qquad (1.2)$$

In section 2, we derive lower record values from two-sided power distribution (TSP) and its cdf. In section 3, some distributional properties (moments, survival function, hazard function, entropy) of the TSP distribution have been presented. In section 4, we discuss on numeric values of means, variances covariance and coefficients of skewness and kurtosis of the TSP.

2. TWO-SIDED POWER RECORD VALUES

Let $X_{L(1)}, X_{L(2)}, \cdots, X_{L(n)}$ be the lower record values from a sequence of $\{X_i\}$ identically independently distributed from two-sided power distribution. Then the probability density function of lower record value $X_{L(n)}$ is given by

$$f_n(x) = \frac{\left[H(x)\right]^{n-1}}{\Gamma(n)} f(x) \qquad , -\infty \le x \le \infty \tag{2.1}$$

where $H(x) = -\ln F(x) \qquad , 0 < F(x) < 1$

Now the pdf of lower record value $X_{L(n)}$ from two-sided power distribution

$$h(x) = \frac{f(x)}{S(x)} = 2\theta x^{-3}\left(\exp\left(\theta \big/ x^2\right) - 1\right)^{-1}, \; n = 1, 2, 3, \cdots \tag{2.3}$$

Note if $n = 1$ then the distribution of record value is the distribution of the parent TSP. By using (2.2) the cdf F_n of the nth record value from the two-sided power distribution is given by

$$F_n(x) = \begin{cases} \dfrac{1}{\Gamma(n)}\Gamma(n, x') & , a \le x \le m \\ 1 - \dfrac{1}{\Gamma(n)}\gamma(n, y') & , m < x \le b \end{cases} \tag{2.3}$$

where $\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$ is the upper incomplete gamma function and $\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt$ is the lower incomplete gamma function.

$$x' = -\ln\left((x-a)^2 \big/ (m-a)(b-a)\right) \text{ and } y' = -\ln\left(1 - (b-x)^2 \big/ (b-a)(b-m)\right)$$

The possible shapes of the graphs of pdf (2.2) and cdf (2.3) of the nth lower record value $X_{L(n)}$ of the two-sided power distribution when $n = 2,3,4,5$ are provided

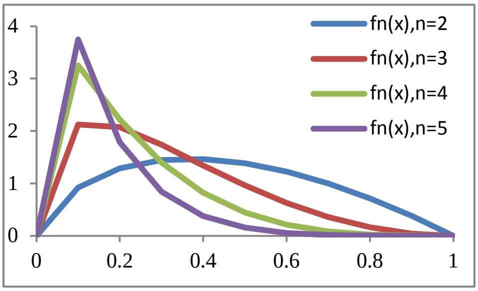


Fig. 2.1: pdf plot for $a=0, b=1, m=1, n=2,3,4,5$

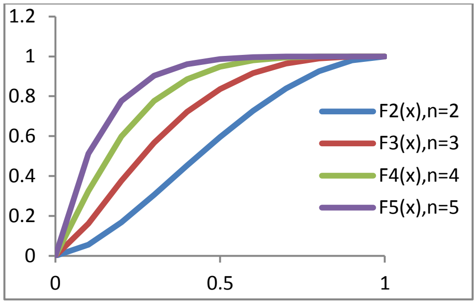


Fig. 2.2: cdf plot for $a=0, b=1, m=1, n=2,3,4,5$

3. DISTRIBUTIONAL PROPERTIES OF TWO-SIDED POWER RECORD VALUE DISTRIBUTION

In this section we will derive the moments, mode, survival function, hazard function and entropy of the lower record value from the two-sided power distribution (TSP).

3.1 Moments

The first four single moments of the two-sided power lower record value distribution with pdf (2.1) are given by,

$$\mu_1' = E(x) = \left[a\Gamma(n,\theta) + b\gamma(n,\psi) + (m-a)(b-a)(2/3)^n \Gamma(n, 3\theta/2) - \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} (1+k)^{-n} \gamma(n,(1+k)\psi) \right] \Big/ \Gamma(n) \tag{3.1}$$

$$\mu_2' = \frac{1}{\Gamma(n)} \begin{bmatrix} a^2\Gamma(n,\theta) + b^2\gamma(n,\psi) + (m-a)(b-a)2^{-n}\Gamma(n,2\theta) \\ +2a\sqrt{(m-a)(b-a)}(2/3)^n \Gamma(n,3\theta/2) \\ (b-m)(b-a)\left[\gamma(n,\psi) - 2^{-n}\gamma(n,2\psi)\right] \\ -2b\sqrt{(b-m)(b-a)} \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} (1+k)^{-n} \gamma(n,(1+k)\psi) \end{bmatrix} \tag{3.2}$$

$$\mu_3' = \frac{1}{\Gamma(n)} \begin{bmatrix} a^3\Gamma(n,\theta) + b^3\gamma(n,\psi) + 3a(m-a)(b-a)2^{-n}\Gamma(n,2\theta) \\ +3b(b-m)(b-a)\left[\gamma(n,\gamma) - 2^{-n}\gamma(n,2\psi)\right] \\ +2^n\sqrt{(m-a)(b-a)}\left[3^{1-n}a^2\Gamma(n,3\theta/2) + (m-a)(b-a)5^{-n}\Gamma(n,5\theta/2)\right] \\ -\sqrt{(b-m)(b-a)} \\ \sum_{k=0}^{\infty} (-1)^k (1+k)^{-n} \gamma(n,(1+k)\psi)\left[3b^2\binom{1/2}{k} + (b-m)(b-a)\binom{3/2}{k}\right] \end{bmatrix} \tag{3.3}$$

$$\mu_3' = \frac{1}{\Gamma(n)}\left[\begin{array}{l} a^4\Gamma(n,\theta)+b^4\gamma(n,\psi)+2^{n+2}a\sqrt{(m-a)(b-a)}\left[3^{-n}a^2\Gamma(n,3\theta/2)+5^{-n}(m-a)(b-a)\Gamma(n,5\theta/2)\right] \\ \quad +(m-a)(b-a)\left[6a^2 2^{-n}\Gamma(n,2\theta)+(m-a)(b-a)3^{-n}\Gamma(n,3\theta)\right]+6b^2(b-m)(b-a) \\ \left[\gamma(n,\psi)-2^{-n}\gamma(n.2\psi)\right]+\left((b-m)(b-a)\right)^2\left[\gamma(n,\psi)+3^{-n}\gamma(n,3\psi)-2^{1-n}\gamma(n,2\psi)\right] \\ \quad -4b\sqrt{(b-m)(b-a)}\sum_{k=0}^{\infty}(-1)^k(1+k)^{-n}\gamma(n,(1+k)\psi)\left[b^2\binom{1/2}{k}+(b-m)(b-a)\binom{3/2}{k}\right] \end{array}\right] \tag{3.4}$$

where $\theta = -\ln\left((m-a)/(b-a)\right)$ and $\psi = -\ln\left(1-(b-m)/(b-a)\right)$

By taking $a=0, b=1, m=1$ in equations (3.2), (3.3) and (3.4) we get the coefficients of skewness and kurtosis are as follows

$$\beta_1 = 2^{2n}\left(\frac{1}{5^n}-\frac{3}{6^n}+\frac{2^{2n}}{3^{3n}}\right)^2 \Bigg/ \left(2^{-n}-\frac{2^n}{3^n}\right)^3 \tag{3.5}$$

$$\beta_2 = 3^{-n}\left(1-\frac{2^{2(n+1)}}{5^n}+\frac{2^{2n+1}}{3^{n-1}}-\frac{2^{4n}}{3^{3n-1}}\right) \Bigg/ \left(2^{-n}-\frac{2^n}{3^n}\right)^2 \tag{3.6}$$

Now the joint pdf of $X_{L(r)}$ and $X_{L(s)}$

$$H_x = \sum_{x=1}^{K} h_x \neq -\ln S_x.\ r<s \tag{3.7}$$

Variances and covariance of (2.2) and (3.7) are

$$E\left(X_{L(r)}, X_{L(s)}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{r,s}(x,y)\,dxdy \tag{3.8}$$

For $a=0, b=1, m=1, r=s=1,2,3,4,5. \therefore r<s$

Table 1
Means of the lower record value from two-sided power distribution

n	**1**	**2**	**3**	**4**	**5**
μ_n	2/3	4/9	8/27	16/81	32/243

Table 2 Variances and covariance of the lower record values from two-sided power distribution

n \ ***m***	1	2	3	4	5
1	0.056				
2	0.037	0.053			
3	0.025	0.035	0.037		
4	0.016	0.023	0.024	0.023	
5	0.011	0.015	0.016	0.015	0.014

3.2 Survival Function and Hazard Function

The survival function and hazard function of the nth lower record value for the pdf (2.2) and cdf (2.3) are respectively, given by

$$S_n(x) = \begin{cases} \left(\Gamma(n) - \Gamma(n, x')\right) \big/ \Gamma(n) & a \le x \le m \\ \Gamma(n, y') / \Gamma(n) & , m < x \le b \end{cases}$$

and

$$h_n(x) = \begin{cases} \left[-\ln\left((x-a)^2/(m-a)(b-a)\right)\right]^{n-1} \left(2(x-a)/(m-a)(b-a)\right) \Big/ \left(\Gamma(n) - \Gamma(n, x')\right), & a \le x \le m \\ \left[-\ln\left(1-(b-x)^2/(b-m)(b-a)\right)\right]^{n-1} \left(2(b-x)/(b-m)(b-a)\right) \Big/ \Gamma(n, y') & , m < x \le b \end{cases}$$

where $\Gamma(a, x)$, is the upper incomplete gamma function and $\gamma(a, x)$, is the lower incomplete gamma function. The possible shapes of the survival function and hazard function are shown by graph for $n = 2,3,4,5$. From Fig. 3.1 we see the survival function is decreasing and positively skewed with longer right tail. From Fig. 3.2 we see that hazard function is increasing and then decreasing function and negatively skewed with longer left tail.

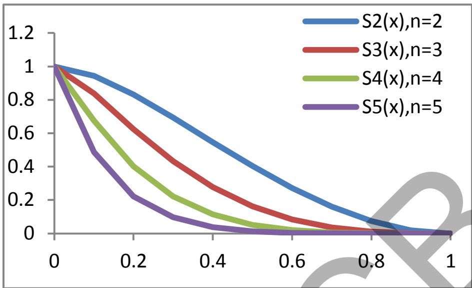


Fig. 3.1: Survival function plot for $a = 0, b = 1, m = 1, n = 1,2,3,4,5$

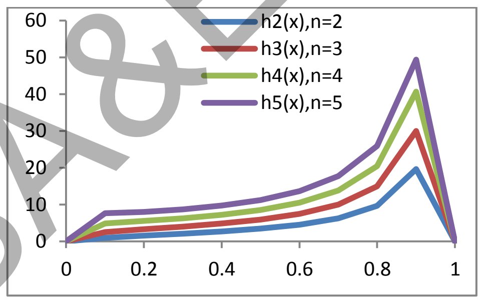


Fig. 3.2: Hazard function plot for $a = 0, b = 1, m = 1, n = 1,2,3,4,5$

3.3 Entropy

The entropy is defined as a measure of uncertainty or randomness of a random phenomenon. Shannon (1948) introduced the mathematical foundation of entropy (information theory). The entropy formula contains the expected information or uncertainty of probability distribution. If X be a random variable from a continuous probability density function, then

$$H(x) = -\int_{-\infty}^{\infty} f(x) \ln f(x) dx$$

and the entropy of the lower record values $X_{L(n)}$ is given by,

$$H_n(x) = -\int_{-\infty}^{\infty} f_n(x) \ln f_n(x) dx$$

$$= \ln\Gamma(n) - (n-1)\int_{\infty}^{\infty} \ln\left[H(x)\right] \frac{\left[H(x)\right]^{n-1}}{\Gamma(n)} f(x) dx - \int_{\infty}^{\infty} \ln f(x) \frac{\left[H(x)\right]^{n-1}}{\Gamma(n)} f(x) dx$$

Now the entropy of the two-sided power lower record value distribution is as follows

$$H_n(x) = 2\ln\Gamma n - \mathrm{I}(X) - \Psi(X)$$

where

$$\mathrm{I}(X) = \frac{(n-1)}{n^2\Gamma(n)}\left[\begin{array}{l}\theta^n\,{}_2F_2(n,n;n+1,n+1;-\theta)\,n\ln(\theta)\left\{\Gamma(n+1)-n\Gamma(n,\theta)\right\} \\ \quad + n\ln(\psi)\left\{\Gamma(n+1)-n\Gamma(n,\psi)\right\} - \psi^n\,{}_2F_2(n,n;n+1,n+1;-\psi)\end{array}\right]$$

$$\Psi(X) = \frac{1}{\Gamma(n)}\left[\begin{array}{c}\ln\left(2\big/\sqrt{(m-a)(b-a)}\right)\Gamma(n,\theta)-\Gamma(n+1,\theta)/2 \\ +\ln\left(2\big/\sqrt{(b-m)(b-a)}\right)\gamma(n,\psi) \\ -\sum_{k=1}^{\infty}\frac{1}{k}(1+k)^{-n}\,\gamma\left(n,(1+k)\psi\right)/2\end{array}\right]$$

Here we will present graph for entropy for $n = 1,2,3,4,\ldots,10$. From Fig. 3.3 we see that as n increasing the values of entropy are decreasing.

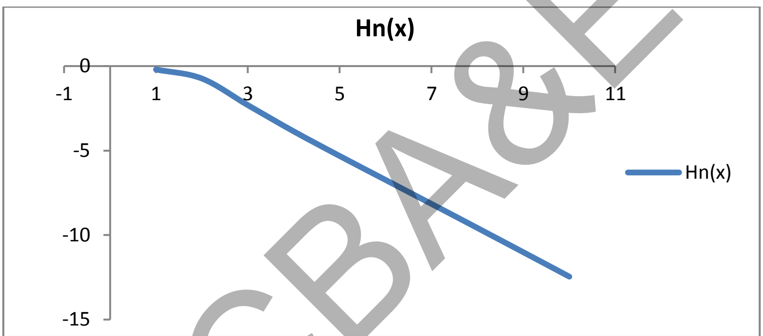


Fig. 3.3: Entropy Graph for $a = 0, b = 1, m = 1, n = 1,2,3,4,\ldots,10.$

4. CONCLUDING REMARKS

In this paper, we have discussed the distribution of lower record values when the parent distribution is the two-sided power distribution. The associated cdf, pdf, moments, survival function, hazard function, entropy etc. have been derived along with graphs to describe the shapes of respective function. The joint distribution of lower record values is derived and numeric values of means, variances, and covariance are also derived for different values of 'n'. The associated graphs of pdf shows that with the distribution of two-sided power record values is positively skewed with longer right tail, and plot shows the survival function is decreasing with 'n' increasing and plot of hazard function shows increase while decreasing function and negatively skewed with long left tail. We hope this paper will contribute a useful participation for the enhancement of research in the theory of record values.

REFERENCES

1. Ahsanullah, M. (1986). Record values from a rectangular distribution. *Pak. J. Statist*., 2, 1-5.
2. Ahsanullah, M. (1992). Record values of independent and identically distributed continuous random variables. *Pak. J. Statist*., 8, 9-34.
3. Ahsanullah, M. (2004). *Record values – theory and applications.* Inc. Dallas. Lanham. Boulder. New York.
4. Ahsanullah, M. (2009). Records and concomitants. *Bull. Malays. Math. Sci. Soc*., 32(2), 101-117.
5. Ahsanullah, M. (2010). Some characterization of exponential distribution by upper record values. *Pak. J. Statist*., 26, 69-75.
6. Ahsanullah, M. and Bhoj, D.S. (1996). Record values of extreme value distributions and a test for domain of attraction of type I extreme value distribution. *Sankhya*, 58, 151-158
7. Ahsanullah, M. and Houchens, R.L. (1989). A note on record values from a Pareto distribution. *Pak. J. Statist.*, 5, 51-57.
8. Arslan, G. and Ahsanullah, M. (2005). On characteristics properties of the Uniform Distribution. *Sankhya*, 67, 715-721.
9. Chandler, K.N. (1952). The distribution and frequency of record values. *J. Roy. Statist. Soc.*, 14, 220-228.
10. Dorp, J.R. and Kotz, S. (2002). A novel extension of the triangular distribution and its parameter estimation. *The Statistician*, 51, 63-79.
11. Gupta, R.C. and Ahsanullah, M. (2004). Some characterization results based on the conditional expectation of a function of non-adjacent order statistics (record values). *Ann.Inst. Statist. Math*., 56,721-732.
12. Ihara, S. (1993). Information theory for continuous systems. World Science Publishing Co. Pte. Ltd.
13. Johnson, D. (1997).The triangular distribution as a proxy for the beta distribution in risk analysis. *Statistician*, 46, 387-398.
14. Kacker, R.N. and Lawrence, J.E. (2007). Trapezoidal and Triangular distributions for type B evaluation of standard uncertainty. *Metrologia*, 44, 117-127.
15. Khan, R.U. and Zia, B. (2009). Recurrence relation for single and product moments of record values from Gompertz distribution and a characterization. *World Applied Sciences*, 7(10), 1331-1334.
16. Nadarajah, S. (2005). On the two-sided power distribution. *Metrika*, 61, 309-321.
17. Raqab, M.Z. (2000) on the moments of the record values. *Commun. Statist.-Theory Meth.* 29, 1631-1647.
18. Raqab, M.Z. and Ahsanullah, M. (2003). On moments generating function of records from extreme value distribution. *Pak. J. Statist*., 19, 1-13.
19. Shakil, M. and Ahsanullah, M. (2011). Record values of the ratio of Rayleigh random variables. *Pak. J. Statist*., 27, 307-325.
20. Shannon, C.E. (1948). A mathematical theory of communication. *Bell System Tech. J.,* 27, 379-423.
21. Sultan, K.S., AL-Dayian, G.R. and Mohammad, H.H. (2008). Estimation and prediction from gamma distribution based on record values. *Elsevier*, 52, 1430-1440.

PREDICTING THE SEQUENCE OF FAILURE EVENTS IN A REPAIRABLE SYSTEM*

Anwar Khalil Sheikh[1], Mohammad Saleem[1] and Munir Ahmad[2]

[1] Department of Mechanical Engineering, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia.

[2] Faculty of Mathematical Science, The University of Lahore Lahore, Pakistan

ABSTRACT

The intensity of the random arrivals of failure is a reflection of the imstantaneours rate of deterioration of a system. In this paper non-homogeneous Poisson process is used. in conjunction with log-linear process intensity, to model the failures of a mechanical or aging hype system as function of time. The reliability and failure growth characteristics of such a system are discussed by using appropriate estimation of the process parameters.

KEY WORDS

Reliability, repairable stystem, non-homoetneous Poisson, log-linear process intensity.

1. REPAIRABLE SYSTEM

By a repairable system we mean such a system which can experience several failures during its entire useful life, and after every failure event, some sort of repair action can restore all required functions of the system. Cumulative number of failures, $N(t)$, as a function of *t*, for such a system is illustrated in Figure I. Figure I represents the cumulate number of repairable failures occurred among the many solder connections in a circuit board subjected to accelerated life testing.

In several cases such as leak arrivals in pipelines (Figure 2), or level of road damage observed on a large highway as function of time (road age), etc., the growth of cumulative number of failures may have a well-defined underlying trend.

To deal with such a failure pattern in a probabilistic frame work, one can envision a population distribution of the cumulative number of failures (repairs) at age t, since different systems of the same type (i.e., a set of similar pipelines, turbines, heat pumps, fans, etc.) will accumulate different number of failures (repairs) by age t. Some accumulate no repair, some one repair, some two repairs, etc. Figure 3 shows the discrete distribution of the cumulative number of repairs (or failures) per system age, t. The distribution mean $N(\bar{r})$, is called the "Mean Cumulative Number of Failure Function of the population. Alternatively, it may he referred as, Mean Cumulative Number of Repairs (or Failures) or Recurrence Function", MCRF.

*Published in Pak. J. Statist., (2001), Vol. 17(2)

2. INTENSITY F FAILURE EVENTS

The intensity of failure events $\lambda(t)$ is defined as the derivative

$$\lambda(t) = \frac{d\bar{N}(t)}{dt} = \lim_{\Delta t \Rightarrow 0} \frac{\bar{N}(t+\Delta t) - \bar{N}}{\Delta t} \tag{2.1}$$

and is often called as "Intensity Function" of the failure process. Other names of the function $\lambda(t)$ in statistical literature are "Instantaneous Repair Rate Function" or "Recurrence Rate Function". In this paper, we will refer $\lambda(t)$ simply as Intensity Function. Figures 2 and 3 show that $\lambda(t)$ quantifies the average number of failures per unit time per system, for example failures per month per heat pump, etc. $\lambda(t)$ should not be confused with "hazard function $h(t)$ which is a reliability index of a 'non repairable system', (Usually a component)". The hazard function $h(t) = f(t)/\{1-F(t)\}$ has an entirely different definition, meaning and use. In statistical literature many authors have failed to make this distinction and thus created a lot of confusion and have obtained erroneous conclusions [1,4]. This confusion is the greatest for the renewal theory and the appropriate function characterizing the rate of expected number of renewals is known as Renewal Rate Function. $\omega(t)$ which is given by

$$\omega(t) = f(t) + \int_0^1 \omega(1-\tau)\, df(\tau)\, d\tau \tag{2.2}$$

and in general $\omega(t) \neq \lambda(t)$. Ascher and Feingold [1984] have carefully pointed out the differences between the Recurrence Rate Function of repaired systems (i.e., failure intensity), and the hazard function of non repaired system.

3. STOCHASTIC PROCESS OF FAILURE ARRIVALS

3.1 Simple Homogeneous Poisson Process

The poisson process is the simplest model for recurrent events such as failures (or repairs), and its failure intensity is constant, whose reciprocal is called the Mean Time between Failures (MTBF) in reliability analysis [See Figure 4]. Underlying assumptions of a homogeneous poisson process are:

1. $N(0) = 0$
2. $N(t) = k, t \geq 0,$ has an independent increment, i.e., failures in non-overlapping intervals are independent random variables.
3. $P\{[N(t+\Delta t) - N(t)] \geq 2\} = 0\Delta(t)\Delta t$; being very small.
4. $P\{[N(t+\Delta t) - N(t)] > 1\} = \lambda\Delta t + 0\Delta t$.

Since λ is considered independent of time in the above assumption, therefore it will result into a simple (homogeneous) poisson process [4]. The number of failures at any time *t*, have the following probability:

$$P\left[N(t)=k\right]=\frac{(\lambda t)^k}{K!}\exp[-\lambda t]=\frac{\left(\bar{N}(t)\right)^k}{K!}\exp\left[-\bar{N}(t)\right] \quad (3.1)$$

where

$$\bar{N}(t)=\lambda t \quad V\left[N(t)\right]=\lambda t$$

Coefficient of variation of number of failures is:

$$K\left\{N(t)\right\}=\frac{\sqrt{V\left(N(t)\right)}}{\bar{N}(t)}=\frac{\sqrt{\lambda t}}{\lambda t}=\frac{1}{\sqrt{\lambda t}}$$

3.2 Non Homogeneous Poisson's Process (NHPP)

For most products $\lambda(t)$ is not constant (Figure 1 & Figure 2) and its reciprocal cannot be interpreted as MTBF. Rather it is better to think only in terms of $\lambda(t)$, which has a valid meaning whether constant or not. Evans [1985] has discussed this point in more details. There are a number of parametric relationships which can be used to model $\lambda(t)$, depending upon the type of the system and the environment in which the system operates. In several instances for example, in case of leakage (failures) in a pipeline [Figure 2], we can characterize $\lambda(t)$ as a log-linear function, i.e.;

$$\ln\lambda(t)=\ln\frac{\alpha}{\theta}+\frac{t}{\theta},$$

where α and θ are the parameter of the log-linear failure intensity process, Thus

$$\lambda(t)=\frac{\alpha}{\theta}\exp\left[\frac{t}{\theta}\right] \quad (3.2)$$

and

$$\bar{N}(t)=\int_0^1 \lambda(\tau)\,d\tau=\alpha\left\{\exp\left[\frac{1}{\theta}\right]-1\right\} \quad (3.3)$$

Since $\lambda(t)$ is now a function of time, hence the assumption (iv) in section 3.1 will be expressed as

(iv) $P\left\{\left[N(t+\Delta t)-N(t)\right]>1\right\}=\lambda(t)\Delta t+0\Delta t$

or if we define a modified time scale

$$\tau=\int_0^1 \lambda(x)\,dx,$$

then $\lambda(\tau)=d(\tau)/d\tau=1,\ and\ \bar{N}(\tau)=\tau$. Which means that on a modified time scale the above condition will lead to the following probability statement, as an extension of simple (homogeneous) Poisson's process:

$$P\left\{N(t)=K\right\}=P\left\{N(\tau)=K\right\}=\frac{\left(\bar{N}(t)\right)^k}{K!}\exp\left[-\bar{N}(t)\right]$$

$$P\left\{N(t)=K\right\}=\frac{\left[\int_0^1 \lambda(x)dx\right]^k}{K!}\exp\left[-\int_0^1 \lambda(x)dx\right] \tag{3.4}$$

or

$$P\left\{N(t)=K\right\}=\frac{t^k}{K_!}\mathrm{exp}\left[-t\right]=\frac{\left\{\alpha\left[\mathrm{exp}(t/0)-1\right]\right\}^k}{K_!}\mathrm{exp}\left\{-\alpha\left[\exp\left(\frac{t}{\theta}\right)-1\right]\right\} \tag{3.5}$$

and

$$P\left\{N(t)\le K\right\}=\sum_{j=0}^{k} P\left\{N(t)=j\right\} \tag{3.6}$$

The probability of finding no leak in time *t*, (also known as the reliability of pipeline in view of first leak) is given by

$$P\left\{N(t)=0\right\}=\exp\left\{-\alpha\left[\exp\left(\frac{t}{\theta}\right)-1\right]\right\}=R(t)=R(t;\alpha,\theta) \tag{3.7}$$

Figure 5 illustrates the non homogeneous Poisson process both on actual time scale (t) and modified time scale τ.

4. MAXIMUM LIKELIHOOD ESTIMATORS OF PARAMETERS

Figure 6 Illustrate the type of information which we must have, to estimate the parameters α and θ. The maximum likelihood function can be defined by first writing a likelihood function $L\left(t_{1,}t_2,...,t_K\right)$ [Cox and Lewis (1968)].

$$L=R(t_K)\prod_{t=1}^{K}\lambda(t_i) \tag{4.1}$$

Differentiating log likelihood function with respect to α and θ equating it to zero. Thus for time location t_K, the estimate of parameters is [7].

$$\hat{0}=t_K\sum_{t=2}^{k} t_1 \tag{4.2}$$

for $k=2,3,4,....$, and

$$\hat{\alpha}=k\left\{\exp\left[\frac{-t_K}{\hat{0}}\right]\right\} \tag{4.3}$$

5. VERIFICATION OF THE PREDICTION METHODOLOGY

Several data sets were used to check the adequacy of the proposed method of failure predictions. In all such cases underlying trend was possible to characterize by log-linear intensity function, and in each case the proposed method provides very satisfactory result. In general the range of error in actual and predicted value was between 2 to 4% after a few initial failures. For illustrative purpose one set of result is presented here.

a) Data by Bradford [1970] was analyzed. (Table 1). By considering only, first n observations $n>1$, we estimated α and θ from MLE equation given in Section 4, and Calculated $n+1,\ n+2,\ n+3,....,$, etc. values of failure times. The following observations concerning the predicted and observed values:
   i) Even after first 4 observations the predictions become quite reasonable.
   ii) After first 8 observations the error in the prediction reduces significantly.

b) The parameters α and θ behave as illustrated in Figure 7 and 8. Both parameter α and θ stabilized around a steady state value θ_s and α_s. These values can he used for a long range future forecast about the failure events. These steady state values can also be used to study the economic trade offs (i.e.; replacement versus maintenance strategies), of a repairable system.

6. CONCLUSIONS

1. Repairable system can be modelled as a non-homogeneous poisson process, if the underlying pattern of failure arrivals can be expressed as a well defined parametric model of failure intensity.
2. Log-linear intensity model is one possible failure growth of characterization. The maximum likelihood estimation of future events based on this characterization gives very accurate prediction. From an engineering point of view these results are quite good.
3. Often the parameters of the failure growth model $N(t)=\int_0^1 \lambda(t)dt$ stabilizes around there steady state values and the error of prediction reduces significantly. These steady state values of the parameters can be used in economic decision making regarding repairable systems.

Table 1
Estimation of time to future leaks using Log-linear intensity process. Data from Bradford [3]

K	T	Time (Months) (Estimated) n										
		2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	17.00	-	-		-	-	-	-	-	-	-	-
2	22.00	-	-	-	-	-	-	-	-	-	-	-
4	25.00	23.25		-	-	-	-	-	-	-	-	-
6	29.00	23.73	25.91	-		-	-	-	-		-	-
7	32.00	24.75	29.01	29.95	-	-	-	-	-	-	-	
8	33.00	25.13	30.35	32.36	33.00	-	-	-	-	-	-	-
9	34.00	25.46	31.51	34.01	35.12	33.83	-	-	-		-	-
10	35.00	25.76	32.54	35.48	36.65	35.50	34.73	-.	-	-	-	-
12	35.00	26.02	33.46	36.80	38.03	36.80	36.08	35.69	-	-	-	-
13	36.00	26.48	35.07	39.21	40.45	39.08	38.27	37.82	36.02	-	-	-
14	38.00	26.68	35.77	40.14	41.52	40.09	39.24	38.76	37.48	36.47	-	-
16	39.00	26.86	36.43	41.09	42.52	41.02	40.14	39.63	38.43	37.71	38.53	-
17	40.00	27.19	37.61	42.81	44.32	42.72	41.76	41.21	40.15	39.3&	40.64	39.48
18	41.00	27.35	38.15	43.60	45.14	43.49	42.50	, 41.94	40.93	40.11	41.44,	41.08
19	42.00	27.49	38.66	44.34	45.91	44.22	43.20	42.62	41.67	40.82	42.20	41.91

$k =$ Number of leaks, $t =$ Actual Time (Months) $n =$ Number of Observations

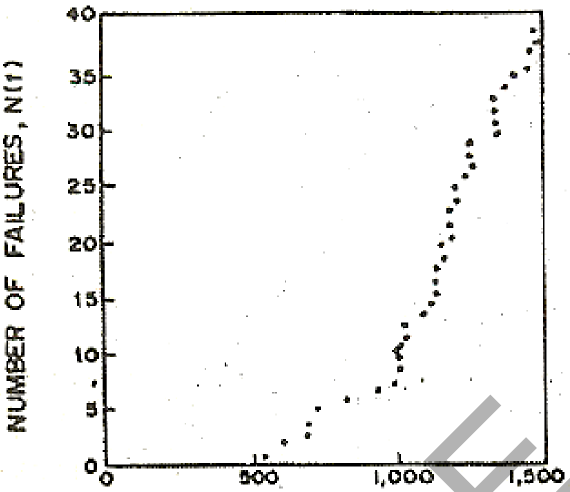
Table 2
Estimation of time to Future leaks using Log-linear intensity process. Data from NACE [2]

K	T	Time (Months) (Estimated) n								
		4	5	6	7	8	9	10	9	10
-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	-
35.78	-	-	-	-	-	-	-	-	-	-
33.60	38.33	-	-	-	-	-	-	-		-
34.41	38.05	42.61	-	-	-	-	-	-	-	-
34.37	38.87	42.28	43.22	-	-	-	-	-	-	-
34.65	39.54	43.33	42.98	44.01	-	-	-	-	-	-
34.88	40.10	44.22	43.81	43.87	44.88	-	-	-	-	-
35.08	40.61	44.98	44.54	44.53	44.72	45.80	-	-	-	-
35.26	41.02	45.66	45,17	45.16	45.36	45.66	47.86	-	47.86	-
35.41	42.41	46.27	45.75	45.72	46.93	46.25	47.77	48.80	47.77	48.80

$k =$ Number of leaks, $t =$ Actual Time (Months) $n =$ Number of Observations

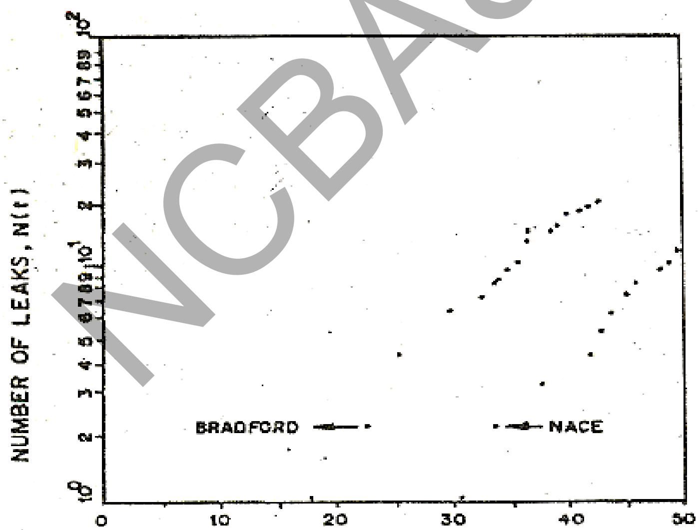
REFERENCES

1. Ascher, H. and Feingld, H. (1984). *Repairable System Reliability*, Marcel Dekker, New York, N.Y.
2. Bradrord, S.A. (1970). Corrosion Failure Expectancy: A Critical Examination, *Material Protection and Performance*, 9(7), 13.
3. Cox, D.R. and Lewis, D.A.W. (1968). *The Statistical Analysis of Series of Events*, Latimar Trend & Co. Ltd.
4. Evans, R.A. (1985). MTBF Reliability Review.
5. NACE BASIC Corrosion Course (1982). Publication of National Association of Corrosion Engineers, 6-13.
6. Nelson, Wayne (1988). Graphical Analysis of System Repair Data. *Journal of Quality Technology*, Vol. 20, No.1.



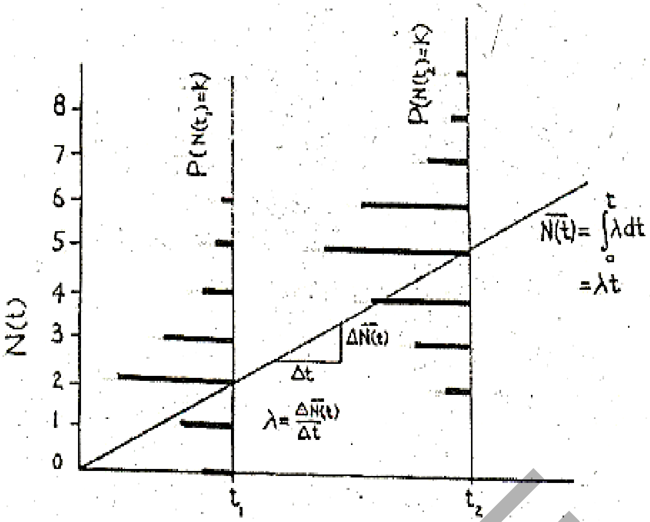
HOURS, t

Fig. 1: The cumulative number of repairable failures, N(t) by time, t.



TIME (MONTHS), t

Fig. 2: Number of leak in Pipelines, N(t) by time, t.



Age, t

Fig. 3: Cumulative number of repair, N(t) per system age, t.

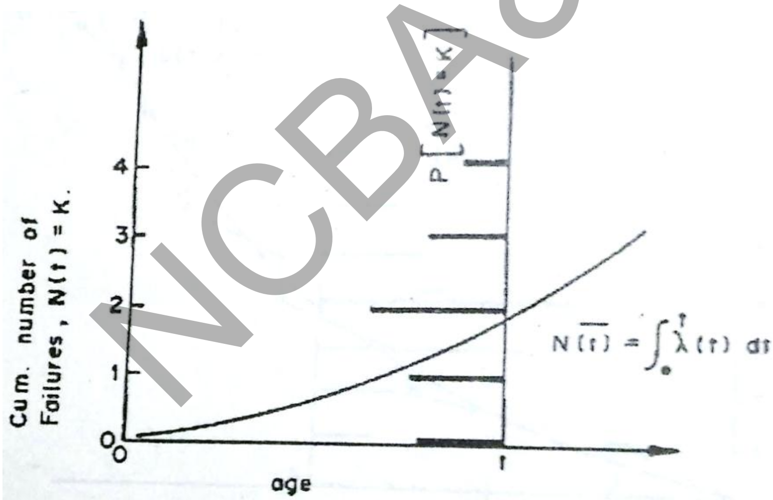
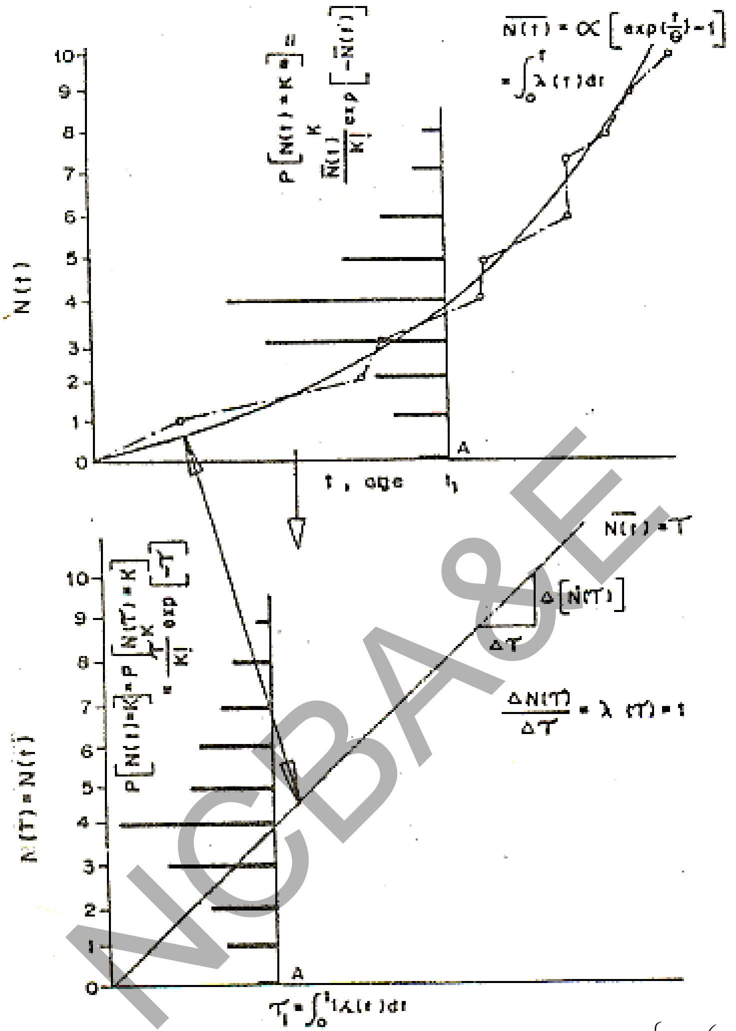
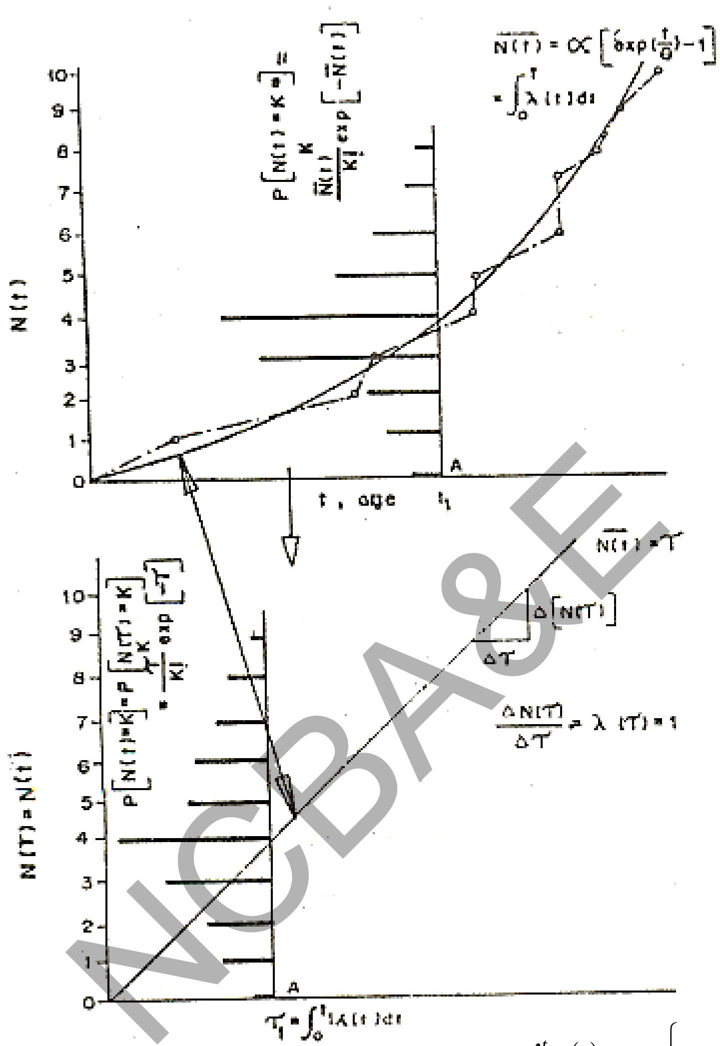


Fig. 4: Cumulative number of failures, $N(t) = k$ at age, t.



Transformed Age, $\tau = \int_0^t \lambda(t)\, d = \propto \left\{ \exp\left(\frac{t}{\theta}\right) - 1 \right\}$

Fig. 5: Non-homogeneous Poisson Process, N(t) on actual time, t and on modified time, τ



Transformed Age, $\tau = \int_0^t \lambda(t)\, d = \propto \left\{ \exp\left(\frac{t}{\theta}\right) - 1^0 \right\}$

Fig. 6: N(t) Versus Age, t

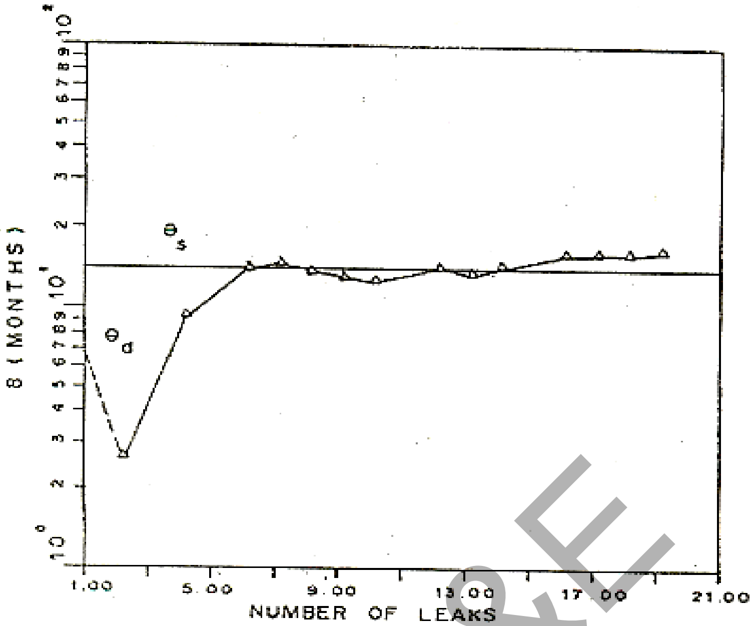


Fig. 7: Time versus Number of Leaks

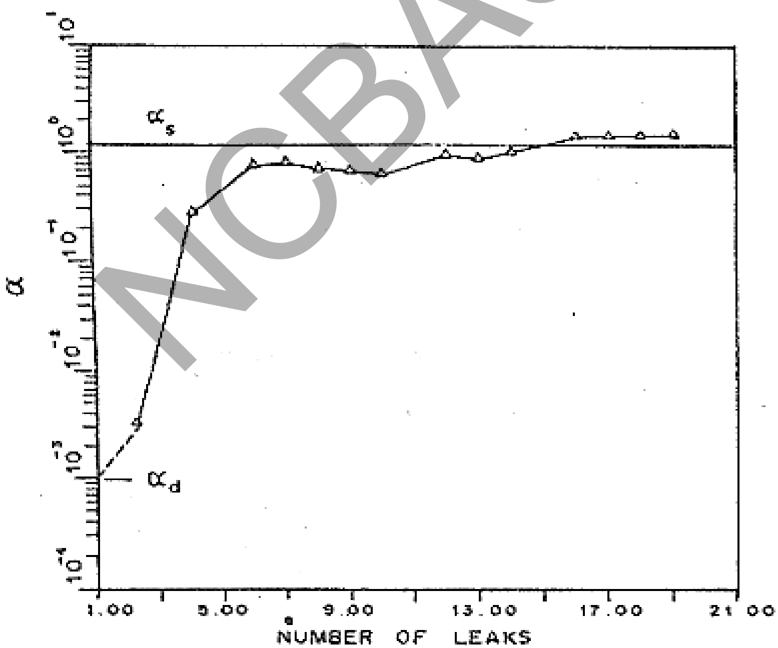


Fig. 8: α Versus Number of Leaks

ON THE APPLICATION OF SADDLE POINT METHOD IN THE DETERMINATION OF EXACT AND ASYMPTOTIC MOMENTS*

M.H. Kazi and **Munir Ahmad**
Department of Mathematical Sciences, University of Petroleum and Minerals
Dhahran, Saudi Arabia

ABSTRACT

The method of saddle-point was first introduced by Flower (1936) in to statistical mechanics and latter Jeffreys (1948) and Cox (1948) used the technique in some areas of statistics. The method may be used to determine exact moments of positive powers of random variables and asymptotic moments of negative powers of continuous random variables. Moments of positive powers of random variables are well-known, but negative movements of continuous random variables are not discussed much in literature. However, negative movements of discrete random variables have been obtained by many authors. In this paper we discussed the method and apply it to obtain exact and asymptotic moments of the powers of random variable having some exponential-type distributions.

KEY WORD

Diffuse probability density, moments of reciprocals, asymptotic expressions, normal, gamma, Rayleigh, inverted Rayleigh distributions.

1. INTRODUCTION

The problem of estimation of the reciprocals often arise in many situations, for instance, in econometrics, biological sciences, survey sampling and engineering sciences, notably in life-testing(see Zellner, 1978; Cox and Tio, 1973; Srivastsva and Bhatnagar, 1981; Bartholomew, 1957; and Epstein *et al*., 1953, 1954).

Moments of the power of reciprocals of discrete random variables have been investigated in the literature (see Chao and Strawderman, 1972; Grab and Savage, 1954; Mendenhall and Lehman, 1960; Stephen, 1945; Tiku, 1964; Kumar, and Consul, 1979; Govindarajulu, 1962; Rider, 1962; and Stephan, 1945), and recently expectation and variance of the reciprocals of continuous random variables have been approximated by Srivastava and Bhatnagar (1981) and Zellner (1978). The moments of the reciprocals of some random variables do not exist. Feller (1971) has remarked that the expectation of the square reciprocal of a normal random variable with mean zero and variance σ^2 does not exist. Zellner (1978) studies in brief the bimodality of the posterior probability distribution of the reciprocal of mean. Zellner (1978) considered the model $y = \mu + \varepsilon_i$ where the ε_i 's are NID$\left(0, \sigma^2\right)$ with the value of σ^2 assumed known with the diffuse prior probability density function $p(\mu)$ for μ, given by $p(\mu) \propto$ constant, $-\infty < \mu < \infty$.

*Published in Pak. J. Statist., (1985), Vol. 1(1).

The posterior pdf for μ is

$$p(\theta/\sigma,\bar{y}) \propto \theta^{-2} \exp\left[-\frac{z^2}{2}\left(\frac{\theta-\tilde{\theta}}{\theta}\right)^2\right],$$

where $\tilde{\theta}=1/\bar{y}, \theta=1/\mu$ and $z=\sqrt{n}\bar{y}/\sigma$. The posterior sampling distribution of θ^2 is bimodal and does not possess finite moments.

Srivastava and Bhatnagar (1981) have given some estimators of θ which possess finite moments. It is of interest to investigate the moments of the reciprocal of mean asymptotically. In this paper, we use saddle point method to obtain exact and asymptotic expressions for moments of the positive integral powers of the reciprocals of random variable having an exponential family of distributions. We illustrate the method by applying it to normal and Rayleigh random variables. The saddle point method is also used to derive asymptotic expressions for higher moments of the maximum likelihood estimator of the reciprocal of mean and the results are compared with those obtained by Srivastava and Bhatnagar (1981), Zellner (1979) and Zellnerand Park (1979).

2. THE SADDLE-POINT METHOD

Consider a random variable $X=Y^{-m}$ where Y is $f(y;\theta)$ and m is any positive integer. Properties of the probability density function of Y when X is a normal random variable are discussed by Gusev and Roshchin (1975). The $r\underline{\text{th}}$ moments about origin of the random variable X when Y is normal with mean μ and variance σ^2 is

$$\mu_r' = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} x^r u(x)\,dx, \tag{2.1}$$

where

$$u(x) = x^{-(1+1/m)} \exp\left[-\frac{1}{2\sigma^2}\left(x^{-1/m}-\mu\right)^2\right]. \tag{2.2}$$

The function $u(x)$ appears to have a singularity at $x=0$. The integral (2.1) is divergent as such, but we can find asymptotic expression for the integral for small values of σ using the steepest descent method which enables us to pick up the dominant contribution to the integral from the neighborhood of the saddle point. For the details of the saddle point method with applications to statistics, reference may be made to Daniel (1954). We confine ourselves to describe only the salient features of the method.

Consider the integral

$$1=\int_c g(z)e^{\rho h(z)}dz. \tag{2.3}$$

where c is the path of integration in the complex z-plane along the real axis and the functions $g(z)$ and $h(z)$ are functions of the complex variable z, not necessarily analytic, which as a special case may involve only real values of z. In order to evaluate the integral asymptotically for large values of ρ, the path of integration is deformed to satisfy the following conditions:

i) The path passes through a zero z_0 (called saddle point) of $h'(z)$.

ii) The imaginary part of $h(z)$ is constant on the path.

If we write $h(z) = h_1 + ih_2$ where h_1 and h_2 are real functions, h_2 is constant on a path of steepest descent, then the dominant part of the asymptotic expansion arises from the part of the path near the highest saddle-point. If the path c is deformed to pass through the saddle-point, then the integral will be obtained in the neighborhood of the saddle point. The saddle point is obtained by solving $dh/dz = 0$ and the path of integration (2.3) will be the locus of the points determined by the equation

$$h(z) = h(z_0) - s^2, -\infty < s < \infty. \tag{2.4}$$

The saddle point corresponds to the value $s = 0$. The integral (2.3) taken over c is now replaced by the integral of the same integrand over the new path of integration given by the equation (2.4) which transforms z to s through $\phi(s) \equiv g(z)\frac{dz}{ds}$ and the dominant contribution to the integral now stems from the vicinity of the saddle point.

The integral (2.3) is written as

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} e^{\rho[h(z_0)-s^2]}\phi(s)\,ds, \\ &= e^{\rho h(z_0)} \int_{-\infty}^{\infty} e^{-\rho s^2}\phi(s)\,ds. \end{aligned} \tag{2.5}$$

For large values of ρ, only small values of s will contribute significantly to the integral.

Expanding $\phi(s) = \phi(0) + s\phi^{(1)}(0) + \frac{s^2}{\alpha!}\phi^{(2)}(0) + \ldots + \frac{s^k}{k!}\theta^{(k)}(0) + \ldots$ in a series of powers of s, substituting in (2.5), and integrating over s and using the formula

$$\int_{-\infty}^{\infty} s^m e^{-\rho s^2} ds = \begin{cases} 0, & \text{when } m \text{ is odd} \\ \sqrt{2\pi}\,\dfrac{m!\left(\sqrt{2\rho}\right)^{-m-1}}{2^{m/2}(m/2)!}, & \text{when } m \text{ is even} \end{cases}$$

We obtain the following asymptotic expansion of the integral for large values of ρ:

$$1 = \exp\left[\rho h\left(z_0\right)\right]\left(\pi / \rho\right)^{\frac{1}{2}}\left[\phi(0) + \frac{1}{4\rho}\phi^{(2)}(0) + \ldots\right], \tag{2.6}$$

where

$$\phi^{(k)}(0) = \frac{d^k}{ds^k}\left[\phi(s)\right]_{s=0}, \quad k = 0,\ 1,\ 2, \ldots$$

The expression (2.6) is terminated when $\phi^{(k)}(0) = 0$ for some values of k and may become exact in some cases as shown in section 3.

3. MOMENTS OF ONE PARAMETER EXPONENTIAL DISTRIBUTION

A one-parameter family of distributions is

$$f(z;\theta) = B(\theta)\ g_1(z)\ \exp\left[\rho(\theta) h(z)\right]\ , z \in R_z. \tag{3.1}$$

The higher moments of the random variables and the reciprocals of the random variables be rewriting (3.1)

$$f_1(z,\theta) = B^{-1}(\theta)\ f(z,\theta) = g_1(z)\ \exp\left[\rho h(z)\right]$$

are given by

$$\mu_r' = E\left(z^{-1}\right)^r = \int_{R_z} z^{-r} g_1(z)\, e^{\rho h(z)} dz. \tag{3.2}$$

Let

$$g(z) = z^{-r} g_1(z).$$

The integral (3.2) is written as in (2.5)

$$\mu_r' = \int_{-\infty}^{\infty} e^{\rho\left[h(z_0) - s^2\right]}\phi(s)\, ds$$

and is expressed as in (2.6) giving exact or asymptotic moments whether $\phi^{(k)}(0) = 0$ or not for some values of k.

Some one-parameter probability distributions such as normal, gamma, exponential and Rayleigh, etc., may belong to one-parameter exponential family for which exact or asymptotic moments may be expressed in terms of functions of moments of normal random variables. Some examples are given below to illustrate the method.

3.1 Exact Moments of Normal Random Variable

If X is a normal random variable with mean, μ andvariance, σ^2, then rth moment is given by

$$\mu_r' = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} z^r \exp\left[-\frac{1}{2\sigma^2}(z-\mu)^2\right] dz. \qquad (3.3)$$

In this case, we rewrite (3.3) as

$$\sigma\mu_r' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^r \exp\left[-\frac{1}{2\sigma^2}(z-\mu)^2\right] dz.$$

If

$$g(z) = \frac{z^r}{\sqrt{2\pi}},$$

$$h(z) = -\frac{1}{2}(z-\mu)^2$$

and

$$\rho = 1/\sigma^2$$

then for small values of σ, ρ is large. The saddle point is $z_0 = \mu$ and also $h(z_0) = 0$. The path of integration in (2.4) is given by

$$z = \mu + \sqrt{2}\, s$$

and the function $\phi(s)$ is

$$\phi(s) = \frac{z^r}{\sqrt{2}}\sqrt{2} = \frac{z^r}{\sqrt{\pi}}$$

$$= \frac{1}{\sqrt{\pi}}\left(\mu + \sqrt{2}\, s\right)^r.$$

Using the saddle point method and substituting these values in (2.6), we have

$$\sigma\mu_r^{!} = \exp\left[\rho h(z_0)\right]\left(\frac{\pi}{\rho}\right)^{\frac{1}{2}}\left[\phi(0) + \frac{1}{4\rho}\phi^{(2)}(0) + \ldots\right],$$

$$\phi^{(k)}(0) = \frac{d^k}{ds^k}\left[\phi(s)\right]_{s=0},\ k = 0, 1, 2, \ldots$$

$$= \frac{1}{\sqrt{\pi}} r(r-1)\ldots(r-k+1)\mu^{r-k}$$

where $k = r$,

$$\phi^{(r)}(0) = \frac{r!}{\sqrt{\pi}}$$

$$\phi^{(k)}(0) = 0 \quad for\, k > r.$$

Thus, the exact expression for μ_r' is given by

$$\mu_r' = \frac{1}{\sigma}\sum \frac{\sqrt{2\pi}}{2^k k!}\frac{\phi^{(2k)}(0)}{\left(\sqrt{2\rho}\right)^{2k+1}}$$

$$= \frac{1}{\sigma}\sum_{k=0}^{[r/2]} \frac{\mu^{r-2k}}{k!\rho^{k+1}} \tag{3.4}$$

Let $r = 2m$, then

$$\mu_{2m}' = \mu^{2m}\sum_{k=0}^{m} \frac{(\sigma/\mu)^{2k}}{k!}. \tag{3.5}$$

3.2 Asymptotic Moments of Power of Reciprocal of Normal Random Variable

In case of the integral (2.1), we have

$$g(z) = \frac{z^{-mr}}{\sqrt{2\pi}\sigma}, \text{ where } z = r^{-1/m}$$

$$h(z) = -\frac{1}{2}(z-\mu)^2$$

$$\rho = 1/\sigma^2.$$

For small values of σ, ρ is large. The saddle point is $z_0 = \mu$ and also $h(z_0) = 0$. The path of integration in (2.4) is given by

$$z = \left(\mu + \sqrt{2}s\right),$$

and the function $\phi(s)$ is given by

$$\phi(s) = \frac{1}{\sqrt{\pi}\sigma}\left(\mu + \sqrt{2}s\right)^{-mr}$$

Using the saddle point method and substituting these values in (2.6), we have

$$\mu_r' = \mu^{-mr}\left[1 + \sum_{j=1}^{\infty}\frac{(mr)_{2j}}{2^j (j!)}\left(\frac{\sigma}{\mu}\right)^{2j}\right] \tag{3.6}$$

where

$$(a)_k = a(a+1)\ ...(a+k-1).$$

3.3 Exact Moments of Rayleigh Random Variable

Given the Rayleigh distribution,

$$f(z) = \frac{z-\mu}{\sigma^2}\exp\left[-\frac{1}{2\sigma^2}(z-\mu)^2\right], \quad z \ge \mu.$$

The $r\underline{th}$ moment about origin is

$$\mu_r' = \frac{1}{\sigma^2}\int_{\mu}^{\infty} z^r \left(z-\mu\right)\exp\left[-\frac{1}{2\sigma^2}\left(z-\mu\right)^2\right] dz. \tag{3.7}$$

Let

$$1 = \int_{\mu}^{\infty} z^r \left(z-\mu\right) e^{-1/2\sigma^2\left(z-\mu\right)^2}\, dz$$

$$h\left(z\right) = -\frac{1}{2}\left(z-\mu\right)^2, h'\left(z\right) = -\frac{2}{2}\left(z-\mu\right) = 0 \Longrightarrow z_0 = \mu.$$

$$\rho = \frac{1}{\sigma^2}$$

$$g\left(z\right) = z^r\left(z-\mu\right)$$

$$h\left(z\right) = h\left(z_0\right) - s^2$$

$$\frac{1}{2}\left(z-\mu\right)^2 = +s^2$$

$$z-\mu = +\sqrt{2}s.$$

Taking the +ve root only as $z \geq \mu$, the saddle point should be $\geq \mu$ and

$$z = \mu + \sqrt{2}s.$$

$$\phi\left(s\right) g\left(s\right)\frac{dz}{ds} = \left(\mu+\sqrt{2}s\right)^r\left(\sqrt{2}s\right)$$

$$\phi\left(0\right) = 0$$

$$\phi''\left(0\right) = 2r\mu^{r-1} + 2r\mu^{r-1} = 4r\mu^{r-1}$$

$$\phi^{iv}\left(0\right) = 16r\left(r-1\right)\left(r-2\right)\mu^{r-3}$$

$$\phi^{vi} = 2\left(r\right)_5\mu^{r-5}$$

and so on,

$$\phi^{\left(2k\right)}\left(0\right) = 2^{2k}\left(r\right)_{2k-1}\mu^{r-2k+1}$$

$$\mu_r' = \frac{1}{\sigma^2}\left(\sum_{k=0}^{\infty}\frac{\sqrt{2\pi}}{2^k\,\Gamma\left(k\right)}\frac{\phi^{\left(2k\right)}\left(0\right)}{\left(\sqrt{2\rho}\right)^{2k+1}}\right)$$

$$= \sqrt{\pi}\sum_{k=0}^{\infty}\left(\frac{\left(r\right)_{2k-1\mu^{r-2k+1}}}{\Gamma\left(k\right)}\sigma^{2k-1}\right). \tag{3.8}$$

When $\mu = 0$ then

$$\mu'_r = \sqrt{\pi} / 2r!\sigma^r / \Gamma(r+1/2). \tag{3.9}$$

The mean and variance of Rayleigh distribution when $\mu = 0$ are

$$\mu'_1 = \sqrt{\frac{\pi}{2}}\sigma$$

and

$$\mu_2 = 2\sigma^2\left(1-\frac{\mu}{4}\right).$$

3.4 **Asymptotic Moments of Inverted Rayleigh Random Variable**

Theinverted Rayleigh distribution is obtained from

$$f(x) = \frac{x-\mu}{\sigma^2} e^{-(x-\mu)^2/2\sigma^2}, \quad x \geq \mu. \tag{3.10}$$

By letting $z = \frac{1}{x}$, we obtain

$$f(z) = \frac{1-\mu z}{\sigma^2 z^3} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right], \qquad z \geq \frac{1}{\mu}.$$

The $r\underline{th}$ moment about origin is

$$\mu'_r = \int_0^\infty z^r \frac{(1-\mu z)}{\sigma^2 z^3} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] dz. \tag{3.11}$$

Let

$$h(z) = -\frac{1}{2}\left(\frac{1}{z}-\mu\right)^2, h'(z) = 0 \text{ gives } z = \frac{1}{\mu}.$$

Thus the transformation is

$$z = \left(\mu + \sqrt{2}s\right)^{-1}$$

Now consider

$$1 = \int z^{r-3}(1-\mu z)\exp\left[-\frac{1}{2\sigma^2}\left(\frac{1}{z}-\mu\right)^2\right] dz.$$

Now

$$\phi(s) = z^{3-r}\,(1-\mu z)\frac{-\sqrt{2}}{(\mu+\sqrt{2}s)}$$

$$= \left(\mu+\sqrt{2}s\right)^{-r+3}\left(1-\frac{u}{\mu+\sqrt{2}s}\right)\frac{-\sqrt{2}}{\left(\mu+\sqrt{2}s\right)^{2}}$$

$$\phi(s) = -\left(\mu+\sqrt{2}s\right)^{-r}\,2s$$

$$\phi(0) = 0$$

$$\phi''(0) = 4\sqrt{2}r\mu^{-r-3}$$

$$\varphi^{iv}(0) = 16\sqrt{2}\,(r)_3\,\mu^{-r-3}$$

$$\mu'_r = \frac{1}{\sigma^2}\sum_{k=0}^{\infty}\frac{\sqrt{2}\pi}{\Gamma(k)}\frac{(r)_{2k-1}\,\mu^{-r-2k+1}}{\left(\sqrt{2}\rho\right)^{2k+1}}$$

$$= \sqrt{2}\pi\sum_{k=0}^{\infty}\frac{r^{(2k-1)}}{\Gamma(k)}\mu^{-r-2k+1}\sigma^{2k-1} \qquad (3.12)$$

As an illustration of the usage of the method we consider the estimation of the inverse of mean and compare our results with those obtained by Srivastava and Bhatnagar (1981) who consider a similar problem.

The maximum likelihood estimate of $1/\mu$ is $1/\bar{x}$ which does not possess finite moments. Srivastava and Bhatnagar (1981), Zellner (1978) and others have recently discussed the estimation of $1/\mu$. Srivastava and Bhatnagar (1981) considered the estimator

$$t_k = n\bar{x}\,/\left(n\bar{x}^2 + ks^2\right) \quad for \quad k > 0 \qquad (4.1)$$

where $\bar{x}$ and s^2 are unbiased estimators of population mean μ and variance σ^2 respectively of a normal population. They obtained $E(t_k)$ and $E\left(t_k^2\right)$. The moments of t_k exist for $k > 0$ and for small values of k or large values of n. t_k is an approximate estimate of $1/\mu$. Following Srivastava and Bhatnagar (1981) notations, Ahmed *et al.* (1982a,b) find explicit expressions for the rth moment of S-B estimator when (i) $r = 2m$ and $r = 2m-1$:

$$\mu'_{2m} = \frac{1}{\mu^{2m}} \frac{(n/2\theta)^m e^{-n/2\theta}}{\Gamma\left(\frac{n-1}{2}\right)} \sum_{a=0}^{\infty} \sum_{j=0}^{\infty} \frac{\binom{2m+\alpha-1}{\alpha}\left(1-\frac{1}{n-1}\right)^{\alpha} \Gamma(a)\beta(b,c)}{\Gamma\left(j+\frac{1}{2}\right) j!} \cdot \left(\frac{n}{2\theta}\right)^j \quad (4.3)$$

where $a = j - 2m + \frac{n}{2}$, $b = j + 2m + \frac{3}{2}$ and $c = a + \frac{n}{2} - \frac{1}{2}$ and

$$\mu'_{2m+1} = \frac{1}{\mu^{2m+1}} \frac{(n/2\theta)^m e^{-n/2\theta}}{\Gamma\left(\frac{n-1}{2}\right)} \sum_{a=0}^{\infty} \sum_{j=0}^{\infty} \binom{2m+a-2}{a}\left(1-\frac{k}{n-1}\right)^{a}.$$

$$\frac{\Gamma(a-1)\beta(b-2,c)}{\Gamma\left(j+\frac{3}{2}\right) j!}\left(\frac{n}{2\theta}\right)^j. \quad (4.4)$$

If $m=1$, we obtain S-B expressions for $E(t_k)$ and $E\left(t_k^2\right)$.

Similar results can be obtained when the variance σ^2 is unknown. If σ^2 is unknown then σ^2 is replaced by its unbiased estimator s^2. Ahmed *et al.* (1982a) find the asymptotic expression for the *r*th moment of the randomvariable $(\bar{x})^{-1}$ for large n using the formula (2.6). Here

$$g(\bar{x}) = \frac{\bar{x}}{\sqrt{2}\pi\sigma^2},$$

$$h(\bar{x}) = -\frac{1}{2\sigma^2}(\bar{x}-\mu)^2,$$

and

$$\rho = n,$$

The saddle point is $\bar{x}_0 = \mu$ and also $h(\bar{x}_0) = 0$. The transformation from $\bar{x}$ to s is given by

$$x = \left(\mu + \sqrt{2}s\right)$$

and the function $\phi(s)$ is given by

$$\varphi(s) = \frac{1}{\sqrt{\pi}\sigma}\left(\mu + \sqrt{2}s\right)^{-r}$$

Substituting these values in (2.6), we have for large n

$$\mu'_r = \mu^{-r}\left[1 + \sum_{j=1}^{\infty} \frac{(r)_{2j}}{2^j n^j}\left(\frac{\sigma}{\mu}\right)^{2j}\right] \tag{4.7}$$

where $(a)_k = a(a+1)...(a+k-1)$. The rth moment about origin is not finite unless n is large. Using the first two terms of (4.7), we have

$$\mu'_r = \mu^{-r}\left[1 + \frac{r(r+1)}{2n}\left(\frac{\sigma}{\mu}\right)^2 + \frac{r(r+1)(r+2)(r+3)}{8n^2}\left(\frac{\sigma}{\mu}\right)^4\right], \tag{4.8}$$

If $r = 1$ and 2, the results are identical to the S-B estimator when $k \to 0$.

If μ and σ^2 are unknown, σ/μ can be replaced either by their unbiased estimators or by the consistent estimator of σ/μ which is the coefficient of variation of the observations.

REFERENCES

1. Ahmad, Munir and Kazi, M.H. (1982a). A note on the estimation of variance of a normal population. (submitted).
2. Ahmad, Munir and Kazi, M.H. (1982b). On the estimation of moments of inverse of means. (submitted).
3. Ahmad, Munir, Kazi, M.H. and Sheikh, A.K. (1982). Moments of the powers of reciprocals of the normal random variables. (submitted).
4. Bartholomew, A.J. (1957). A problem in life testing. *J. Amer. Statist. Assoc.*, 52, 350-55.
5. Box, G.E.P. and Tio, C.C. (1973). *Bayesian Inference in Statistical Analysis*. Addison Wesley Pub. Co. Inc., p. 87
6. Chao, M.T. and Strawderman, W.E. (1972). Negative moments of positive random variables. *J. Amer. Statist. Assoc.*, 429-31.
7. Cox, D.R. (1948). A note on the asymptotic distribution of the range. *Biometrika*, 35, 311-15.
8. Daniels, H.E. (1954). Saddle point approximations in statistics. *Ann. Math. Statist.*, 25, 631-650.
9. Epstein, B. and Sobel, M. (1954). Some theorems relevant to life testing. *Ann. Math. Statist.*, 25, 373-81.
10. Epstein, B. and Sobel, M. (1953). Life testing. *J. Amer. Statist. Assoc.*, 48, 486-520.
11. Feller, W. (1971). *An Introduction to Probability Theory and Its Applications*, Vol.II. John Wiley & Sons, New York.
12. Fowler, R.H. (1936). *Statistical Mechanics*. Cambridge University Press, Cambridge.
13. Govindarajulu, Z. (1962). The reciprocal of the decapitated negative binomial variable. *J. Amer. Statist. Assoc.*, 57, 906-13.
14. Gusev, A.S. and Roshchin, I.M. (1975). Calculation of the scatter of fatigue lives of machine components with random loading. *Vestnik Mashinostroeniya*, 55(10), 3-6. (U.D.C. 620.178.3).
15. Grab, E.L. and Savage, I.R. (1954). Tables of the expected value of 1/X for positive

Bernoulli and Poisson variables. *J. Amer. Statist. Assoc*., 49, 169-77.

16. Jeffreys, H. (1948). *Theory of Probability*. Oxford University Press.
17. Kumar, Ashish and Consul, P.C. (1979). Negative moments of a modified power series distribution and bias of the maximum likelihood estimator. *Commun. Statist. Theor. Meth*. A, 8(2), 151-166.
18. Mendenhall, W. and Lehmann, E.H., Jr. (1960). An approximation to the negative moments of the positive binomial useful in life testing. *Technometrics*, 2, 227-41.
19. Rider, P.R. (1962). Expected values and standard deviations of the reciprocal of a variable from a dicapitated negative binomial distribution. *J. Amer. Statist. Assoc.*, 57, 439-45.
20. Shenton, L.R. and Bowman, K. (1963). Higher moments of a maximum likelihood estimate. *J. Royal Statist. Soc.*, B, 25, p. 305.
21. Srivastava, V.K. and Bhatnagar, S. (1981). Estimation of the inverse of mean. *J. Statist. Planning and Inference*, 5, 329-334.
22. Stephan, F.F. (1945). The expected value and variance of the reciprocal and other negative powers of a positive Bernoullianvariate, *Ann. Math. Statist.*, 16,50-61.
23. Tiku, M.L. (1964). A note on the negative moments of a truncated Poisson variate. *J. Amer. Statist. Assoc.*, 39, 1220-24.
24. Zellner, A. (1978). Estimation of functions of population means and regression coefficients including structural coefficients: A minimum loss (MELO) approach. *J. Econometrics*, 8, 127-158.

GENERALIZED FOURIER TRANSFORMS AND APPLICATIONS IN PROBABILITY AND STATISTICS*

M.A. Chaudhry and **Munir Ahmed**
Department of Mathematical Sciences
King Fahd University of Petroleum & Minerals
Dhahran, Saudi Arabia

ABSTRACT

In this paper we introduce the theory of generalized functions developed by L-Schwartz (1950-51) and discuss the generalized Fourier Transform and its applications in probability and statistics. The notion of the generalized weight functions for orthogonal polynomials is introduced. The use of the generalized function technique in determining the probability functions corresponding to the given moment functions is discussed.

Subject Classification: 60-62 and 60 B15.

KEYWORDS

Generalized functions, Fourier Transforms, weight functions and singular functions.

INTRODUCTION

The singular functions have long been used in the fields of Physics and Engineering, although these cannot be properly defined within the framework of classical function theory. The simplest of the singular functions is the delta function. It is common defined as

$$\delta(s) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \tag{1}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \tag{2}$$

According to the classical definition of a function and integrals, the definitions (1) and (2) are inconsistent. There are several extensions and generalization of the concept of a mathematical function, see [3], [5] and [10]. However, we shall briefly discuss here the theory of generalized functions developed by Schwarts [14] and point out its applications in probability in the subsequent sections.

NOTATIONS

$K_a = \left\{x \in R^n : \left|x_2\right| \leq a, 1 \leq i < n\right\}$ is a compact subset of R^n and $\left\|x\right\| = \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)$ is the norm of $x = \left(x_1, \ldots x_n\right)$. Let $k = \left(k_1, k_2, \ldots . k_n\right)$ Z_+^n. Then,

*Published in Pak. J. Statist., (1989)A, 5(1),

we define

$$|k| = k_1 + k_2 + \ldots k_n$$

$$x^k = x_1^{k_1} x_2^{k_2} \ldots . x_n^{k_n}$$

$$D^K = \frac{\partial^{|k|}}{\partial x_1^{k_1} x_2^{k_2} \ldots \partial x_n^{k_n}}$$

Supp f = The support of the function $f(x) =$ the closure of the set of all points x such that $f(x) \neq 0$.

1. THE TEST SPACE, $D(a) \equiv D(K_a)$

An infinitely differentiable function $\phi(x)$ is said to belong to the space $D(a)$ if for each $p = 1, 2, 3, \ldots .$

$$\|\phi\|_p = \operatorname*{Sup}_{|k| \le px} \operatorname{Sup} \left| D^K \phi(x) \right| < \infty$$

$\|.\|_p$; $p = 1, 2, 3, \ldots .$ define a sequence of norms on $D(a)$ and :

$$\|\phi\|_1 \le \|\phi\|_2 \le \ldots \le \|\phi\|_p \le \ldots \tag{1.1}$$

Let $D_p(a)$ be the completion of $D(a)$ with respect to the norm $\|.\|_p$. Then, it follows from (1.1) that:

$$D_1(a) \supseteq D_2(a) \supseteq \ldots D_p(a) \supseteq \ldots . \supseteq D(a)$$

As a matter of fact

$$D(a) = \bigcap_{p=1}^{\infty} D_p(a)$$

and therefore, it is complete countably wormed space [3],[9].

2. THE TEST SPACE *D*

The space ***D*** consists of infinitely differentiable functions outside a compact set (depending upon the function) vanish identically. It can be seen that

$$D = \bigcup_a D(a)$$

$$= \bigcup_a \left(\bigcup_{p=1}^{\infty} D_p(a) \right)$$

REMARKS:

(1) ***D*** is complete [1], [3], [9]
(2) ***D*** is not metrizable [3]
(3) $\{\phi_n\}_{n=1} \subset D$ converges to ϕ iff $D^k\phi_n\left(\forall n = 1,2,3,\ldots.\text{and}\,\forall|k| = 0,1,2,3,\ldots\right)$ vanishes outside the same compact set K_a and $D^k\phi_n \to D^k\phi$ for all $k = 0,1,2,3,\ldots$

EXAMPLE:

Let

$$\phi(x;a) = \begin{cases} \exp\left(\dfrac{-|a|^2}{|a|^2\,|x|^2}\right) & \text{for}\,|x| < |a| \\ 0 & \text{for}\,|x| \geq |a| \end{cases} \tag{2.1}$$

Then

$$\phi(x:a) \in D(a) \subset D \text{ and supp } \phi = [-a,a]$$

Let

$$\psi_n(x) = \frac{1}{n}\phi(x,a)$$

and

$$\xi_n(x) = \frac{1}{n}\phi\left(\frac{x}{n};a\right)$$

Then, sup ψ_n = supp $\phi = [-a,a] = K_a$
and supp $\xi_n = [-na,na]$

Now $\xi_n(x) \xrightarrow{D} 0$. However, $\{\xi_n(x)\}_{n=1}^{\infty}$ does not converge to zero in the sense of the convergence in ***D*** because all of $\xi_n(x)(n = 1,2,3,\ldots)$ do not have the same support.

3. THE DISTRIBUTION SPACE *(D)*'

Let $\mathbb{C}$ be the field of complex numbers. Then, $f\ : D \to \mathbb{C}$ is said to be a continuous linear function if:

(1) $< f, \alpha\phi + \beta\psi >= \alpha < f,\phi > + \beta < f,\psi >$ and

(2) Whenever $\phi_n \xrightarrow{D} \phi, > f,\phi_n > \xrightarrow{\quad} > f,\phi_{\mathbb{C}} >$.

The space of all continuous linear functionals defined on *D* is denoted by *(D)'*. The elements of *(D)'* are called generalized functions. The convergence in *(D)'* is defined as follows:

A sequence $f_n, n = 1, 2, 3, \ldots$ of generalized functions is said to converge to generalized function $f \in (D)'$ and we write:

$$f_n \xrightarrow{(D)'} f \text{ iff } < f_n, \phi > \xrightarrow[\mathbb{C}]{} < f, \phi > \forall \phi \in D \tag{3.1}$$

4. EXAMPLES OF GENERALIZED FUNCTIONS

1. (Regular Generalized Functions):

Let $f(x)$ be a locally integrable function, i.e., $\int_{\Omega} |f(x)| dx < \infty$ for every bounded region Ω in IR^n. Then, the map $f = D \to \mathbb{C}$ defined by:

$$< f, \phi >= \int_{IR} f(x)\phi(x) dx \, \forall \phi \in D$$

defines a continuous linear functional on D and hence *is* an element of $(D)'$. Such type of generalized functions are called regular.

2. (Singular generalized functions):

Let $\delta : D \to \mathbb{C}$ be defined by:

$$< \delta, \phi >= \phi(0) \forall \phi \in D$$

Then. $\delta \in (D)'$. *We* cannot find any locally integrable function $f(x)$ for which $< \delta, \phi >=< f, \phi >= \int_{IR} f(x)\phi(x) dx \forall \phi \in D$.

Suppose there exists some $f(x)$ (locally integrable) such that (10), is satisfied. Take $\phi(x) = \phi(x; a)$ as defined in (2.1). Then,

L.H.S. $=< \delta, \phi >=< \delta(x); \phi(x; a) >= 1/e$

and

R.H.S. $=< f, \phi >= \int_{R} f(x)\phi(x)\phi(x; a) dx \to 0 \;\; as \;\; a \to 0$

which contradicts the assumption.

The type of generalized functions which are not regular (such as δ-distribution) are called singular.

REMARK

The singular generalized functions can be approximated by a sequence (or by a set) of regular generalized functions in the sense of the convergence defined in (3.1).

EXAMPLES

1. It can be seen in [1],[3],[6],[9] and [10] that among other coverging regular generalized functions we also have

   i) $\frac{\sin(nx)}{x} \xrightarrow{D} \delta(x) \ as \ n \to \infty$

   ii) The normal probability function coverges to $\delta(x)$ as $t \to 0^+$, i.e. $\frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right) \xrightarrow{D} \delta(x) \ as \ t \to 0^+$ and

   iii) $\frac{n}{\pi\left[x^2 + n^2\right]} \xrightarrow{D} \delta(x) as \ n \to 0^+$.

2. Consider the Heaviside function $H(x)$ defined by:

   $$H(x) = \begin{cases} 1 & if \ x > 0 \\ 0 & if \ x < 0 \end{cases}$$

   Than $< H, \phi >= \int_0^\infty \phi(x) dx$

   defines a regular generalized function on ***D.***

   One can define a probability function on H as follows:

   Since $H(x - x_j) = 0$ if $x < x_j$ and $H(x - x_j) = 1$ if $x > x_j$,

   Then $P(H(x - x_j) = 1) = F(x)$ and $P(H(x - x_j) = 0) = 1 - F(x)$.

3. Let us define $F^*(x): D \to \mathbb{C}$ by ;

   $$< F^*, \phi > \sum_{n=1}^{\infty} \phi(n)$$

   Then, $F^*(x) = \sum_{n=1}^{\infty} \delta(x - n)$

   is a shifted singular generalized function. It is also called sampling distribution because it gives the information about the function $\phi(x)$ at $x = n$

4. The function $\frac{1}{x}$ does not define a regular generalized function, because $\int_{-\infty}^{\infty} \frac{\phi(x)}{x} dx$ is not convergent for all test functions. Let us define $PV\left(\frac{1}{x}\right)$ as follows.

$$< PV\left(\frac{1}{x}\right), \phi >= PV\int_{-\infty}^{\infty} \frac{\phi(x)}{x} dx$$

$$= \lim_{\in \to 0^+} \int_{|x| \ge \in} \frac{\phi(x)}{x} dx$$

$$= \int_{-\infty}^{\infty} \frac{\phi(x) - \phi(0)}{x} dx$$

The integral in (11) is convergent because $\phi(x)$ is differentiable at $x = 0$. Moreover $PV\left(\frac{1}{x}\right)$ is continuous, [1],[3]. Therefore, $PV\left(\frac{1}{x}\right)$ is a singular generalized function.

5. Let us define

$$\delta_{\pm}(x) = \frac{1}{2}\delta(x) \mp \left(\frac{1}{2\pi}\right) PV\left(\frac{1}{x}\right)$$

Then, $\delta_{\pm}$, are also singular generalized functions on D and are called the Heisenberg distributions. It is shown $[1]$ that:

$$< \delta_{\pm}(x), \phi(x) = \mp \lim_{\in \downarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\phi(x)}{x \pm i \in} dx$$

It follows from (12) and (13) that:

$$\frac{1}{x + i0} = -\pi i \delta(x) + PV\left(\frac{1}{x}\right) x$$

$$\frac{1}{x - i0} = \pi i \delta(x) + PV\left(\frac{1}{x}\right) x$$

(14) and (15) are called the Sokhotski-Plemelj relations.

6. OPERATIONS ON GENERALIZED FUNCTIONS

Since the locally integrable functions are examples of generalized functions it is, therefore, natural to define operations on them that will remain valid for integrable functions.

Let $f, g \in (D)', \alpha \in C$ and $x = Ay - a$ where A is an $n \times n$ matrix with det $A \ne 0$ and a is a constant vector, be a non-singular linear transformation of the space IR^n onto itself. Then, for every $\phi \in D$ the following operations are defined:

(a) Addition: $< f + g, \phi > \; = \; < f, \phi \; < \; + < g, \phi >$

(b) Linear change of variables:

$$< f\left(Ay-a\right),\phi\left(y\right) >= \frac{1}{\left|\det A\right|} < f\left(x\right),\phi\{A^{-1}\left(x+a\right)\} > \qquad (6.1)$$

For a simple *translation, i.e.*, when $A = I$, the unit matrix, (6.1) yields

$$< f\ \left(y-a\right),\ \phi\ \left(y\right) >=\ < f\ \left(x\right),\phi\left(x\ +a\right) >$$

For *a simple scale expansion* $A = cl,\ a = 0,$ (6.1) becomes,

$$< f\ \left(cy\right),\ \phi\ \left(y\right) >= \left(\frac{1}{\left|c\right|^{n}}\right)\ < f\ \left(x\right),\phi\left(x\ \mathrm{c}\right) >$$

For a *simple reflection take* $x = -y,\ \left(c = \ -1\right)$

$$< f\left(-y\right),\ \phi\left(y\right) >=< f\ \left(x\right),\ \phi\ (I\ \mathrm{x}) >$$

For a *simple rotation,* $a = 0, A^{T} = A^{-1}$, *we* have

$$< f\ Ay),\ \phi(y) >= \frac{1}{\left|\det A\right|} < f\left(x\right),\phi\left(A^{T}x\right) >$$

(c) Multiplication by $\psi\left(t\right)C^{\infty}\left(IR'\right)$:

$$< \psi f,\phi >=< f,\psi\phi >$$

e.g.

$$< \psi\delta,\phi >= \psi[0)\phi\left(0\right).$$

Therefore

$$\psi\left(x\right)\delta\left(a\right) = \psi(0)\delta\left(x\right)$$

(d) Differentiation: The generalized derivative $D^{k}f$ of the generalized function $f\ \in\ \left(D\right)'$ is defined as follows:

$$< D^{k}f,\phi >= \left(-1\right)^{|k|} < f,D^{k}\phi > \qquad (6.2)$$

e. g.

$$< \frac{d}{dx}H\left(x\right),\phi\left(x\right) >= -< H\left(x\right),\phi'\left(x\right) >= -\int_{0}^{\infty}\phi'\left(x\right)dx = \phi\left(0\right)$$

Therefore

$$\frac{d}{dx}H\left(x\right) = \delta\left(x\right)$$

In other words, the derivative of the Heaviside function is Dirac's δ-function which was one of the properties of the δ-function deduced by Dirac.

An important consequence of the definition (6.2) is that generalized functions have derivatives of all orders. It reveals an important fact that *continuous* and locally integrable functions are infinitely differentiable in the generalized sense which gives us relief from the difficulties that arise with non-differentiable functions.

7. TENSOR PRODUCT AND CONVOLUTION OF TWO GENERALIZED FUNCTIONS

The tensor product of two generalized functions: $f \in \left(D\left(IR^n\right)\right)'$ and $g \in \left(D\left(IR^m\right)\right)'$ is defined by:

$$< f\left(x\right)\times g\left(y\right),\phi\left(x,y\right) >=< f\left(x\right),< g\left(y\right),\phi\left(x,y\right) >> \ \forall\phi\left(x,y\right)\varepsilon D\left(IR^{n+m}\right).$$

The product $f\left(x\right)\times g\left(y\right)$ belongs to $\left(D\left(IR^{n+m}\right)\right)'$. It is commutative and associative when extended to any finite number of generalized functions.

Let $f\left(x\right)$ and $g\left(x\right)$ be two locally integrable functions on IR^n. Then, their convolution;

$$f\left(x\right)* \mathrm{g}\left(x\right)=\int_{IR} n^{f(t)g(x-t)dt}$$

is also a locally integrable function $[1]$, $[2]$, $[6]$ and $[9]$. Therefore, if defines a functional on *D*, *i.e.*,

$$\begin{aligned} < f\left(x\right)^{*}\ g\left(x\right),\phi\left(x\right) > &= \int\phi\left(x\right)\int\ f\left(t\right)g\left(x - t\right)dt\ dx \\ &= \iint\ f\left(x\right)g\left(y\right)\phi\left(x+y\right)dx\ dy \text{ [By Fubini's Theorem]} \\ &=< f\left(x\right)\ \times g\left(x\right),\ \phi\left(x+y\right) > \end{aligned}$$

The convolution of two generalized functions may be defined in the same manner. The difficulty is that $\phi\left(x+y\right)$ need not have a bounded support even if $\phi\left(x\right)\in D(\mathrm{IR}^{\mathrm{n}})$ $[3]$, $[9]$. To make the definition meaningful, we can put restrictions on *f* and *g* such that the support of $f \times g$ intersects the support of $\phi\left(x+y\right)$ in a bounded set.

DEFINITION:

A generalized function *f* is said to vanish on a set a $\Omega \leq IR^n$ if $< \mathrm{f},\phi > 0$ for all $\phi\in D$ *with* supp $\phi\subset\Omega$. The complement of the union of open sets Ω on which f vanishes is a closed set, called the support of the generalized function f.

If g has a compact support, then the convolution $f * g$ is *well defined* and is given by:

$$< f * g,\ \phi > \ = \ < f(x) \times g(y),\ \eta(y)\ \phi(x+y) >$$

where η is any test function equal to 1 in the neighborhood of the support of g $[2]$, $[6]$ and $[10]$.

It is easy to verify that $\delta^m * f = f^{(m)}$. Thus, if $p(x)$ is polynomial, $P(\delta) * y = f$ is an ordinary differential equation. Another property of the convolution is that if f or g has compact support, then

$$D^k\left(f * g\right) = (D^k f) * g = f * D^k g$$

These properties give a simple proof of the existence theorem for *linear partial differential* equations with constant coefficients:

$$P(D)y = f \tag{7.1}$$

The existence of the fundamental solution, i.e., the existence of solution of $P(D)E = \delta$ is proved in $[3]$, $[6]$ and $[9]$. If f has a compact support, then $y = E * f$ is the generalized solution of (7.1). If follows from the fact that:

$$P(D)\left[E * f\right] = \left(P(D)E\right) * f = \delta * f = f\ .$$

8. THE TEST SPACE S

A complex valued function $\phi(x)$ is said to belong to the space S if it has the following properties:

(1) $\phi(x)$ is infinity differentiable, i.e. $\phi(x)\ C^{\infty}\left(IR^n\right)$

(2) $\phi(x)$ as well as its derivatives of all orders, vanish at infinity faster than the reciprocal of any polynomial, i.e.,

$$\left|\ x^p D^k \phi(x)\ \right| \leqq C_{pk},\ p = 0,\ 1,\ 2,\ \ldots \tag{8.1}$$

where C_{pk} is a constant depending on p, k and ϕ.

A sequence $\left\{\phi_m(x)\right\}_{m=1}$ of test functions is said to converge to $\phi(x)$ in S if for each $|k| = 0, 1, 2, 3, \ldots,$ the sequence $\{D^k \phi_m(x)\}_{m=1}$ converges uniformly to $D^k\phi(x)$ in every bounded region Ω *of* IR^n. This means that constants C_{pk} in (8.1) can be chosen independently of x such that

$$\left| \ x^p (D^k \phi_m - D^k \phi \ \right| < \ C_{pk}$$

for all values of m.

The space S is closed and the testing function space D is dense in S $[1],[3],[9]$ and $[12]$. The dual space of S is denoted by $(S)'$. The elements of $(S)'$ are called distributions of slow growth or tempered distributions. If follows from the definition of convergence in D and in S that a sequence $\{\phi_m(x)\}$ converging to a function $\phi(x)$ in the sense D of also converges to $\phi(x)$ in the sense of S. Accordingly every linear continuous functional on S is also a linear continuous function on D and therefore, $(S)' \subset (D)'$. This inclusion is strict because the distributions which grow too rapidly at infinity are not elements of $(S)'$. For example the regular distribution $f = exp\left(x^2\right) \in (D)'$ but is not a member of $(S)'$.

9. THE FOURIER TRANSFORM

An essential part of the theory of generalized function and its application rests on the concept of the Fourier Transform. If $\phi(x)$ is an absolutely integrable on the real line then, its Fourier Transform is defined by:

$$\hat{\phi}(u) = \int_{-\infty}^{\infty} e^{iux} \phi(x) dx \qquad (9.1)$$

the integral in (9.1) exists, since by assumption $\left| \ \phi(u) \ \right| \ \leq \int_{-\infty}^{\infty} dx \left|\phi(x)\right| < \infty$. If moreover, $\hat{\phi}(u)$ is absolutely integrable, the inverse Fourier Transform is given by;

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \hat{\phi}(u) du \qquad (9.2)$$

It follows that

$$\hat{\hat{\phi}}(x) = 2 - \phi(-x) \qquad (9.3)$$

Now we state a theorem which reveals the characteristics feature of the space $S[1]$ $[3]$ and $[12]$.

Theorem 9.1

The Fouries Transform as defined in (9.1) and its inverse are continuous linear and one-to-one mapping of S onto itself.

DEF (Fourier Transform of Tempered Distribution):

The Fourier Transform $\hat{\tau}(u)$ of a tempered distribution $\tau(x) \in (S)'$ is defined by:

$$< \hat{\tau}(u),\ \phi(u) <=< \tau\ (u), \hat{\phi}(u) >,\ \ \phi \in s \qquad (9.4)$$

The functional on the right hand side (9.4) is well defined because $\hat{\phi}(u) \varepsilon S$.

It is clearly linear and continuous. Hence $\hat{\tau}(u) \varepsilon (S)'$. As a matter of fact we have the following theorem $[3]$,$[6]$,$[9]$ and $[12]$.

Theorem 9.2

The generalized Fourier Transform as defined in (9.4) and its inverse as defined by:

$$< F^{-1}(\tau),\ \phi >=< \tau, F^{-1}(\phi) > \qquad (9.5)$$

are continuous linear and one-to-one mapping of $(S)'$ onto itself.

The definitions (9.4) and (9.5) are consistent with the classical definitions (9.1) and (9.2) whenever the latter are applicable.

The extensions to n-dimensional space of the definitions and results art straight forward. We shall mention the n-dimensional generalization of the results when it is necessary.

10. EXAMPLES

(a) The delta function

$$< \hat{\delta}(x), \phi\ >=< \delta(x), \hat{\phi}, >=< \delta(x), \int_{-\infty}^{\infty} e^{ixy} \phi(y)\ dy = \int_{-\infty}^{\infty} \phi(y) dy =< I\ , \phi >$$

Thus. $\hat{\delta}(x) = 1$

According to (9.3) we have

$$[\hat{1}] = [\hat{\hat{\delta}}] = (2\pi)^n \delta(-x) = (2\pi)^n \delta(x)$$

or

$$F^{-1}[\delta(x)] = \frac{1}{(2\pi)^n}$$

For $n = 1$ this gives the well-known integral representation formula for the delta function:

$$\delta(x) = \frac{1}{2-} F[1] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixy} dy$$

(b) If $\tau \in (S)'$, then,

i) $D^k F[\tau] = F\left[(ix)^k \tau\right]$

ii) $F\left[P\left(\frac{d}{dx}\right)\tau\right] = P(-iu)\, F[\tau]$

iii) $F\left[x^k \tau\right] = \left(-i\frac{d}{du}\right)^k F(\tau)$

iv) $F\left[\tau(x-a)\right] = e^{iau} F(\tau)$

v) $F\left[p\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)\tau\right] p(-iu_1, -u_2, \ldots, -u_m) F[\tau]$

vi) $F\left[P(x_1, x_2, \ldots, x_n)\tau\right] = \mathrm{P}\left(-i\frac{\partial}{\partial u_1}, -i\frac{\partial}{\partial u_2}, \ldots, -i\frac{\partial}{\partial u_m}\right)\mathrm{F}[\tau]$

vii) $F\left[\tau(Ax)\right] = |\det A|^{-1} F\left[\tau\left(A^{T^{-1}}(x)\right)\right]$

For the proof see [1], [9], [11] and [14] .

(c) The Heaviside function, $n = 1$

Since $(x-\xi)\phi(x-\xi) = 0$, it follows that $y_\bullet = \delta(x-\xi)$ is a solution to the homogeneous differential equation

$$(x-\xi)\delta(x-\xi) = 0$$

Moreover, $y_p = a(x)\, PV\left(\frac{1}{x-\xi}\right)$ is a generalized solution of the inhomogeneous differential equation:

$$(x-\xi)\tau(x-\xi) = a(x) \tag{10.1}$$

where $a(x) \in \overset{\infty}{C}(IR)$.

Therefore,

$$y = \delta(x-\xi) + a(x) PV\left(\frac{1}{x-\xi}\right) \tag{10.2}$$

is the solution of (10.1).

Since $\frac{DH}{dx}(x) = \delta(x)$ we find that

$$F\left[\frac{dH}{dx}(x)\right]=(-iu)\hat{H}(u)=1$$

Therefore, by using (10.1) and (10.2) we get:

$$F\ [H\ (x)](u)=C\ \delta(u)+i\,PV\ \left(\frac{1}{u}\right)$$

Changing x to $-x$ in this formula we get:

$$F\left[H(-x)\right]=C\delta(u)-i\ PV\left(\frac{1}{u}\right)$$

$$\therefore\quad F\ \left[H\ (x)\right]+F\ [H\ (-x)]\ =2\ C\ \delta(u) \qquad (10.3)$$

$$\Rightarrow\ F\ \left[H\ (x)\right]+F\left[H(-x)\ \right]=F\ [1]=2\pi\delta(x) \qquad (10.4)$$

From the uniqueness of the Fourier Transform and by equating (10.3) and (10.4) we get $C=\pi$.

$$\therefore\ F\left[H(x\right](u)=\pi\delta(u)+iPV\left(\frac{1}{u}\right) \qquad (10.5)$$

If we write (10.5):

$$\int_{-\infty}^{\infty} H(x)e^{ixu}dx=\pi\delta(u)+i\,PV\left(\frac{1}{u}\right)$$

and separate real and imaginary parts we get :

$$\int_{0}^{\infty}\ cos\ (xu)\ dx=\pi\delta(u)$$

$$\int_{0}^{\infty}\ sin(xu)\ dx=PV\left(\frac{1}{u}\right)$$

(d) The signum function:

Since Sgn $x=H\ (x)-\ H\ (-x)$

$$\Rightarrow F\ \left[sgn\ (x)\right]\ (u)=F\ \left[H\ (x)\right](u)-F[H(-x)](u)$$

$$=2i\ PV\left(\frac{1}{u}\right)$$

Therefore, by inversion formula:

$$F\ [PV\ \left(\frac{1}{u}\right)]\ (x)=\pi\, i\ \text{sgn}\, x$$

$$\Rightarrow F\ \left[PV\left(\frac{1}{u-a}\right)\right]\ (x)=\pi\, i\, e^{iax}\text{sgn}\, x$$

(e) Since $\frac{1}{x^m} = \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}\left(\frac{1}{x}\right)$

$$\Rightarrow \; F\left[\frac{1}{x^m}\right](u) = \frac{(-1)^{m-1}}{(m-1)!}(-iu)^{m-1} F\left[\frac{1}{x}\right](u)$$

$$= i^m \pi \frac{u^{m-1}}{(m-1)!}\operatorname{sgn} u$$

By the translation property:

$$F\left[\frac{1}{(x-a)^m}\right](u) = i^m \frac{u^{m-1}}{(m-1)!} e^{iau} \operatorname{sgn} u$$

11. DISTRIBUTIONAL WEIGHT FUNCTIONS

Let $P_n(x)$ be a polynomial of degree n such that

$$\int_a^b P_n(x)\, p_m(x)\, w(x)\, dx = 0, \; \mathrm{m} \neq n$$

i.e., $\{P.\ (x)\}_{n-1}$ is an orthogonal sequence of polynomials with respect to the weight function $w\ (x)$. The numbers $\mu_n\ (n = 0,1,2,...)$ defined by:

$$\mu_n = \int_a^b x^n w(x)\, dx \tag{11.1}$$

are called moments. The moments play an important role in the theory of orthogonal polynomials. As a matter of fact, every polynomial can be expressed in term of its moments [13].

It follows from (11.1) that if we know the weight function of an orthonormal polynomial sequence, then we can calculate the moments. The theory of generalized functions helps us in solving the invers problem. That is, given the moments, we can determine the corresponding weight functions. For this purpose (11.1) can be written in the functional from as:

$$p_n = < w\ (x),\ x^n > \text{ for all } n = 0,\ 1,\ 2,\ 3,...$$

Let $\psi(x)$ a *real analytic function* whose Taylor series converges to for all x.

Then,

$$< w, \phi > = < w, \sum_{n=0}^{\infty} \frac{\psi^{(n)}(0)}{n!} x^n >$$

$$= \sum_{n=0}^{\infty} \frac{\psi^{(n)}(0)}{n!} w, x^n >$$

Since $(-1)^n \psi^{(n)}(0) = <\delta^{(n)}(x), \psi(x)>$ [By definition of $\delta^{(n)}(x)$]

Therefore,

$$< w, \psi > = < \sum_{n=0}^{\infty} (-1)^n \frac{\mu_n}{n!} \delta^{(n)}(x), \psi >$$

which implies

$$w(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n \delta^{(n)}(x)$$

in the sense of generalized function. It is important to note that when the moments $\{U_i\}_{i=0}^{\infty}$ are those associated with the classical orthogonal polynomiaIs-the Legendre polynomials, the Laguerre polyomials, or the Hermite polynomials-the weight function $w(x)$ yields the same results as the classical weight functions concerning orthogonality and norms. However, when the moment $\{U_i\}_{i=0}^{\infty}$ are those associated with the Jacobi polynomials or the generalized Laguerre polynomials, then w remains a suitable generalized weight function belonging to certain space of generalized function[6],[12] and [13] .

12. APPLICATIONS TO PROBABILITY AND STATISTICS

The theory of generalized functions developed by Schwartz has advantage over the measure theory and singular integral treatments used to explain singular integrals occuring in probability and statistics.

Probability Distributions

Let X be a random variable taking real values in $(-\infty, \infty)$ and $\phi(t)$ be the probability distribution function. The probability distribution $\phi(t)$ is called discrete or continous according to whether *t* takes on discrete or continuous values. For example,

$$\phi(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-(t-\mu)^2/2\sigma^2} dt$$

is well known Gaussian distribution which is continuous, and in this case

$$\phi(t) = \frac{d\phi}{dt} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\sigma^2}$$

It should be noted that $\phi(t)$ is a function, not a functional. Therefore, the probability distribution and generalized functions refer to different mathematical objects.

Suppose that it is certain that the random variable X takes the value x_0. Then:

$$\phi(\mathrm{t}) = 0 \text{ for } t < x_0$$
$$\phi(\mathrm{t}) = 1 \text{ for } t > x_0$$

Thus, $\phi(\mathrm{t}) = \mathrm{H}(\mathrm{t} - \mathrm{x}_0)$ is the Heaviside step function. In this case the probability density $\phi(t)$ does not exist in the ordinary sense. However, in the sense of generalized functions we have

$$\phi(t) = \delta(t - x_0)$$

Similarly, if the random variable X takes the values $x_1, x_2, \ldots, x_n,$ with the probabilities $p_1, p_3, \ldots p_n,$ respectively, such that $\sum_{i=1}^{n} p_i = 1,$ then, the probability distribution $\phi(t)$ is given by:

$$\phi(t) = \sum_{i=1}^{n} p_i H(t - x_i)$$

and the probability density function $\phi(t)$ is the generalized function given by :

$$\phi(t) = \sum_{i=1}^{n} p_i \delta(t - x_i)$$

EXAMPLE

The binomial probability distribution function $\phi(t)$ is defined by:

$$\phi\ (t) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} H(t - k)$$

The probability density function is the generalized functions.

$$\phi(t) = \frac{d}{dt}\phi(t) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \delta(t - k)$$

Characteristic Functions

Given a probability density $\phi(t)$, the characteristic function $x(u)$ defined as ;

$$x(u) = \int_{-\infty}^{\infty} e^{iut} \phi(t)\, dt$$

i.e., $x(u)$ is the Fourier transform of $\phi(t)$. Since $\phi(t)$ is the derivative of the bounded function $\phi(t)$, the characteristic function in (12.1) exists in the generalized sense.

Conversely, given a characteristic function $x(u)$ it follows from (12.1) that *the* probability density would be the inverse Fourier transform of $x(u)$, i.e.,

$$\phi(t) = F^{-1}\{x(u)\} - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iut} x(u)\, du$$

Distributional Fourier transform permits us to treat the discontinuous distributions and the casual distributions alongside of the continuous distributions.

EXAMPLE 1

Let us take $X(u) = e^{i\lambda u}$. Then, from (12.9)

$$\phi(t) = F^{-1}\left\{e^{i\lambda u}\right\} = \delta(t-\lambda)$$

EXAMPLE 2

For the Gaussian distribution the probability density function $\phi(t)$ is given by;

$$\phi(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\mu)2 / 2\sigma^2}$$

Therefore,

$$x(u) = F^{-1}\left\{\phi(t)\right\} = \exp\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right)$$

For the special case of taken to be zero, we have

$$x(u) = \exp\left(-\frac{1}{2}\sigma^2 u^2\right)$$

and

$$\phi(t) = F^{-1}\left\{x(u)\right\} = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}\left(t^2 / \sigma^2\right)}, \qquad -\infty < t < \infty.$$

Probability Fields

Let Ω be the set of elementary events and IB be the class of the subets Ω of a such that

i) The family IB contains the empty set ϕ and the total set Ω.
ii) If $A \in IB$, and α is a real number. Then $\alpha A \in IB$; and
iii) If the sets $A_1, A_2, ..., A_n$ belong to 1B, then their sum belongs to IB.

The probability measure P or IB has the following properties:

i) $P(\Omega) = 1$
ii) If the sets of $A_1, A_2, ..., A_n, \ldots$ are mutually disjoint, that is, if $A_j \cap A_i = \phi$ for $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

The system (Ω, IB, P) is called the probability field.

A random variable x is a mapping $x; \Omega \to IR$ such that

$$X^{-1}\left((-\infty, t) = \left\{w \varepsilon \Omega : x(w) \varepsilon (-\infty, t)\right\}\right) \varepsilon IB.$$

We associated with the random variable X a probability distribution function**,** $\phi(t)$ as follows:

$$\phi(t) = P\left(x^{-1}(-\infty, t)\right) = p(x(w) < t)$$

The probability distribution function $\phi(t)$ is locally integrable and hence generates a regular distribution ϕ defined by;

$$< \phi, \psi >= \int_{-\infty}^{\infty} \phi(t)\psi(t)\,dt; \forall \psi(t) D$$

Therefore, $\phi(D)'$.

On the other hand,

$$< \phi', \psi >= - < \phi, \psi, = -\int_{-\infty}^{\infty} \phi(t)\psi'(t)\,dt$$
$$= -\left[\phi(t)\psi(t)\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \psi(t)\frac{d\phi(t)}{dt}dt$$
$$= 0 + \int_{-\infty}^{\infty} \psi(t)\,d\phi(t)$$
$$=< \phi, \psi >$$

So the probability density it $\phi(t)$ is the distributional derivative of the probability distribution $\phi(t)$.

Now we can define the classical quantities in the following way;

1) The expectation value is:

$$E(x) =< t, \phi >= \int_{-\infty}^{\infty} t\phi(t)\,dt = \int_{-\infty}^{\infty} t\,d\phi(t) = \int_{\Omega} x(w)\,dp(w)$$

2) The variance is:

$$\sigma^2 =< \left(t - E(x)\right)^2 . \phi >= \int_{-\infty}^{\infty} \left\{t - E(x)\right\}^2 d\phi(t)$$
$$= \int_{-\Omega} \left\{x(w) - E(x)\right\}^2 dp(w)$$

3) The non-central *m*th moment is :

$$< t^m, \phi >= \int_{-\infty}^{\infty} t^m d\phi(t) = \int_{\Omega} \left[x(w)\right]^m dp(w).$$

4) The central *m*th moment is :

$$< \left(t - E(x)\right)^m, \phi >= \int_{-\infty}^{\infty} \left\{t - E(x)\right\}^m d\phi(t) = \int_{\Omega} = \left\{x(w) - E(x)\right\}^m dp(w)$$

In this notation we can define the characteristic function $x(u)$ by:

$$x(u) = E\left(e^{itu}\right)$$

$$=< e^{itu}, \phi >$$

and the Inverse Fourier Transform by

$$\phi =< e^{-itu}, x(u) >$$

The foregoing concepts can be extended to a finite system of random variables $X_1, X_2, ...X_n$ This system may be considered as a mapping from the set Ω into the n-dimensional space IR^n. Such a mapping is called an n-dimensional random variable, the probability distribution is now:

$$\phi(t_1, t_2, ...t_n) = P(x_1(w) < t_1, x_2(w) < t_2, ...X_n(t) < t_n)$$

The moments are given by the formula :

$$\mu_{|k|} = \int_{IR^n} t^k d\phi = \int_{IR^n} t_1^{k_1} t_2^{k_2} ...t_n^{k_n} d\phi$$

$$\int_\Omega \{(x_1(w))\}^{k_1} (x_2(w))^{k_2} ....(x_n(w))^{k_n} \, dP(w)$$

where $|k| = k_1 + k_2 + ... + k_n$.

ACKNOWLEDGEMENT

The authors are thankful to the KFUPM for the excellent research and computing facilities,

REFERENCES

1. Bremmerman, H.J. (1968). *Distributions, Complex Variables and Fourier Transforms*, Addison Wesley.
2. Butner, P.L. and Nessel, R.L. (1971). *Fourier Analysis and Approximation*, Vol. 1, Academic Press.
3. Gel'fand, I.M. and Shilov, G.E. (1964). *Generalized Functions*, Vol. 2, Academic Press, New York.
4. Gel'fand, I.M. and Vilenkin, M. Ya (1964). Generalized Functions, Vol, 4, Academic Press, New York.
5. Erdel'yi, A. (1966). *Operational Calculus and Generalized Functions*, New York: Holt. Rinehart Co.
6. Kanwal, R.P. (1983). *Generalized Functions: Theory and Technique*, Academic Press.
7. Gistescu, R. and Marinescu, G. (1973). *Applications of Theory of Distributions*, Editura Academici, Bucharest.
8. Roose, B.W. (1969). *Analytic Functions and Distributions in Physics and Engineering*, Wiley, New York.
9. Zemanian, A.H. (1965). *Distribution theory and Transform Analysis*, New York: McGraw Hill.
10. Zemanian, A.H. (1968). *Generalized Integral Transformations*, New York:

lnterscience.
11. Vladimirov, V.S. (1971). *Equations of Mathematical Physics*, Marcel Dekker, Inc., New York.
12. Chihara, T.S. (1979). *An introduction to orthogonal polynomials*, Gordon & Breach, New York.
13. Morton, R.D. and Krall, A.M. (1978). Distributional weight functions for orthogonal polynomials. *SIAM Journal on Mathematical Analysis*, 9(4), 604-626.

A GENERALIZATION OF HYPER GEOMETRIC AND GAMMA FUNCTIONS*

G. Mustafa Habibullah[1], **Abdus Saboor**[1] and **Munir Ahmad**[1]

[1] National College of Business Administration and Economics
Lahore, Pakistan.
Email: mustafa1941@yahoo.com; drmunir@ncbae.edu.pk

[2] Department of Mathematics, Kohat University of Science
and Technology, Kohat, Pakistan.
Email: dr.abdussaboor@kust.edu.pk; saboorhangu@gmail.com

ABSTRACT

Al-Saqabi et al. [4] defined a gamma-type function and its probability density function involving a confluent hypergeometirc function Φ_1 of two variables [7], where

$$\Phi_1\left(a,b;c;-\alpha x^{-\delta},\beta x^{\delta}\right)=\sum_{l,k=0}^{\infty}\frac{(a)_{k+l}\,(b)_k\left(-\alpha x^{-\delta}\right)^k\left(\beta x^{\delta}\right)^l}{(c)_{l+k}\;k!\,l!},\quad \left|\alpha x^{-\delta}\right|<1$$

and discussed some of its statistical functions. We propose extension of Φ_1 by introducing more parameters in following form:

$$\varphi\left(a,\frac{b}{2},\frac{b+1}{2},c;\frac{d}{2},\frac{d+1}{2},e,f;-\alpha x^{-\delta},\beta x^{\delta}\right)$$

$$=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}\left(\frac{b}{2}\right)_m\left(\frac{b+1}{2}\right)_m(c)_n\left(-\alpha x^{-\delta}\right)^m\left(\beta x^{\delta}\right)^n}{\left(\frac{d}{2}\right)_m\left(\frac{d+1}{2}\right)_m(e)_n\,(f)_n\,m!\,n!},\quad \left|\alpha x^{-\delta}\right|<1.$$

We then define gamma-type function involving newly defined hypergeometric function of two variables and discuss its probability density function along with some of its associated statistical functions. We use inverse Mellon transform technique to derive closed form of gamma-type function and moment generating function.

KEYWORDS

Gamma function; inverse Mellon transform; hypergeometric function of two variables; moment generating function; moments.

1. INTRODUCTION

Kobayashi [11] considered a generalized gamma function, $\Gamma_m(u,v)$. Galue et al. [8] generalized Kobayashi [11] gamma function by introducing Gauss hypergeometric function in it. Agarwal and Kalla [1] defined and studied a generalized gamma

*Published in Pak. J. Statist. (2013), Vol. 29(2).

distribution. They used a modified form of the generalized gamma function of Kobayashi [11, 12]. Ghitany [9] discussed additional properties for gamma function defined by Agarwal and Kalla [1]. Al-Musallam and Kalla [2, 3] extended gamma function by involving Gauss hypergeometric function. Al-Musallam and Kalla [2, 3] and Kalla et al. [10] then discussed some of its properties. Provost et al. [14], Saboor and Ahmad [16] and Saboor et al. [17] discussed such generalizations.

The remainder of this section is devoted to the inverse Mellin transform technique, which is central to the derivation of the closed form of gamma-type function and the moment generating function of the gamma-type distribution.

If $f(x)$ is a real piecewise smooth function that is defined and single valued almost everywhere for $x>0$ and such that $\int_0^\infty x^{k-1}\left|f(x)\right|dx$ converges for some real value k, then $M_f(s)=\int_0^\infty x^{s-1}f(x)dx$ is the Mellin transform of $f(x)$. Whenever $f(x)$ is continuous, the corresponding the inverse Mellin transform is

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}M_f(s)ds \tag{1.1}$$

which together with $M_f(s)$; constitute a transform pair. The path of integration in the complex plane is called the Bromwich path where Bromwich path is a part of integration in the complex plane running from $c-i\infty$ to $c+i\infty$, where c is a real positive number chosen so that the path lies to the right of all singularities of the analytic. Equation (1.1) determines $f(x)$ uniquely if the Mellin transform is an analytic function of the complex variable s for $c_1\le\Re(s)=c\le c_2$ where c_1 and c_2 are real numbers and $\Re(s)$ denotes the real part of s. In the case of a continuous nonnegative random variable whose density function is $f(x)$, the Mellin transform is its moment of order $(s-1)$ and the inverse Mellin transform yields $f(x)$. Letting

$$M_f(s)=\frac{\left\{\prod_{j=1}^{m}\Gamma\left(b_j+s\right)\right\}\left\{\prod_{i=1}^{n}\Gamma\left(1-a_i-s\right)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-s\right)\right\}\left\{\prod_{i=n+1}^{p}\Gamma\left(a_i+s\right)\right\}}, \tag{1.2}$$

where m,n,p,q are nonnegative integers such that $0\le n\le p,\ 1\le m\le q,$ are positive number and $a_i, i=1,\dots,p,\ b_j,\ j=1,\dots,q$, are complex number such that $-\left(b_j+v\right)\ne\left(1-a_i+\lambda\right)$ and $v,\lambda=0,1,2,\dots,\ j=1,\dots,m$, and $i=1,\dots,n$, the G-function can be defined as follows in terms of the inverse Mellin transform of $M_f(s)$:

$$f(x)=G_{p,q}^{m,n}\left(x\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_f(s)x^{-s}ds, \tag{1.3}$$

where $M_f(s)$ is as defined in (1.2) and the Bromwich path $(c-i\infty, c+i\infty)$ separates the points $s=-(b_j+\nu)$, $j=1,\dots,m$, $\nu=0,1,2,\dots$, the poles of $\Gamma(b_j+s)$, $j=1,\dots,m$, from the points $s=(1-a_i+\lambda)$, $i=1,\dots,n$, $\lambda=0,1,2,\dots,$ the poles of $\Gamma(1-a_i-s)$, $i=1,\dots,n$. Thus, one must have

$$\mathfrak{Max}_{1\le j\le m}\Re\{-b_j\} < c < \mathfrak{Min}_{1\le i\le n}\Re\{1-a_i\}. \tag{1.4}$$

The integral (1.3) converges absolutely when $m+n-\frac{1}{2}(p+q)>0$.

Moreover,

$$G_{p,q}^{m,n}\left(x\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_p\end{matrix}\right.\right)=G_{q,p}^{n,m}\left(\frac{1}{x}\left|\begin{matrix}1-b_1,\dots,1-b_q\\ 1-a_1,\dots,1-a_p\end{matrix}\right.\right). \tag{1.5}$$

For example, when $p=q$, the G-function is defined for $0<x<1$, and the identity (1.5) can be used to evaluate the hypergeometric functions for $x>1$. For the main properties of the G-function as well as applications to various disciplines, the reader is referred to Mathai [13].

2. NEW φ FUNCTION

We introduce an extension of hypergeometric function of two variables in following form:

$$\varphi\left(a,\frac{b}{2},\frac{b+1}{2},c;\frac{d}{2},\frac{d+1}{2},e,f;-\alpha x^{-\delta},\beta x^{\delta}\right)$$

$$=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}\left(\frac{b}{2}\right)_m\left(\frac{b+1}{2}\right)_m(c)_n\left(-\alpha x^{-\delta}\right)^m\left(\beta x^{\delta}\right)^n}{\left(\frac{d}{2}\right)_n\left(\frac{d+1}{2}\right)_m(e)_n(f)_n\, m!n!}$$

$$=\sum_{n=0}^{\infty}\frac{(a)_n(c)_n\left(\beta x^{\delta}\right)^n}{(e)_n(f)_n\, n!}\sum_{m=0}^{\infty}\frac{(a+n)_m\left(\frac{b}{2}\right)_m\left(\frac{b+1}{2}\right)_m\left(-\alpha x^{-\delta}\right)^m}{\left(\frac{d}{2}\right)_m\left(\frac{d+1}{2}\right)_m m!}, \tag{2.1}$$

where $\left|\alpha x^{-\delta}\right|<1$.

Using Lemma 5, [15, p.22], (2.1) becomes

$$(\alpha)_{2k}=2^{2k}\left(\frac{\alpha}{2}\right)_k\left(\frac{\alpha+1}{2}\right)_k.$$

(2.1) becomes,

$$\varphi\left(a,\frac{b}{2},\frac{b+1}{2},c;\frac{d}{2},\frac{d+1}{2},e,f;-\alpha x^{-\delta},\beta x^{\delta}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(a)_n(c)_n\left(\beta x^{\delta}\right)^n}{(e)_n(f)_n\,n!}\sum_{m=0}^{\infty}\frac{(a+n)_m(b)_{2m}\left(-\alpha x^{-\delta}\right)^m}{(d)_{2m}\,m!}. \tag{2.2}$$

$$=\frac{\Gamma(d)}{\Gamma(b)\Gamma(d-b)}\sum_{n=0}^{\infty}\frac{(a)_n(c)_n\left(\beta x^{\delta}\right)^n}{(e)_n(f)_n\,n!}\sum_{m=0}^{\infty}\frac{(a+n)_m\left(-\alpha x^{-\delta}\right)^m}{m!}\int_0^1 t_1^{\,b+2m-1}\left(1-t_1\right)^{d-b-1}dt_1$$

$$=\frac{\Gamma(d)}{\Gamma(b)\Gamma(d-b)}\sum_{n=0}^{\infty}\frac{(a)_n(c)_n\left(\beta x^{\delta}\right)^n}{(e)_n(f)_n\,n!}\int_0^1 t_1^{\,b-1}\left(1-t_1\right)^{d-b-1}\sum_{m=0}^{\infty}\frac{(a+n)_m\left(-\alpha x^{-\delta}\,t_1^{\,2}\right)^m}{m!}dt_1. \tag{2.3}$$

$$=\frac{\Gamma(d)}{\Gamma(b)\Gamma(d-b)}\sum_{n=0}^{\infty}\frac{(a)_n(c)_n\left(\beta x^{\delta}\right)^n}{(e)_n(f)_n\,n!}\int_0^1 t_1^{\,b-1}\left(1-t_1\right)^{d-b-1}\left(1+\alpha x^{-\delta}\,t_1^{\,2}\right)^{-a-n}dt_1$$

$$=\frac{\Gamma(d)}{\Gamma(b)\Gamma(d-b)}\int_0^1 t_1^{\,b-1}\left(1-t_1\right)^{d-b-1}\left(1+\alpha x^{-\delta}\,t_1^{\,2}\right)^{-a}\sum_{n=0}^{\infty}\frac{(a)_n(c)_n\left(\dfrac{\beta x^{\delta}}{1+\alpha x^{-\delta}\,t_1^{\,2}}\right)^n}{(e)_n(f)_n\,n!}dt_1. \tag{2.4}$$

$$=\frac{\Gamma(d)}{\Gamma(b)\Gamma(d-b)}\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\frac{\Gamma(f)}{\Gamma(c)\Gamma(f-c)}$$
$$\times\int_0^1\int_0^1\int_0^1 t_1^{\,b-1}t_2^{\,a-1}t_3^{\,c-1}\left(1-t_1\right)^{d-b-1}\left(1-t_2\right)^{e-a-1}\left(1-t_3\right)^{f-c-1}\left(1+\alpha x^{-\delta}\,t_1^{\,2}\right)^{-a}$$
$$\exp\left(\frac{\beta x^{\delta}t_2t_3}{1+\alpha x^{-\delta}\,t_1^{\,2}}\right)dt_1dt_2dt_3, \tag{2.5}$$

since

$$\sum_{n=0}^{\infty}\frac{(a)_n(c)_n\left(\dfrac{\beta x^{\delta}}{1+\alpha x^{-\delta}\,t_1^{\,2}}\right)^n}{(e)_n(f)_n\,n!}$$
$$=\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\frac{\Gamma(f)}{\Gamma(c)\Gamma(f-c)}\int_0^1\int_0^1 t_2^{\,a-1}t_3^{\,c-1}\left(1-t_2\right)^{e-a-1}\left(1-t_3\right)^{f-c-1}$$
$$\exp\left(\frac{\beta x^{\delta}t_2t_3}{1+\alpha x^{-\delta}\,t_1^{\,2}}\right)dt_2dt_3. \tag{2.6}$$

3. A GAMMA-TYPE FUNCTION

We define a following gamma-type function involving newly defined hypergeometric function of two variables φ.

$$H\left(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda\right)$$

$$=\int_0^\infty x^{\lambda-1}e^{-px^\delta}\varphi\left(a,\frac{b}{2},\frac{b+1}{2},c;\frac{d}{2},\frac{d+1}{2},e,f;-\alpha x^{-\delta},\beta x^\delta\right)dx, \tag{3.1}$$

where,

$$\mathrm{Re}\left(p\right)>0,\mathrm{Re}\left(\delta\right)>0,\mathrm{Re}\left(\lambda-1\right)>0,\left|Arg\left(\alpha\, x^{-\delta}\right)\right|<\pi.$$

Using (2.1), one has

$$H\left(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda\right)$$

$$=\int_0^\infty x^{\lambda-1}e^{-px^\delta}\sum_{n=0}^\infty\frac{(a)_n(c)_n\left(\beta x^\delta\right)^n}{(e)_n(f)_n\,n!}\sum_{n=0}^\infty\frac{(a+n)_m\left(\frac{b}{2}\right)_m\left(\frac{b+1}{2}\right)_m\left(-\alpha x^{-\delta}\right)^m}{\left(\frac{d}{2}\right)_m\left(\frac{d+1}{2}\right)_m m!}dx \tag{3.2}$$

$$=\sum_{n=0}^\infty\frac{(a)_n(c)_n(\beta)^n}{(e)_n(f)_n\,n!}\int_0^\infty x^{\lambda+\delta n-1}e^{-px^\delta}\,{}_3F_2\left(a+n,\frac{b}{2},\frac{b+1}{2};\frac{d}{2},\frac{d+1}{2};-\alpha\, x^{-\delta}\right)dx. \tag{3.3}$$

Note that

$$\int_0^\infty x^{\lambda+\delta n-1}e^{-px^\delta}\,{}_3F_2\left(a+n,\frac{b}{2},\frac{b+1}{2};\frac{d}{2},\frac{d+1}{2};-\alpha\, x^{-\delta}\right)dx$$

$$=\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(a+n)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\frac{1}{2\pi i}\int_0^\infty x^{\lambda+\delta n-1}e^{-px^\delta}$$

$$\times\int_{c-i\infty}^{c+i\infty}\frac{\left(\alpha\, x^{-\delta}\right)^s\Gamma(-s)\Gamma(a+n+s)\Gamma\left(\frac{b}{2}+s\right)\Gamma\left(\frac{b+1}{2}+s\right)}{\Gamma\left(\frac{d}{2}+s\right)\Gamma\left(\frac{d+1}{2}+s\right)}ds\,dx$$

$$=\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(a+n)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}$$

$$\times\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(\alpha)^s\Gamma(-s)\Gamma(a+n+s)\Gamma\left(\frac{b}{2}+s\right)\Gamma\left(\frac{b+1}{2}+s\right)}{\Gamma\left(\frac{d}{2}+s\right)\Gamma\left(\frac{d+1}{2}+s\right)}$$

$$\times\int_0^\infty x^{\lambda+\delta(n-s)-1}e^{-px^\delta}\,dx\,ds$$

$$=\frac{1}{\delta\, p^{n+\lambda/\delta}}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(a+n)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}$$

$$\times\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(\alpha\, p)^s\,\Gamma(-s)\Gamma(a+n+s)\Gamma\left(\frac{b}{2}+s\right)\Gamma\left(\frac{b+1}{2}+s\right)\Gamma\left(n+\frac{\lambda}{\delta}-s\right)}{\Gamma\left(\frac{d}{2}+s\right)\Gamma\left(\frac{d+1}{2}+s\right)}\,ds\,.$$

Hence,

$$\int_0^\infty x^{\lambda+\delta n-1}e^{-px^\delta}\,{}_3F_2\left(a+n,\frac{b}{2},\frac{b+1}{2};\frac{d}{2},\frac{d+1}{2};-\alpha\, x^{-\delta}\right)dx$$

$$=\frac{1}{\delta\, p^{n+\lambda/\delta}}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(a+n)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}$$

$$\times\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(-s)\Gamma(a+n+s)\Gamma\left(\frac{b}{2}+s\right)\Gamma\left(\frac{b+1}{2}+s\right)\Gamma(n+\lambda/\delta-s)}{\Gamma\left(\frac{d}{2}+s\right)\Gamma\left(\frac{d+1}{2}+s\right)}\left(\frac{1}{\alpha\, p}\right)^{-s}ds\,,$$

$$=\frac{1}{\delta\, p^{n+\lambda/\delta}}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(a+n)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}G_{4,3}^{3,2}\left(\frac{1}{\alpha\, p}\left|\begin{matrix}1-n-\frac{\lambda}{\delta},1,\frac{d}{2},\frac{d+1}{2}\\ a+n,\frac{b}{2},\frac{b+1}{2}\end{matrix}\right.\right). \qquad (3.4)$$

where $\operatorname{Re}(p)>0,\operatorname{Re}(\delta)>0,\operatorname{Re}((n-s)\delta+\lambda)>0$.

Equivalently, in light of (1.5), one has

$$H(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda)$$

$$=\frac{1}{\delta\, p^{\lambda/\delta}}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(a)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\sum_{n=0}^{\infty}\frac{(c)_n(\beta/p)^n}{(e)_n(f)_n\,n!}G_{3,4}^{2,3}\left(\alpha\, p\left|\begin{matrix}1-a-n,1-\frac{b}{2},1-\frac{b+1}{2}\\ 0,n+\frac{\lambda}{\delta},1-\frac{d}{2},1-\frac{d+1}{2}\end{matrix}\right.\right).$$

(3.5)

Since by Slater's theorem [13], on can express Meijer G-function as a sum of residues in terms of generalized hypergeometric functions ${}_pF_{q-1}$

$$G_{p,q}^{m,n}\left(z\left|\begin{matrix}a_1,a_2,\ldots\ldots,a_p\\ b_1,b_2,\ldots\ldots,b_q\end{matrix}\right.\right)=\sum_{h=1}^{m}\frac{\prod_{j=1}^{m}\Gamma\left(b_j-b_h\right)^{*}\prod_{j=1}^{m}\Gamma\left(1+b_h-a_j\right)z^{b_h}}{\prod_{j=m+1}^{q}\Gamma\left(1+b_h-b_j\right)^{*}\prod_{j=n+1}^{p}\Gamma\left(a_j-b_h\right)}$$

$$\times\,{}_pF_{q-1}\left(\begin{matrix}1+b_h-a_1,1+b_h-a_2,\ldots\ldots.,1+b_h-a_p\\ ;(-1)^{p-m-n}\,z\\ 1+b_h-b_1,1+b_h-b_2,\ldots\ldots.,1+b_h-b_p\end{matrix}\right),\qquad(3.6)$$

where $p<q,$ or $p=q$ and $|z|<1$, and for the poles to be distinct no pair among $b_j,\ j=1,2,\ldots..,m$, may differ by an integer or zero. The asterisks in (3.6) remind us to ignore the contribution with index $j=h$. For $m=2,\ n=p=3,\ q=4,\ a_1=1-a-n,$ $a_2=1-\frac{b}{2},\ a_3=1-\frac{b+1}{2},\ b_1=0, b_2=n+\frac{\lambda}{\delta},\ b_3=1-\frac{d}{2},\ b_4=1-\frac{d+1}{2}$, we have from (3.6).

$$H\left(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda\right)=\frac{1}{\delta\,p^{\lambda/\delta}}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(a)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\sum_{n=0}^{\infty}\frac{(c)_n\left(\beta/p\right)^n}{(e)_n\,(f)_n\,n!}$$

$$\left(\Gamma\left(n+\frac{\lambda}{\delta}\right)\frac{\Gamma(a+n)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}\times{}_3F_3\left(a+n,\frac{b}{2},\frac{b+1}{2};-n-\frac{\lambda}{\delta}+1,\frac{d}{2},\frac{d+1}{2};p\alpha\right)\right.$$

$$+\frac{(\alpha\,p)^{n+\lambda/\delta}\,\Gamma(a+2n+\lambda/\delta)\Gamma(-n-\lambda/\delta)\Gamma\left(\frac{b}{2}+n+\lambda/\delta\right)\Gamma\left(\frac{b+1}{2}+n+\lambda/\delta\right)}{\Gamma\left(\frac{d}{2}+n+\lambda/\delta\right)\Gamma\left(\frac{d+1}{2}+n+\lambda/\delta\right)}$$

$$\left.\times{}_3F_3\left(a+2n+\frac{\lambda}{\delta},\frac{b}{2}+n+\frac{\lambda}{\delta}+\frac{1}{2},\frac{b}{2}+n+\frac{\lambda}{\delta};\frac{1}{2}+\frac{\lambda}{\delta}+n+\frac{d}{2},\frac{d}{2}+n+\frac{\lambda}{\delta},n+\frac{\lambda}{\delta}+1;p\alpha\right)\right).$$

After some simple steps, we get

$$H\left(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda\right)$$

$$=\frac{1}{\delta}\left(\frac{\Gamma(\lambda/\delta)}{p^{\lambda/\delta}}\sum_{n=0}^{\infty}\frac{(a)_n(\lambda/\delta)_n(c)_n(\beta/p)^n}{(e)_n\,(f)_n\,n!}\right.$$

$$\times{}_3F_3\left(a+n,\frac{b}{2},\frac{b+1}{2};-n-\frac{\lambda}{\delta}+1,\frac{d}{2},\frac{d+1}{2};p\alpha\right)$$

$$+\frac{\alpha^{\lambda/\delta}\Gamma(d)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n(c)_n(\alpha\beta)^n}{(e)_n(f)_n\,n!}\frac{\Gamma(a+2n+\lambda/\delta)\Gamma(-n-\lambda/\delta)\Gamma\big(b+2(n+\lambda/\delta)\big)}{\Gamma(a+n)\Gamma\big(d+2(n+\lambda/\delta)\big)}$$

$$\times{}_3F_3\left(a+2n+\frac{\lambda}{\delta},\frac{b}{2}+n+\frac{\lambda}{\delta}+\frac{1}{2},\frac{b}{2}+n+\frac{\lambda}{\delta};\ \frac{1}{2}+\frac{\lambda}{\delta}+n+\frac{d}{2},\frac{d}{2}+n+\frac{\lambda}{\delta},n+\frac{\lambda}{\delta}+1;p\alpha\right)\Bigg). \tag{3.7}$$

3.1 Particular Cases

- For $\alpha=0$, we get from (3.2) and (3.7)

$$H(a,c,e,f;0,\beta,\delta;p,\lambda)$$

$$=\int_0^{\infty}x^{\lambda-1}e^{-px^{\delta}}\sum_{n=0}^{\infty}\frac{(a)_n(c)_n\left(\beta x^{\delta}\right)^n}{(e)_n(f)_n\,n!}dx$$

$$=\int_0^{\infty}x^{\lambda+\delta n-1}e^{-px^{\delta}}\,{}_2F_2\left(a,c;e,f;\beta x^{\delta}\right)dx$$

$$=\frac{1}{\delta}\left(\Gamma\left(\frac{\lambda}{\delta}\right)p^{-\frac{\lambda}{\delta}}\,{}_3F_2\left(a,\frac{\lambda}{\delta},c;e,f;\beta/p\right)\right). \tag{3.8}$$

For $c=f$,

$$H(a,e;0,\beta,\delta;p,\lambda)=\frac{1}{\delta}\left(\Gamma\left(\frac{\lambda}{\delta}\right)p^{-\frac{\lambda}{\delta}}\,{}_2F_1\left(a,\frac{\lambda}{\delta};e;\beta/p\right)\right). \tag{3.9}$$

R.H.S of (3.9) is generalized gamma function considered by Al-Zamel [5].

- For $\beta=0$, we obtain from (3.2) and (3.7)

$$H(a,b,c,d;\alpha,0,\delta;p,\lambda)$$

$$=\int_0^{\infty}x^{\lambda-1}e^{-px^{\delta}}\,{}_3F_2\left(a,\frac{b}{2},\frac{b+1}{2};\frac{d}{2},\frac{d+1}{2};-\alpha x^{-\delta}\right)dx$$

$$=\frac{1}{\delta}\left(\frac{\Gamma(\lambda/\delta)}{p^{\lambda/\delta}}\,{}_3F_3\left(a,\frac{b}{2},\frac{b+1}{2};-\frac{\lambda}{\delta}+1,\frac{d}{2},\frac{d+1}{2};p\alpha\right)\right.$$

$$+\frac{\alpha^{\lambda/\delta}\Gamma(d)}{\Gamma(b)}\frac{\Gamma(a+\lambda/\delta)\Gamma(-\lambda/\delta)\Gamma\big(b+2(\lambda/\delta)\big)}{\Gamma(a)\Gamma\big(d+2(\lambda/\delta)\big)}$$

$$\times\,{}_3F_3\left(a+\frac{\lambda}{\delta},\frac{b}{2}+\frac{\lambda}{\delta}+\frac{1}{2},\frac{b}{2}+\frac{\lambda}{\delta};\ \frac{1}{2}+\frac{\lambda}{\delta}+\frac{d}{2},\frac{d}{2}+\frac{\lambda}{\delta},\frac{\lambda}{\delta}+1;p\alpha\right)\Bigg). \tag{3.10}$$

If we set $b = d$ in (3.10) to get

$$H\left(a,c;\alpha,0,\delta;p,\lambda\right)$$

$$= \int_0^{\infty} x^{\lambda-1} e^{-px^{\delta}} {}_1F_0\left(a;-;-\alpha x^{-\delta}\right)dx$$

$$= \frac{1}{\delta}\left(\frac{\Gamma\left(\lambda/\delta\right)}{p^{\lambda/\delta}} {}_1F_1\left(a;-\frac{\lambda}{\delta}+1;p\alpha\right)\right.$$

$$\left. + \frac{\alpha^{\lambda/\delta}\Gamma\left(a+\lambda/\delta\right)\Gamma\left(-\lambda/\delta\right)}{\Gamma\left(a\right)} {}_1F_1\left(a+\frac{\lambda}{\delta};\frac{\lambda}{\delta}+1;p\alpha\right)\right). \tag{3.11}$$

If we put $\alpha = \beta = 0$, $p = 1/k$ and $\delta = k$ in (3.1) then we obtain

$$\int_0^{\infty} x^{\lambda-1} e^{-x^k/k} dx = \Gamma_k\left(\lambda\right),$$

which is k-gamma function [6].

3.2 The Incomplete Gamma-Type Function

The incomplete bivariate gamma-type function is defined as

$$\int_0^{x} u^{\lambda-1} e^{-pu^{\delta}} \varphi\left(a,\frac{b}{2},\frac{b+1}{2},c;\frac{d}{2},\frac{d+1}{2},e,f;-\alpha u^{-\delta},\beta u^{\delta}\right)du$$

$$= H_0^x\left(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda\right), \tag{3.12}$$

and the complimentary incomplete gamma-type function is defined as

$$\int_x^{\infty} u^{\lambda-1} e^{-pu^{\delta}} \varphi\left(a,\frac{b}{2},\frac{b+1}{2},c;\frac{d}{2},\frac{d+1}{2},e,f;-\alpha u^{-\delta},\beta u^{\delta}\right)du$$

$$= H_x^{\infty}\left(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda\right), \tag{3.13}$$

4. A GAMMA-TYPE PROBABILITY FUNCTION USING A NEWLY DEFINED HYPER GEOMETRIC FUNCTION OF TWO VARIABLES

We define the following gamma-type probability density function involving the newly defined hyper geometric function of two variables specified by (2.1):

$$f\left(x\right) = C x^{\lambda-1} e^{-px^{\delta}} \varphi\left(a,\frac{b}{2},\frac{b+1}{2},c;\frac{d}{2},\frac{d+1}{2},e,f;-\alpha x^{-\delta},\beta x^{\delta}\right),\ x>0, \tag{4.1}$$

where $C^{-1} = H\left(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda\right)$, $\lambda - 1 > 0,\ p > 0,\ \alpha < p,\ \delta > 0, \left|Arg\left(\alpha\right)\right| < \pi.$

As $f\left(x\right) \geq 0$, $\lim_{x\to 0} f\left(x\right) = 0$; $\lim_{x\to\infty} f\left(x\right) = 0$ and $\int_0^{\infty} f\left(x\right)dx = 1$, $f\left(x\right)$ defines a bona fide probability density function.

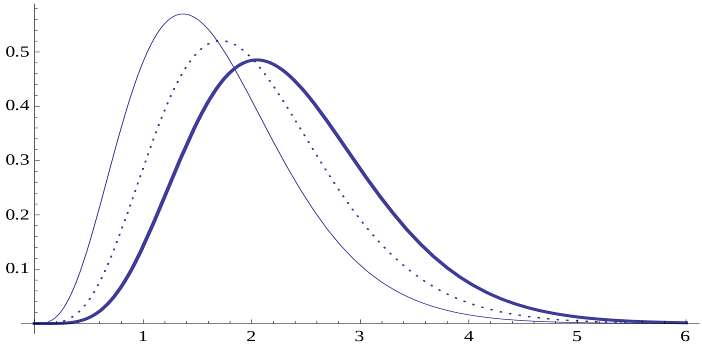


Fig. 4.1: The gamma-type pdf. $a = 1.3$; $b = 0.6$; $c = 2.7$; $\beta = 0.4$; $d = 3.4$; $e = 2.4$; $h = 4.6$; $g = 3.4$; $f = 1.7$; $\alpha = 0.3$; $p = 2.2$; $\delta = 1.2$; $\lambda = 4.2$ (solid line); $\lambda = 5.2$ (dotted line); $\lambda = 6.2$ (thick line).

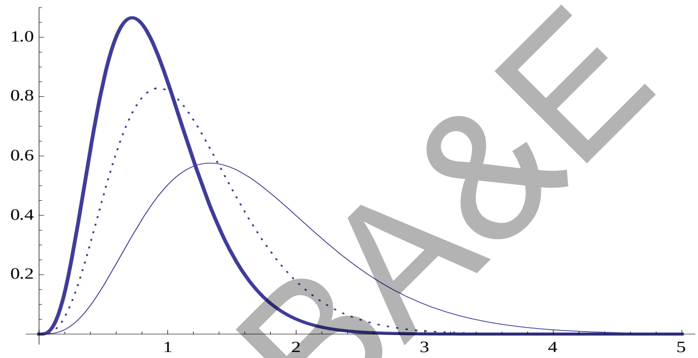


Fig. 4.2: The gamma-type pdf. $a = 1.3$; $b = 0.6$; $c = 2.7$; $\lambda = 4.1$; $d = 3.4$; $e = 2.4$; $h = 4.6$; $g = 3.4$; $f = 1.7$; $\alpha = 0.3$; $\beta = 0.4$; $\delta = 1.2$; $p = 2.2$ (solid line); $p = 3.2$ (dotted line); $p = 4.2$ (thick line).

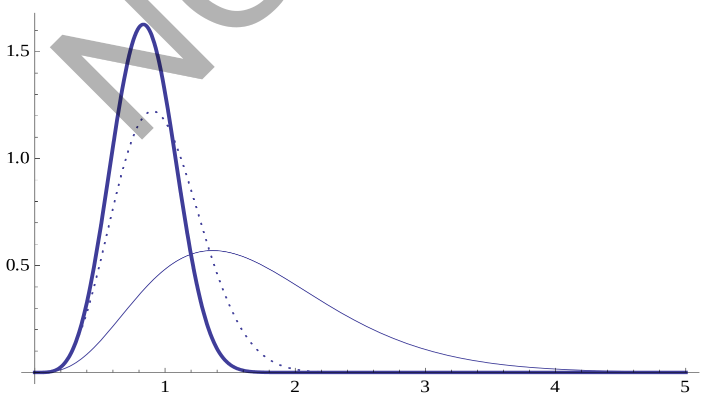


Fig. 4.3: The gamma-type pdf. $a = 1.3$; $b = 0.6$; $c = 2.7$; $\lambda = 4.2$; $d = 3.4$; $e = 2.4$; $h = 4.6$; $g = 3.4$; $f = 1.7$; $\alpha = 0.3$; $\beta = 0.4$; $p = 2.2$; $\delta = 1.2$ (solid line); $\delta = 2.2$ (dotted line); $\delta = 3.2$ (thick line).

Figure 4.1, Figure 4.2 and Figure 4.3 illustrate how the parameters λ, p and δ effect the gamma-type distribution.

4.1 Particular Cases of Probability Function

I. If we set $\alpha = 0$ and $c = f$, (4.1) becomes,

$$f(x) = \frac{\delta\, p^{\lambda/\delta} x^{\lambda-1} e^{-p x^{\delta}}}{\Gamma(\lambda/\delta)} \frac{{}_1F_1\left(a; e; \beta x^{\delta}\right)}{{}_2F_1\left(a, \lambda/\delta; e; \beta/p\right)},$$

which is modified form of pdf defined by Al-Zamel [5].

II. If we set $\beta = 0$ and $b = d$, (4.1) becomes,

$$f(x) = \frac{x^{\lambda-1} e^{-p x^{\delta}}\, {}_1F_0\left(a; -; -\alpha x^{-\delta}\right)}{\eta},$$

where,

$$\eta = \frac{1}{\delta}\left(\frac{\Gamma(\lambda/\delta)}{p^{\lambda/\delta}}\, {}_1F_1\left(a; -\frac{\lambda}{\delta}+1; p\alpha \right). + \frac{\alpha^{\lambda/\delta}\Gamma(a+\lambda/\delta)\Gamma(-\lambda/\delta)}{\Gamma(a)}\, {}_1F_1\left(a+\frac{\lambda}{\delta}; \frac{\lambda}{\delta}+1; p\alpha \right)\right)$$

5. STATISTICAL FUNCTIONS

Closed form representations of the moment generating function of a gamma-type random variable which is denoted by X, as well as the associated moments are provided in this section.

The Moment Generating Function of X

The moment generating function for density function is defined as

$$M_X(t) = E\left(e^{tX}\right) = \int_0^{\infty} e^{tx} f(x)\, dx$$

The inverse Mellin transform technique will be used to derive the moment generating function of the gamma-type distribution. Let X be a random variable whose pdf is specified by (4.1).

The moment generating function with respect to the distribution specified by (4.1), is given by

$$M_X(t) = C\int_0^{\infty} e^{tx} x^{\lambda-1} e^{-p x^{\delta}} \varphi\left(a, \frac{b}{2}, \frac{b+1}{2}, c; \frac{d}{2}, \frac{d+1}{2}, e, f; -\alpha x^{-\delta}, \beta x^{\delta} \right) dx .$$

$$= C\sum_{n=0}^{\infty} \frac{(a)_n (c)_n (\beta)^n}{(e)_n (f)_n\, n!} \int_0^{\infty} x^{\lambda+\delta n-1} e^{-p x^{\delta}} e^{tx}\, {}_3F_2\left(a+n, \frac{b}{2}, \frac{b+1}{2}; \frac{d}{2}, \frac{d+1}{2}; -\alpha x^{-\delta} \right) dx . \quad (5.1)$$

Letting $\delta=1$ and using (3.8), (5.1) becomes

$$M_X(t)=\frac{C\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(a)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\sum_{n=0}^{\infty}\frac{(c)_n(\beta)^n}{(e)_n(f)_n\,n!}$$

$$\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(\alpha)^s\,\Gamma(-s)\Gamma(a+n+s)\Gamma\left(\frac{b}{2}+s\right)\Gamma\left(\frac{b+1}{2}+s\right)}{\Gamma\left(\frac{d}{2}+s\right)\Gamma\left(\frac{d+1}{2}+s\right)}\int_0^{\infty}x^{\lambda+(n-s)-1}e^{-(p-t)x}dx\,ds$$

$$=\left(\frac{1}{p-t}\right)^{\lambda}\frac{C\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(a)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\sum_{n=0}^{\infty}\frac{(c)_n\left(\frac{\beta}{p-t}\right)^n}{(e)_n(f)_n\,n!}$$

$$\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\left(\alpha(p-t)\right)^s\,\Gamma(-s)\Gamma(a+n+s)\Gamma\left(\frac{b}{2}+s\right)\Gamma\left(\frac{b+1}{2}+s\right)\Gamma(\lambda+n-s)}{\Gamma\left(\frac{d}{2}+s\right)\Gamma\left(\frac{d+1}{2}+s\right)}ds\,,$$

where, $\operatorname{Re}(p)<\operatorname{Re}(t)$, $\operatorname{Re}(\lambda)>\operatorname{Re}(s)$.

Hence,

$$M_X(t)=\left(\frac{1}{p-t}\right)^{\lambda}\frac{C\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(a)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}$$

$$\times\sum_{n=0}^{\infty}\frac{(c)_n\left(\frac{\beta}{p-t}\right)^n}{(e)_n(f)_n\,n!}\;G_{4,3}^{3,2}\left(\frac{1}{\alpha(p-t)}\left|\begin{matrix}1-n-\lambda,1,\frac{d}{2},\frac{d+1}{2}\\ a+n,\frac{b}{2},\frac{b+1}{2}\end{matrix}\right.\right).$$

Equivalently, in light of (1.5), one has

$$M_X(t)=\left(\frac{1}{p-t}\right)^{\lambda}\frac{C\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(a)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}$$

$$\times\sum_{n=0}^{\infty}\frac{(c)_n\left(\frac{\beta}{p-t}\right)^n}{(e)_n(f)_n\,n!}G_{3,4}^{2,3}\left(\alpha(p-t)\left|\begin{matrix}1-a-n,1-\frac{b}{2},1-\frac{b+1}{2}\\ 0,n+\frac{\lambda}{\delta},1-\frac{d}{2},1-\frac{d+1}{2}\end{matrix}\right.\right). \tag{5.2}$$

The Moments

The r^{th} moment about the origin of a continuous real random variables X with density function, $f(x)$ defined by

$$u_r' = \int_0^\infty x^r f(x)\,dx .$$

For the density function defined in (4.1), we have

$$\mu_r' = CH(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda+r), \qquad (5.3)$$

where $C^{-1} = H(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda)$.

Variance

The variance for the distribution of X is given by

$$V(X) = C\big(H(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda+2) - CH^2(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda+1)\big), \qquad (5.4)$$

The Factorial Moments

Factorial moments for probability density function defined in (4.1) are as follows

$$\begin{aligned} &E\big(X(X-1)(X-2)\ldots(X-i+1)\big) \\ &\quad \equiv E\left(X^i + \alpha_1 X^{i-1} + \alpha_2 X^{i-2} + \ldots + \alpha_{i-1} X\right) \\ &\quad = \sum_{k=0}^{i-1} \alpha_k (-1)^k E\left(X^{i-k}\right), \end{aligned} \qquad (5.5)$$

where α_k is integer, $\alpha_k \neq 0$; which satisfy the first identity

Now,

$$\begin{aligned} &E\big(X(X-1)(X-2)\ldots(X-i+1)\big) \\ &\quad \equiv \sum_{k=0}^{i-1} \alpha_k (-1)^k CH(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda+i-k). \end{aligned} \qquad (5.6)$$

The Negative Moments

Negative moments is defined as

$$E\left[\left(\frac{1}{X^r}\right)\right] = \int_0^\infty \frac{1}{X^r} f(x)\,dx , \qquad (5.7)$$

using (5.5), we derive negative moments for density function defined in (4.1)

$$E\left(\frac{1}{X^r}\right) = C\,H(a,b,c,d,e,f;\alpha,\beta,\delta;p,\lambda-r) \;,\; \lambda > r+1. \qquad (5.8)$$

REFERENCES

1. Agarwal, S.K. and Kalla, S.L. (1996). A generalized gamma distribution and its application in reliability. *Commun. in Statist.: Theo. and Meth.*, 25(1), 201-210.
2. Al-Musallam, F. and Kalla, S. (1997). Asymptotic expansions for generalized gamma and incomplete gamma functions, *Appl. Anal*., 66, 173-187.
3. Al-Musallam, F. and Kalla, S.L. (1998). Further results on a generalized gamma function occurring in diffraction theory. *Integral Transforms and Special Functions*, 7, 175-190.
4. Al-Saqabi, B.N., Kalla, S.L. and Tuan, V.K. (2003). Unified probability density function involving a confluent hypergeometric function of two variables. *Appl. Math. Comput*., 146, 135-152.
5. Al-Zamel. (2001). On a generalized gamma-type distribution with τ-confluent hypergeometric function. *Kuwait J. Sci. Eng.*, 28(1), 25-36.
6. Diaz, R. and Pariguan, E. (2007). On hypergeometric functions and Pochhammer K-symbol. *Divulgaciones Maemàticas*, 15(2), 179-192.
7. Erdélyi, A., Magnus, W.; Oberhettinger, F. and Tricomi, F.G. (1953). *Higher Transcendental Functions* Vol. I, McGraw-Hill Book Co, New York.
8. Galue, L., Alobaidi, S. and Kalla, S. (2001). Further generalizations of Kobayashi's gamma function. *Serdica Math. J*., 27, 303-316.
9. Ghitany, M. (1998). On a recent generalization of gamma distribution. *Commun. in Statist.: Theo. and Meth.*, 27(1), 223-233.
10. Kalla, S.L., Al-Saqabi, B.N. and Khajah, H.G. (2001). A unified form of gamma-type distributions. *Appl. Math. Comput.*, 118, 175-187.
11. Kobayashi, K. (1991a). On a generlized gamma function occurring in diffraction theory. *J. Phys. Soc. Jpn.*, 60, 1501-1512.
12. Kobayashi, K. (1991). Plane way diffraction by a strip. Exact and asymptotic solutions. *J. Phys. Soc. Jpn.*, 60, 1891-1905.
13. Mathai, A. (1993). *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*. Oxford University Press, Oxford.
14. Provost, S.B., Saboor, A. and Ahmad, M. (2011). The gamma-Weibull distribution. *Pak. J. Statist.*, 27(2), 111-113.
15. Rainville, E.D. (1960). *Special Functions.* The Macmillan Company, New York.
16. Saboor, A. and Ahmad, M. (2012). Bivariate generalized gamma-type probability function using a hypergeometric function of two variables. *Pak. J. Statist.*, 28(1), 81-91.
17. Saboor, A., Provost, S.B. and Ahmad, M. (2012). The moment generating function of a bivariate gamma-type distribution. *Appl. Math. Comput.*, 218(24), 11911-11921.

CHARACTERIZATIONS OF A PROBABILITY FUNCTION USEFUL IN SIZE MODELING*

Munir Ahmad and **M. Aslam Chaudhry**
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran- 31261, Saudi Arabia

ABSTRACT

A relationship between the inverse Gaussian density function and the density function introduced by Chaudhry and Ahmad is exploited to find some characterizations of the new density function.

1. INTRODUCTION

Chaudhry and Ahmad (1992) have recently introduced a new probability density function (pdf)

$$f(x) = 2\sqrt{\frac{\alpha}{\pi}}\exp\left[-\left(\sqrt{\alpha}x - \sqrt{\beta}x^{-1}\right)^2\right] \quad \alpha > 0, \beta \geq 0, x > 0 \qquad (1)$$

found useful in size modeling.

The mode of the function in (1) is $\mu_0 = (\beta/\alpha)^{1/4}$. The substitution of $\beta = \alpha\mu_0^4$ in (1) leads to

$$f(x) = 2\sqrt{\frac{\alpha}{\pi}}\exp\left(2\alpha\mu_0^2\right)\exp\left[-\alpha\mu_0^2\left(\left(x/\mu_0\right)^2 - \left(\mu_0/x\right)^2\right)\right],$$
$$\alpha > 0, \mu_0 > 0, x > 0. \qquad (2)$$

Here, μ_0 is the location parameter and $\alpha\mu_0^2$ is the shape parameter.

It may be noted that if a random variable X follows the inverse Gaussian distribution (see Johnson and Kotz, 1970), then the pdf of the $1/\sqrt{X}$ will be the probability distribution (1) (see Chaudhry and Ahmad (1992)). In this paper we exploit this relationship of the new pdf with the inverse Gaussian pdf to find some of its characterizations.

2. CHARACTERIZATIONS

Some characterizations of the new probability function are stated in the following theorems.

*Published in Pak. J. Statist. (1993), Vol. 9(1)A.

Theorem 1.

Suppose $E\left(X^{-2}\right), E\left(X^{-4}\right), E\left(X^{2}\right)$ and $\left[E\left(X^{-2}\right)\right]^{-1}$ exist and are different from zero. Then the necessary and sufficient condition that the variates Y and Z follow the probability distribution (1) is that $Y=\sum X^{-2}$ and $Z=\sum X^{2}-n^{2}\left(\sum X^{-1}\right)^{-1}$ are independently distributed.

Theorem 2.

Let X_0, X_1 and X_2 be three independent random variables and let

$$W_1=\frac{1}{2}\left(X_1^2-X_0^2\right)$$

$$W_2=\frac{1}{2}\left(X_2^2-X_1^2\right)$$

Then the necessary and sufficient condition that X_0, X_1 and X_2 be identical new pdf (1) random variables is that $\left(W_I,\ W_2\right)$ has the joint probability distribution

$$g\left(w_1, w_2\right)=\frac{2}{\pi}\frac{\left(w_1+\sqrt{1+w_1^2}\right)\left(w_2+\sqrt{1+w_2^2}\right)}{\left(1+w_1^2\right)\left(1+w_2^2\right)\left[1+\left(w_1+\sqrt{1+w_1^2}\right)^2+\left(w_2+\sqrt{1+w_2}\right)^2\right]^{1/2}}$$

$$-\infty<w_1<\infty, -\infty<w_2<\infty.$$

Proof:

The proofs of the theorems (1) and (2) are involved. However, they can be traced following Khatri (1962) and the fact that the probability distribution of $1/X^2$ is inverse Gaussian when X follows the pdf (I) (see Johnson and Katz, 1970 and Chaudhry and Ahmad, 1992).

3. ACKNOWLEDGEMENTS

The authors are indebted to referees for helpful comments and to the King Fahd University of Petroleum and Minerals for providing excellent research facilities.

4. REFERENCES

1. Chaudhry, M. Aslam and Ahmad, Munir (1992). On a probability function useful in size modeling. Accepted for publication in *Canadian Journal of Forest Research*. Technical Report No. 132, Department of Statistics, King Fahd University of Petroleum and Minerals, Dhahran, S. Arabia p. 14.
2. Ahmad, M. and Chaudhary, M.A. (1992). On a new probability function useful in size modelling. *Forestry Canada*, 1, 27-33.
3. Khatri, C.G. (1962). A characterization of the inverse Gaussian distribution. *Ann. Math. Stat.*, 33, 800-803.
4. Johnson, N.L. and Kotz, S. (1970). *Continuous Univariate Distributions-1*, Vol. 2, Houghton Miffin Co., Boston, U.S.A.

ON SIZE-BIASED GEETA DISTRIBUTION*

Khurshid Ahmad Mir[1] and **Munir Ahmad**[2]
[1] Department of Statistics, Govt. Degree College (Boys)
Baramulla, Kashmir, India. Email: khrshdmir@yahoo.com
[2] National College of Business Administration and Economics
Lahore, Pakistan. Email: drmunir@brain.net.pk

ABSTRACT

In this paper, a size-biased Geeta distribution (SBGET) is defined. Recurrence relations for central moments and the moments about origin are given. Different estimation methods for the parameters of the model are also discussed. R- Software has been used for making a comparison among the three different estimation methods and with the simple Geeta distribution.

KEY WORDS

Geeta distribution; Size-biased Geeta distribution; Moment estimator; Maximum likelihood estimator; Bayes estimator.

Mathematics Subject Classification: 62E15.

1. INTRODUCTION

The weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded; instead they are recorded according to some weight function. When the weight function depends on the lengths of the units of interest, the resulting distribution is called length biased. More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, resulting distribution is called size-biased. Size-biased distributions are a special case of the more general form known as weighted distributions. First introduced by Fisher (1934) to model ascertainment bias, these were later formalized in a unifying theory by Rao (1965). These distributions arise in practice when observations from a sample are recorded with unequal probability.

If the random variable X has distribution $f(x;\theta)$, with unknown parameter θ, then the corresponding size-biased distribution is of the form

$$f^*(x;\theta) = \frac{x^c f(x;\theta)}{\mu_c'}, \tag{1.1}$$

$$\mu_c' = \int x^c f(x;\theta)\,dx \qquad \text{for continuous case}$$

and

$$\mu_c' = \sum x^c f(x;\theta) \qquad \text{for discrete case.} \tag{1.2}$$

*Published in Pak. J. Statist. (2009), Vol. 25(3).

When $c = 1$ and 2, we get the simple size-biased and area-biased distributions respectively.

In this paper, a size-biased Geeta distribution (SBGET) is defined. Recurrence relations for central moments and the moments about origin are obtained. The estimates have been obtained by employing the moments, maximum likelihood and Bayesian method of estimation. In order to make a comparative analysis among the three estimation methods for the parameter of the size-biased Geeta distribution (SBGET), one of the standard software packages R-Software is used which is meant for data analysis and graphics. Also, Comparison is made with the simple Geeta distribution.

2. GEETA DISTRIBUTION

Consul (1990a) defined the Geeta distribution over the set of all positive integers with the probability mass function as

$$P[X = x] = \frac{1}{\beta x - 1}\binom{\beta x - 1}{x}\alpha^{x-1}(1-\alpha)^{\beta x - x}; \qquad x = 1, 2 \ldots$$

$$1 < \beta < \alpha^{-1} \text{ and } 0 < \alpha < 1. \qquad (2.1)$$

The Geeta distribution has a maximum at $x = 1$ and is L-shaped for all values of α and β. It may have a short tail or a long tail depending upon the values of α and β. Its mean and variance are given by

$$\mu = (1-\alpha)(1-\alpha\beta)^{-1} \qquad (2.2)$$

$$\mu_2' = \frac{\alpha(1-\alpha)(\beta-1) + (1-\alpha)^2(1-\alpha\beta)}{(1-\alpha\beta)^3}$$

$$\sigma^2 = (\beta-1)\alpha(1-\alpha)(1-\alpha\beta)^{-3} = \mu(\mu-1)(\beta\mu-1)(\beta-1)^{-1}.$$

The family of Geeta probability models belongs to the classes of the modified power series distributions (MPSD) and the Lagrangian series distributions. Consul (1990b) also expressed it as a location-parameter probability distribution given below:

$$P_1[X = x] = \frac{1}{\beta x - 1}\binom{\beta x - 1}{x}\left[\frac{\mu - 1}{(\beta\mu - 1)}\right]^{x-1}\left[\frac{\mu(\beta-1)}{\beta\mu - 1}\right]^{\beta x - x}; \; x = 1, 2, 3 \ldots. \qquad (2.3)$$

3. SIZE-BIASED GEETA DISTRIBUTION (SBGET)

A size-biased Geeta distribution (SBGET) is obtained by applying the weights x^c, where $c = 1$ to the Geeta distribution (2.1).

We have from (2.1) and (2.2)

$$\sum_{x=1}^{\infty} x \cdot P(X = x) = (1-\alpha)(1-\alpha\beta)^{-1}.$$

This gives the size-biased Geeta distribution (SBGET) as

$$P_2[X=x]=(1-\alpha\beta)\binom{\beta x-2}{x-1}\alpha^{x-1}(1-\alpha)^{\beta x-x-1}\ ;\ x=1,2\ldots$$

$$1<\beta<\alpha^{-1} \text{ and } 0<\alpha<1 \qquad (3.1)$$

3.1 Moments of SBGET

The r[th] moment $\mu_r'(s)$ of SBGET (3.1) about origin is obtained as

$$\mu_r'(s)=E(X^r)=\sum_{x=1}^{\infty}x^r P_2[X=x] \qquad ; r=1,2\ldots$$

$$\mu_r'(s)=\sum_{x=1}^{\infty}x^r(1-\alpha\beta)\binom{\beta x-2}{x-1}\alpha^{x-1}(1-\alpha)^{\beta x-x-1}. \qquad (3.2)$$

Obviously $\mu_0'(s)=1$ and for r ≥ 1

$$\mu_r'(s)=\frac{(1-\alpha\beta)}{(1-\alpha)}\sum_{x=1}^{\infty}x^{r+1}\frac{1}{x}\binom{\beta x-2}{x-1}\alpha^{x-1}(1-\alpha)^{\beta x-x-1}$$

$$=\frac{(1-\alpha\beta)}{(1-\alpha)}\sum_{x=1}^{\infty}x^{r+1}P[X=x]$$

$$\mu_r'(s)=\frac{(1-\alpha\beta)}{(1-\alpha)}\mu_{r+1}', \qquad (3.3)$$

where μ_{r+1}' is the $(r+1)^{th}$ moments about origin of Geeta distribution (2.1).

The moments of SBGET can be obtained by using relation (3.3) as

$$\mu_1'(s)=\text{Mean }=\frac{\left(1-2\alpha+\alpha^2\beta\right)}{\left(1-\alpha\beta\right)^2}, \qquad (3.4)$$

and similarly Variance of SBGET (3.1) is given as

$$\mu_2(s)=\frac{2(\beta-1)\alpha(1-\alpha)}{\left(1-\alpha\beta\right)^4}. \qquad (3.5)$$

The higher moments of SBGET (3.1) about origin can also be obtained if the higher moments of Geeta distribution are known.

4. RECURRENCE RELATIONS FOR THE MOMENTS ABOUT ORIGIN OF SIZE-BIASED GEETA DISTRIBUTION

The recurrence relation can be obtained by differentiating (3.2) as

$$\frac{\partial\ \mu_r'(s)}{\partial\alpha} = \sum_{x=1}^{\infty} x^r \binom{\beta x-2}{x-1}\left[\frac{\partial}{\partial\alpha}\left\{\alpha^{x-1}(1-\alpha\beta)(1-\alpha)^{\beta x-x-1}\right\}\right]$$

$$= \sum_{x=1}^{\infty} x^r \binom{\beta x-2}{x-1}(1-\alpha\beta)\alpha^{x-1}(1-\alpha)^{\beta x-x-1}\left[\frac{(x-1)(1-\alpha)-\alpha(\beta x-x-1)}{\alpha(1-\alpha)}\right]$$

$$-\beta\sum_{x=1}^{\infty} x^r \binom{\beta x-2}{x-1}\alpha^{x-1}(1-\alpha)^{\beta x-x-1}$$

$$= \frac{(1-\alpha\beta)}{\alpha(1-\alpha)}\sum_{x=1}^{\infty} x^{r+1} \binom{\beta x-2}{x-1}(1-\alpha\beta)\alpha^{x-1}(1-\alpha)^{\beta x-x-1}$$

$$+\frac{(2\alpha-1)}{\alpha(1-\alpha)}\sum_{x=1}^{\infty} x^{r} \binom{\beta x-2}{x-1}(1-\alpha\beta)\alpha^{x-1}(1-\alpha)^{\beta x-x-1}$$

$$-\beta\sum_{x=1}^{\infty} x^r \binom{\beta x-2}{x-1}\alpha^{x-1}(1-\alpha)^{\beta x-x-1}$$

$$= \frac{(1-\alpha\beta)}{\alpha(1-\alpha)}\mu_{r+1}'(s)+\frac{(2\alpha-1)}{\alpha(1-\alpha)}\mu_r'(s)-\frac{\beta}{(1-\alpha\beta)}\mu_r'(s)$$

$$\mu_{r+1}'(s) = \frac{\alpha(1-\alpha)}{(1-\alpha\beta)}\frac{\partial\mu_r'(s)}{\partial\alpha}+\frac{\alpha\beta(1-\alpha)}{(1-\alpha\beta)^2}\mu_r'(s)-\frac{(2\alpha-1)}{(1-\alpha\beta)}\mu_r'(s). \tag{4.1}$$

The above recurrence relation can be used for getting the higher moments of the model (3.1)

4.1 Recurrence Relations for Central Moments of Size-Biased Geeta Distribution

We define the k-th central moment μ_k of size-biased Geeta distribution (SBGET) as

$$\mu_k == \sum_{x=1}^{\infty}(x-\mu)^k(1-\alpha\beta)\binom{\beta x-2}{x-1}\alpha^{x-1}(1-\alpha)^{\beta x-x-1}.$$

Differentiating with respect to α, we get

$$\frac{\partial\ \mu_k}{\partial\alpha} = \sum_{x=1}^{\infty}\binom{\beta x-2}{x-1}\frac{\partial}{\partial\alpha}\left[\left\{\alpha^{x-1}(1-\alpha\beta)(1-\alpha)^{\beta x-x-1}(x-\mu)^k\right\}\right]$$

$$\frac{\partial\mu_k}{\partial\alpha} = \frac{(1-\alpha\beta)}{\alpha(1-\alpha)}\mu_{k+1}-\frac{\beta}{(1-\alpha\beta)}+\frac{(1-\alpha\beta)\mu\mu_k}{\alpha(1-\alpha)}+\frac{(2\alpha-1)\mu_k}{\alpha(1-\alpha)}-k\frac{\partial\mu}{\partial\alpha}\mu_{k-1}.$$

The above expression gives the recurrence formula

$$\mu_{k+1} = \frac{\alpha(\alpha-1)}{(1-\alpha\beta)}\frac{\partial\mu_k}{\partial\alpha} + \frac{k\alpha(1-\alpha)}{(1-\alpha\beta)}\frac{\partial\mu}{\partial\alpha}\mu_{k-1}$$
$$-\left[\frac{(2\alpha-1)(1-\alpha\beta)-\alpha(1-\alpha)\beta+\mu(1-\alpha\beta)^2}{(1-\alpha\beta)^2}\right]\mu_k .$$

The above recurrence relation can be used for getting the higher central moments of the model (3.1).

5. ESTIMATION METHODS

In this section, we discuss the various estimation methods for size-biased Geeta distribution and verify their efficiencies.

5.1 Method of Moments

In the method of moments replacing the population mean and variance by the corresponding sample mean and variance, we have

$$\bar{x} = \frac{(1-2\alpha+\alpha^2\beta)}{(1-\alpha\beta)^2} \tag{5.1}$$

and

$$s^2 = \frac{2(\beta-1)\alpha(1-\alpha)}{(1-\alpha\beta)^4}. \tag{5.2}$$

On simplifying (5.1), we get

$$\alpha^2\beta(\beta\bar{x}-1)-2\alpha(\beta\bar{x}-1)+\bar{x}-1=0 .$$

Solving above equation for α, we get the estimate of α in terms of $\bar{x}$ and β. Substituting that value in equation (5.2) and using the iterative method with the help of R-software, we get the estimate for β.

5.2 Method of Maximum Likelihood Estimation

The likelihood function of SBGET (3.1) is given as

$$L(x;\alpha,\beta) = \prod_{i=1}^{n}\begin{pmatrix}\beta x_i - 2\\ x_i - 1\end{pmatrix}(1-\alpha\beta)^n\,\alpha^{\sum_{i=}^{n} x_i - n}\,(1-\alpha)^{\beta\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i - n}$$

$$L(x;\alpha,\beta) = C(1-\alpha\beta)^n\,\alpha^{y-n}\,(1-\alpha)^{\beta y - y - n}. \tag{5.3}$$

Where $y=\sum_{i=1}^{n} x_i$ and $C = \prod_{i=1}^{n}\begin{pmatrix}\beta x_i - 2\\ x_i - 1\end{pmatrix}$

The log likelihood function is given as

$$\text{Log L}= \sum_{i=1}^{n}\log\binom{\beta x_i-2}{x_i-1}+\left(n\bar{x}-n\right)\log\alpha$$
$$+\left(\beta n\bar{x}-n\bar{x}-n\right)\log\left(1-\alpha\right)+n\log\left(1-\alpha\beta\right) \tag{5.4}$$

The likelihood equations are given as

$$\frac{\partial\log L}{\partial\alpha}=\frac{\left(n\bar{x}-n\right)}{\alpha}-\left(\frac{\beta n\bar{x}-n\bar{x}-n}{1-\alpha}\right)-\frac{n\beta}{\left(1-\alpha\beta\right)}=0 \tag{5.5}$$

$$\frac{\partial\log L}{\partial\beta}=\log nx+n\bar{x}\log\left(1-\alpha\right)-\frac{n\alpha}{\left(1-\alpha\beta\right)}=0\,. \tag{5.6}$$

For the numerical solution of above two likelihood equations, we operated with the iterative method of (NLM) function in R- software and the estimates of α and β are reflected in Tables 1.1 and 1.2.

5.3 Bayesian Estimation of Parameter of Size- Biased Geeta Distribution (SBGET)

Since $0<\alpha<1$, therefore we assume that prior information about α when β is known from beta distribution as

$$\text{Thus } f\left(\alpha\right)=\frac{\alpha^{a-1}\left(1-\alpha\right)^{b-1}}{B\left(a,b\right)};\ 0<\alpha<1,\ a>0,\ b>0. \tag{5.7}$$

The posterior distribution from (5.3) and (5.7) can be written as

$$\prod\left(\alpha/y\right)=\frac{\left(1-\alpha\beta\right)^{n}\alpha^{y+a-n-1}\left(1-\alpha\right)^{\beta y-y+b-1-n}}{\int_0^1\left(1-\alpha\beta\right)^{n}\alpha^{y+a-n-1}\left(1-\alpha\right)^{\beta y-y+b-1-n}d\alpha}\,.$$

The Bayes estimator of α is given as

$$\hat{\alpha}=\int_0^1\alpha\prod\left(\alpha/\text{y}\right)d\alpha$$

$$\hat{\alpha}=\frac{\int_0^1\left(1-\alpha\beta\right)^{n}\alpha^{y+a-n}\left(1-\alpha\right)^{\beta y-y+b-n-1}d\alpha}{\int_0^1\left(1-\alpha\beta\right)^{n}\alpha^{y+a-n-1}\left(1-\alpha\right)^{\beta y-y+b-n-1}d\alpha}, \tag{5.8}$$

where $\int_0^1\left(1-\alpha\beta\right)^{n}\alpha^{y+a-n}\left(1-\alpha\right)^{\beta y-y+b-n-1}d\alpha$

$$=\Gamma\left(y+a-n+1\right)\Gamma\left(\beta y+b-y-n\right){}^{2}F_{1}\left[-n,y+a-n+1,\beta y+a+b-2n+1,\beta\right] \tag{5.9}$$

and $$\int_0^1 (1-\alpha\beta)^n \ \alpha^{y+a-n-1} (1-\alpha)^{\beta y - y + b - n - 1} \, d\alpha$$

$$= \Gamma(y+a-n)\ \Gamma(\beta y + b - y - n)\ {}^2F_1[-n, y+a-n, \beta y + a + b - 2n, \beta]. \qquad (5.10)$$

Putting these values in equation in (5.8), the Bayes estimator of α is obtained as

$$\hat{\alpha} = \frac{(y+a-n)\ \ {}^2F_1[-n, y+a-n+1, \beta y + a + b - 2n + 1, \beta]}{{}^2F_1[-n, y+a-n, \beta y + a + b - 2n, \beta]}. \qquad (5.11)$$

6. COMPUTER SIMULATION AND CONCLUSIONS

It is very difficult to compare the theoretical performances of different estimators proposed in the previous section. Therefore, we perform extensive simulations to compare the performances of the different methods of estimation mainly with respect to their biases and mean squared errors (MSE's), for different sample sizes and of different parametric values. Regarding the choice of values of (a, b) in the Bayes estimator (5.11), there was no information about their values except that they are real and positive numbers. Therefore 25 combinations of values of (a, b) were considered for a, b=1, 2,3,4,5 and those values of a, b were selected for which the Bayes estimator $\hat{\alpha}$ has minimum variance. It was found that for a=b=5, the Bayes estimator has minimum variance and χ^2 values between the simulated sample frequencies and the estimated Bayes frequencies were the least. Data given in tables (1.1) and (1.2) are the zero-truncated data of P.Garman (1923) and Student (1907) on counts of the number of European red mites on apple leaves and yeast blood cell counts observed per square. In table-1.1, comparison is made between the different methods of estimation for the parameter of size-biased Geeta distribution and it was observed that the Bayes estimator provides us the better fit against MLE or moments estimator. Also, table-1.2 reveals that the size-biased Geeta distribution provides a better result in comparison to simple Geeta distribution.

Table 1.1

No. of mites per leaf	**Leaves Observed**	**Expected frequency**		
		Mom	**MLE**	**Bayes**
1	38	37.96	37.95	37.98
2	17	16.34	16.56	16.92
3	10	9.43	9.53	9.83
4	9	9.10	9.15	8.97
5	3	2.46	2.73	2.95
6	2	1.96	1.97	2.01
7	1	1.14	0.98	0.99
≥ 8	0	1.61	1.13	0.35
Total	80	80.00	80.00	80.00
$\hat{\alpha}$		0.02	0.24	0.56
$\hat{\beta}$		2.3	2.53	2.0
χ^2		0.254	0.057	0.017

Table 1.2

No. of cells per square	Observed No. of squares	Expected frequency	
		GET	SBGET
1	128	126.58	127.34
2	37	36.12	36.92
3	18	17.34	17.62
4	3	2.58	2.93
5	1	1.34	0.97
≥ 6	0	3.04	1.22
Total	187	187.00	187.00
$\hat{\alpha}$		0.01	0.43
$\hat{\beta}$		2.58	1.98
χ^2		0.255	0.028

ACKNOWLEDGEMENTS

The authors are highly thankful to the referee and the editor for their valuable suggestions.

REFERENCES

1. Consul, P.C. (1990a). Geeta distribution and its properties. *Commun. Statist. Theory and Meth.*, 19, 3051-3068.
2. Consul, P.C. (1990b). Two stochastic models for the Geeta distribution. *Commun. Statist. Theory and Meth.*, 19, 3699-3706.
3. Fisher, R.A. (1934). The effects of methods of ascertainment upon the estimation of frequencies. *Ann. Eugenics*, 6, 13-25.
4. Garman, P. (1923). The European red mites in Connecticut apple orchards. *Connecticut Agri. Experi. Station Bulletin*, 252, 103-125.
5. Rao, C.R. (1965). On discrete distributions arising out of methods of ascertainment. In: Patil, G.P. (ed.) *Classical and Contagious Discrete Distributions*. *Statistical Publishing Society, Calcutta,* 320-332.
6. Student (1907). On the error of counting with a haemacytometer. *Biometrika*, 5, 351-360.

ON MOMENTS OF WEIGHTED MEAN OF TWO SAMPLE CORRELATION COEFFICIENTS*

Syed Anwer Hasnain[1] and **Munir Ahmad**[2]
National College of Business Administration and Economics
Lahore, Pakistan
Email: [1]anwerleo@yahoo.com
[2]drmunir@brain.net.pk

ABSTRACT

In this paper we have derived the distribution of weighted mean of two correlation coefficients r_1 and r_2 and obtained its moments using Bessel function and confluent hyper geometric series function.

1. INTRODUCTION

Correlation was explored much before the 20th century. Recently Paul (1988, 1988 a) discussed the estimation and testing the significance of a common correlation coefficient. Bhatti (1990) developed the moment generating function of the mean distribution of correlation coefficients and computed the upper tail area but did not provide its moments (see Aboukalam, 1997).

In this paper we have derived the distribution of weighted mean of two correlation coefficients r_1 and r_2 and obtained its moments using Bessel function and confluent hyper geometric function.

2. WEIGHTED MEAN OF TWO CORRELATION COEFFICIENTS r_1 AND r_2

By considering the two independent sample correlation coefficients r_1 and r_2 with arbitrary weights a_1 and a_2, the moment generating function (mgf) of the weighted mean is defined as $\phi_{(\bar{r}_w)}(t)$, where $\bar{r}_w = \dfrac{a_1 r_1 + a_2 r_2}{a_1 + a_2}$.

The m.g.f. of the sum of two independent random variables is

$$\phi_{(\bar{r}_w)}(t) = \phi_{r_1}\left(c_1 t\right).\phi_{r_2}\left(c_2 t\right),$$

where $c_1 = \left(\dfrac{a_1}{a_1 + a_2}\right)$ *and* $c_2 = \left(\dfrac{a_2}{a_1 + a_2}\right)$.

Following Bhatti (1990) the characteristic function of the sample correlation coefficient (r) in terms of Bessel function, is

*Published in Pak. J. Statist. (2013), Vol. 29(1).

$$\phi_{(\bar{r}_w)}(t) = \left[\Gamma(\nu+1) \quad 2^{\nu}(c_1 t)^{-\nu} \quad J_{\nu}(c_1 t)\right]\left[\Gamma(\nu+1) \quad 2^{\nu}(c_2 t)^{-\nu} \quad J_{\nu}(c_2 t)\right]$$

$$= \left[\Gamma(\nu+1)\right]^2 \quad 2^{2\nu}(c_1 c_2)^{-\nu} \quad t^{-2\nu} J_{\nu}(c_1 t) J_{\nu}(c_2 t).$$

Using the relation (Slater, 1960),

$$J_{\nu}(t) = \frac{e^{-it}(t/2)^{\nu}}{\Gamma(\nu+1)} \, {}_1F_1(\nu+1/2; 2\nu+1; 2it),$$

We have

$$\phi(t) = e^{-it(c_1+c_2)} \; {}_1F_1(\nu+1/2; 2\nu+1; 2c_1 it) \; {}_1F_1(\nu+1/2; 2\nu+1; 2c_2 it),$$

replacing it = y

$$\phi(y) = e^{-y(c_1+c_2)} \; {}_1F_1(\nu+1/2; 2\nu+1; 2c_1 y) \; {}_1F_1(\nu+1/2; 2\nu+1; 2c_2 y)$$

The first derivative of the confluent hypergeometric series (Slater, 1960) is

$$\frac{d}{dt}\left[{}_1F_1(a;b;t)\right] = \frac{a}{b} \; {}_1F_1(a+1; b+1; t).$$

The mean (μ) is

$\mu_1 = 0$

The second derivative of $\phi(t)$ at t = 0 is

$$\phi''(t)\big|_{t=0} = (c_1+c_2)^2 - (c_1+c_2)c_1 - (c_1+c_2)c_2$$
$$+ c_1\left\{-(c_1+c_2) + c_1 \frac{\nu+3/2}{\nu+1} + c_2\right\}$$
$$+ c_2\left\{-(c_1+c_2) + c_1 + c_2 \frac{\nu+3/2}{\nu+1}\right\}.$$

or

$$\mu_2' = \frac{1}{2(\nu+1)}\left(c_1^2 + c_2^2\right).$$

Replacing $\nu = (n-3)/2$, $c_1 = \left(\frac{a_1}{a_1+a_2}\right)$ and $c_2 = \left(\frac{a_2}{a_1+a_2}\right)$, we have, as $\mu_1 = 0$ and

$$\mu_2 = \frac{1}{n-1}\left[\frac{a_1^2 + a_2^2}{(a_1+a_2)^2}\right].$$

Similarly μ_3 is

$$\mu_3 - \left(c_1 + c_2\right)^3 + 3\left(c_1 + c_2\right)^2\left(c_1 + c_2\right) - 3\left(c_1 + c_2\right)\left(c_1^2 + c_2^2\right)\frac{\nu + 3/2}{\nu + 1}$$

$$+3c_1 c_2\left(c_1 + c_2\right)\frac{\nu + 3/2}{\nu + 1} - 6c_1 c_2\left(c_1 + c_2\right) + \left(c_1^3 + c_2^3\right)\frac{\nu + 5/2}{\nu + 1}.$$

$$= c_1^3\left(2 - 3\frac{\nu + 3/2}{\nu + 1} + \frac{\nu + 5/2}{\nu + 1}\right) + c_2^3\left(2 - 3\frac{\nu + 3/2}{\nu + 1} + \frac{\nu + 5/2}{\nu + 1}\right) = 0$$

Hence $\mu_3 = 0$

For fourth moment $\phi^{(iv)}(t)$ is,

$$\phi^{(iv)}(t)\Big|_{t=0} = \left(c_1^4 + c_2^4\right)\left(-3 + 6\frac{\nu + 3/2}{\nu + 1} - 4\frac{\nu + 5/2}{\nu + 1} + \frac{\nu + 5/2}{\nu + 1}\frac{\nu + 7/2}{\nu + 2}\right)$$

$$+c_1^2 c_2^2\left(6 - 12\frac{\nu + 3/2}{\nu + 1} + 6\frac{\nu + 3/2}{\nu + 1}\frac{\nu + 3/2}{\nu + 1}\right).$$

$$= \left(c_1^4 + c_2^4\right)\left(\frac{3}{4(\nu + 1)(\nu + 2)}\right) + c_1^2 c_2^2\left(\frac{3}{2(\nu + 1)^2}\right).$$

or

$$\mu_4 = \left(\frac{a_1^4 + a_2^4}{\left(a_1 + a_2\right)^4}\right)\left[\frac{3}{(n + 1)(n - 1)}\right] + \frac{a_1^2 a_2^2}{\left(a_1 + a_2\right)^4}\left[\frac{6}{(n - 1)^2}\right].$$

The skewness and kurtosis are

$$\beta_1 = 0$$

and

$$\beta_2 = 3\left[\frac{n}{n + 1} - \frac{\left(a_1^2 - a_2^2\right)^2}{(n + 1)\left(a_1^2 + a_2^2\right)^2}\right].$$

$\beta_2 = 3$, when $n \to \infty$, and $\frac{a_2}{a_1} \to \infty$.

If $\frac{a_2}{a_1} = 0$, then $\beta_2 = 3\frac{n - 1}{n + 1}$.

REFERENCES

1. Aboukalam, M.A.F. (1997). On Robust Linear Regression of Hepatic extraction of Insulin. *Biometrical J.*, 39, 171-181.
2. Bhatti, M.I. (1990). Null distributions of the Small Sample mean Correlation Coefficient. *Biometrical J.,* 32, 435-440.
3. Paul, S.R. (1988). *Estimation and testing significance for a common correlation coefficient. Comput. Statist. Theo. and Meth.*, A 17, 39-53. In *Biostatistical Analysis*, 3rd Edition by Jerrold H. Zar (1996). Prentice Hall, New Jersey.
4. Paul, S.R. (1989). Test for the equality of several correlation coefficients. *Can. J. Statist.*, 17(2), 217-227.
5. Slater, L.J. (1960). *Confluent Hypergeometric Functions*. Cambridge University Press, Cambridge, UK.

ON SIZE-BIASED GENERALIZED POISSON DISTRIBUTION*

Khurshid Ahmad Mir[1] and **Munir Ahmad**[2]
[1] Department of Statistics, Govt. Degree College Bemina
Srinagar, Kashmir, India. Email: khrshdmir@yahoo.com
[2] National College of Business Administration and Economics
Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

In this paper, a size-biased generalized Poisson distribution is defined. Moments of the distribution are obtained. Estimation for the parameter of the distribution is studied by employing different estimation techniques. R-Software has been used for the purpose of comparison among different estimation methods. Also, efficiency of the model is studied by AIC, BIC and chi-square techniques.

KEY WORDS

Size-biased generalized Poisson distribution; truncated generalized Poisson distribution; Bayes' Estimator; AIC; BIC; Chi-square.

Mathematics Subject Classification: 62E15.

1. INTRODUCTION

The generalized Poisson distributions (GPDs) arise when the populations are Poissonian type having unequal mean and variance. Consul and Jain (1973a) are the early workers who derived a class of discrete distributions of the Poissonian type. The different aspects of these distributions have been studied by Consul and Jain (1973b), Jain (1975), Consul and Shoukri (1985, 1988), Consul (1986), Famoye and Lee (1992), Lee and Famoye (1996), Tuenter (2000), Sheth (1998), Shanumugam (1984) and Nandi et al (1999). The detailed review works on the book authored by Consul (1989) have been done by Kemp (1992), Olkin (1992) and Shimzu (1992).

The probability function of the generalized Poisson distribution (GPD) is defined for $\lambda_1 > 0$ and $|\lambda_2| < 1$ with the probability function as

$$P_1(X = x) = \frac{\lambda_1\left(\lambda_1 + x\lambda_2\right)^{x-1}\exp\left[-\left(\lambda_1 + x\lambda_2\right)\right]}{x!}, \quad x = 0,1,2\ldots \tag{1.1}$$

For $\lambda_2 = 0$, the distribution (1.1) reduces to Poisson distribution. The model (1.1) has been found to be a member of the Consul and Shenton's (1972) family of Lagrangian distributions.

*Published in Pak. J. Statist. (2013), Vol. 29(3).

2. ZERO-TRUNCATED GENERALIZED POISSON DISTRIBUTION (TGPD)

Shoukri and Consul (1989) redefined the distribution (1.1) by taking $\lambda_1 = \alpha$ and $\lambda_2 = \alpha\beta$ as

$$P_2(X = x) = \frac{(1+\beta x)^{x-1}\,\alpha^x \exp\left[-\alpha(1+\beta x)\right]}{x!}; \; x = 0,1,2,\cdots, \alpha > 0, 0 < \beta < \frac{1}{\alpha}. \quad (2.1)$$

The distribution (2.1) can be truncated at x = 0 and is defined with the probability function as

$$P_3(X = x) = \frac{(1+\beta x)^{x-1}\alpha^x \exp\left[-\alpha(1+\beta x)\right]\left(1-e^{-\alpha}\right)^{-1}}{x!};$$

$$x = 1,\ 2..,\ \alpha > 0, 0 < \beta < \frac{1}{\alpha}. \quad (2.2)$$

For $\beta = 0$, the distributions (2.1) and (2.2) reduce to Poisson distribution and David and Johnson's (1952) truncated Poisson distribution. The different aspects of the distribution (2.2) have been studied by Consul and Famoye (1989), Jani and Shah (1981), Hassan and Mir (2007), Hassan et al (2007). A brief list of authors and their works can be seen in Consul (1989), Johnson, Kotz and Kemp (2005) and Consul and Famoye (2006).

In this paper, we have made an attempt to obtain the Bayes' estimator of size-biased generalized Poisson distribution (SBGPD) for one parameter α when other parameter β is assumed to be known. Furthermore, recurrence relations for the moments of the parameter are also obtained. In order to make comparative analysis among different estimation methods for the parameter of the size-biased generalized Poisson distribution (SBGPD), one of the standard software packages R- software is used. Also, the efficiency of the model is compared with truncated Poisson and truncated generalized Poisson distributions.

3. SIZE-BIASED GENERALIZED POISSON DISTRIBUTION (SBGPD)

Rao (1965) has discussed a situation where an event that occurs has a certain probability of being recorded (or included in the sample). Let X denote an integer-valued random variable with the probability function $P(x,\theta)$ and suppose that when $X = x$ occurs, the probability of recording it is $w(x,\lambda)$ depending on the observed value x and the unknown value of the parameter λ. The probability function of the resulting random variable X^w is

$$P^w(X = x) = \frac{w(x)P(x,\theta)}{E\left[w(X)\right]}.$$

If the weight function $w(x) = \left(\lambda_1 + x\lambda_2\right)$, then we obtain the Jain's (1975) linear function Poisson distribution with the probability function as

$$P_4(X=x)=\frac{(1-\lambda_2)(\lambda_1+x\lambda_2)^x\exp\left[-(\lambda_1+x\lambda_2)\right]}{x!};x=0,1,2,..... \quad (3.1)$$

Mir and Ahmad(2009) defined the size-biased generalized Poisson distribution as

$$P_5(X=x)=\frac{(1-\lambda_2)(\lambda_1+x\lambda_2)^{x-1}\exp\left[-(\lambda_1+x\lambda_2)\right]}{(x-1)!};x=1,2,...\ \lambda_1>0,\ |\lambda_2|<1. \quad (3.2)$$

At $\lambda_2=0$, we get the size-biased Poisson distribution.

For the mathematical tractability, the modified form of the model (3.2) can be obtained by putting $\lambda_1=\alpha$ and $\lambda_2=\alpha\beta$ as

$$P_6(X=x)=\frac{(1-\alpha\beta)\alpha^{x-1}(1+\beta x)^{x-1}\exp\left[-\alpha(1+\beta x)\right]}{(x-1)!};$$

$$x=1,\ 2\ldots.\ \alpha>0, 0<\beta<\frac{1}{\alpha}. \quad (3.3)$$

Mishra and Singh (1993) also studied the distribution (3.2).

3.1 Recurrence Relation for the Moments of SBGPD

The rth moment of the size-biased GPD is given by

$$E\left(X^r\right)=\frac{(1-\lambda_2)}{\lambda_1}\sum_{x=0}^{\infty}x^{r+1}P_1(X=x),$$

where $P_1(X=x)$ is the probability function of GPD.

Thus, knowing the moments of GPD we find out the moments of SBGPD.

Using above relation, the mean and variance of SBGPD are

$$\mu_1'=\frac{\lambda_1}{(1-\lambda_2)}+\frac{1}{(1-\lambda_2)^2}, \quad (3.4)$$

$$\mu_2=\frac{2\lambda_2}{(1-\lambda_2)^4}+\frac{\lambda_1}{(1-\lambda_2)^3}. \quad (3.5)$$

3.2 Graphical Representation of Size-Biased Generalized Poisson Distribution (SBGPD)

We present below some probability functions of SBGPD in figures 1,2,3,4 and 5 considering various values of λ_1 and λ_2.

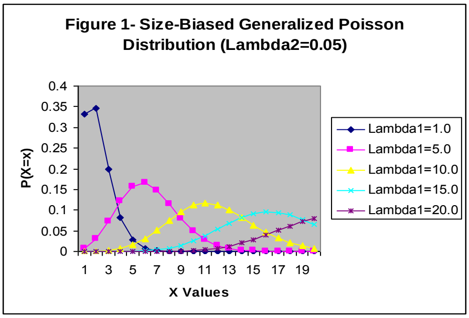


Figure 1- Size-Biased Generalized Poisson Distribution (Lambda2=0.05)

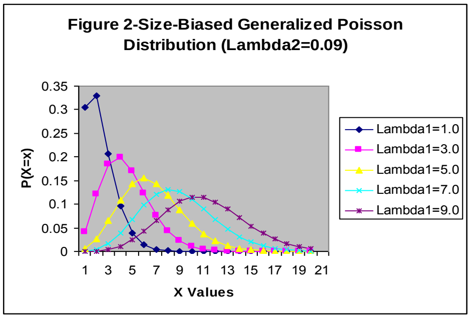


Figure 2-Size-Biased Generalized Poisson Distribution (Lambda2=0.09)

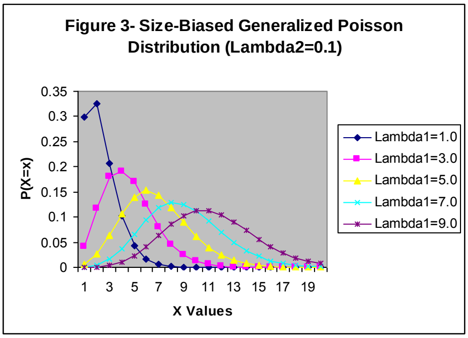


Figure 3- Size-Biased Generalized Poisson Distribution (Lambda2=0.1)

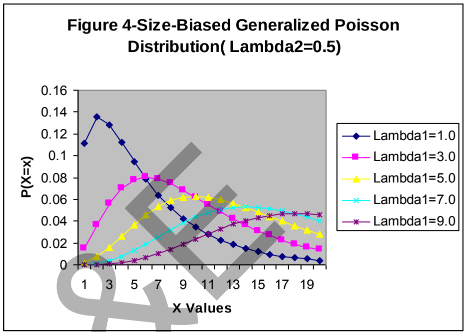


Figure 4-Size-Biased Generalized Poisson Distribution(Lambda2=0.5)

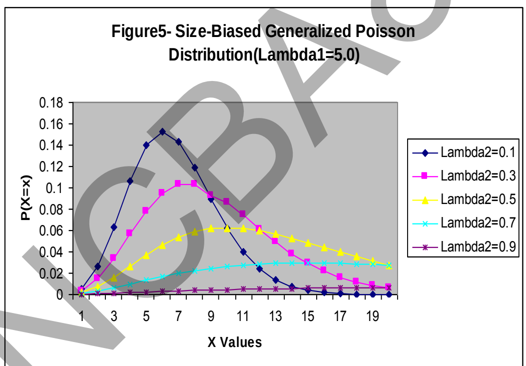


Figure5- Size-Biased Generalized Poisson Distribution(Lambda1=5.0)

In figure 1, for each small value of λ_1, the SBGPD curve changes from L-shaped to symmetric and with the considerable change in the value of λ_1, it becomes positively skewed. In figures 2, 3 and 4, we consider $\lambda_2 = 0.09, 0.1, 0.5$. For each small value of λ_1, the SBGPD curve is unimodal and extremely positively skewed. But it gradually changes to bell-shaped as the value of λ_1 and λ_2 increase. In figure 5, we take $\lambda_1 = 5.0$ and the different values of λ_2. It is observed that the variation in the values of λ_2 alters the shape of the distribution substantially. For larger values of λ_2, the bell-shaped form becomes more flattened.

4. ESTIMATION OF SIZE-BIASED POISSON DISTRIBUTION

The estimation of the parameters of the GPD model have been studied by Consul and Shoukri (1984), Consul and Famoye (1988), Bowman and Shenton (1985) and Famoye and Consul (1990). In this section, the estimation of the parameter of SBGPD is discussed by various estimation techniques and their efficiencies are discussed in section 6.

4.1 Moment Estimation for Size-Biased Generalized Poisson Distribution

By letting $1-\lambda_2=\theta$, in equations (3.4) and (3.5), the mean and variance of SBGPD (3.2) can be expressed as

$$\mu_1' = \frac{\left(\lambda_1\theta+1\right)}{\theta^2} \tag{4.1}$$

$$\mu_2 = \frac{\left[2\left(1-\theta\right)+\lambda_1\theta\right]}{\theta^4} \tag{4.2}$$

This gives an equation in θ as

$$\mu_2\theta^4-\mu_1'\theta^2+2\theta-1=0 \tag{4.3}$$

Replacing μ_1' and μ_2 by the corresponding sample values $\bar{x}$ and S^2 respectively, we get

$$S^2\theta^4-\bar{x}\theta^2+2\theta-1=0 \tag{4.4}$$

It is a polynomial of degree four in θ and can be solved using the Newton-Raphson method to estimate λ_2. An estimate of λ_1 is then obtained as

$$\hat{\lambda}_1 = \frac{\left(\hat{\theta}^2\bar{x}-1\right)}{\hat{\theta}}. \tag{4.5}$$

4.2 Maximum Likelihood Estimation

The log likelihood function of a sample size n from (3.1) is

$$\log L = n\log\left(1-\lambda_2\right)+\left(\sum_{i=1}^{n}x_i-n\right)\log\left(\lambda_1+x\lambda_2\right)-\left(n\lambda_1+\lambda_2\sum_{i=1}^{n}x_i\right)-\sum_{i=1}^{n}\log(x_i-1). \tag{4.6}$$

The log likelihood equations are given as

$$\frac{\partial\log L}{\partial\lambda_1}=\sum_{i=1}^{n}\frac{\left(x_i-1\right)}{\left(\lambda_1+x_i\lambda_2\right)}-n=0 \tag{4.7}$$

$$\frac{\partial\log L}{\partial\lambda_2}=\frac{-n}{\left(1-\lambda_2\right)}+\sum_{i=1}^{n}\frac{\left(x_i-1\right)x_i}{\left(\lambda_1+x_i\lambda_2\right)}-\sum_{i=1}^{n}x_i=0 \tag{4.8}$$

By solving the above equations, we get the maximum likelihood estimates.

4.3 Bayes' Method of Estimation

Let $x_1, x_2, \ldots x_n$ be a random sample from size-biased generalized Poisson distribution (3.3), the corresponding likelihood function is given as

$$L = K\alpha^{y-n} \exp\left[-\alpha\left(n+\beta y\right)\right]\left(1-\alpha\beta\right)^n \tag{4.9}$$

where $K = \prod_{i=1}^{n}\left[\frac{\left(1+\beta x_i\right)}{\left(1-x_i\right)}\right]$, $y = \sum_{i=1}^{n} x_i$.

We assume that before the observations were made, our knowledge about α was only vague. Consequently, the non-informative vague prior of α, $g(\alpha)$ proportional to $\frac{1}{\alpha}$ is applicable to a good approximation.

Thus

$$g(\alpha) = \frac{1}{\alpha}, \quad \alpha > 0 \tag{4.10}$$

The posterior distribution from (4.9) and (4.10) is given as

$$\prod(\alpha / y) = \frac{L\ g(\alpha)}{\int_0^\infty \mathrm{L\ g}(\alpha)\ \mathrm{d}\alpha}$$

$$= \frac{\alpha^{y-n-1}\ \exp\left[-\alpha\left(n+\beta y\right)\right]\left(1-\alpha\beta\right)^n}{\int_0^\infty \alpha^{y-n-1}\ \exp\left[-\alpha\left(n+\beta y\right)\right]\left(1-\alpha\beta\right)^n}.$$

Under squared error loss function, Bayes' estimator of α is given as

$$\hat{\alpha} = \frac{\int_0^\infty \alpha^{y-n}\ \exp\left[-\alpha\left(n+\beta y\right)\right]\left(1-\alpha\beta\right)^n d\alpha}{\int_0^\infty \alpha^{y-n-1}\ \exp\left[-\alpha\left(n+\beta y\right)\right]\left(1-\alpha\beta\right)^n d\alpha}. \tag{4.11}$$

where

$$\int_0^\infty \alpha^{y-n}\ e^{-\alpha\left(n+\beta y\right)}\left(1-\alpha\beta\right)^n d\alpha$$

$$= \frac{\left(-\beta\right)^{n-y-1}\Gamma\left(-1-y\right)\Gamma\left(1+y-n\right)\ {}^1F_1\left[y-n+1, y+2, z\right]}{\Gamma\left(-n\right)}$$

$$+\frac{\left(-\beta\right)^n\Gamma\left(1+y\right)\ {}^1F_1\left[-n,-y,z\right]}{\left(n+\beta y\right)^{y+1}}, \tag{4.12}$$

and

$$\int_0^\infty \alpha^{y-n-1}\, e^{-\alpha(n+\beta y)} \left(1-\alpha\beta\right)^n d\alpha$$

$$= \frac{(-\beta)^{n-y}\,\Gamma(-y)\Gamma(y-n)\ {}^1F_1\left[y-n, y+1, z\right]}{\Gamma(-n)} + \frac{(-\beta)^n\,\Gamma(y)\ {}^1F_1\left[-n, -y+1, z\right]}{(n+\beta y)^y}. \tag{4.13}$$

where $z = \frac{-(n+\beta y)}{\beta}$

Substituting the values from equations (4.12) and (4.13) in equation (4.11), we get the Bayes' estimator of α.

5. NUMERICAL EXPERIMENTS AND DISCUSSIONS

It is very difficult to compare the theoretical performances of different estimators proposed in the previous section. Therefore, we perform extensive simulations to compare the performances of the different methods of estimation mainly with respect to their biases and mean squared errors (MSE's), for different sample sizes and of different parametric values.

5.1 Average Relative Estimates and Average Relative Mean Squared Errors of α.

We consider sample sizes and different values of α. We take $n = 15$, 20, 30, 50,100 and $\alpha = 0.2$, 0.5, 1.0, 2.0. For each combination of n and α, we generate a sample of size n from SBGPD (3.1) and estimate α by different methods. We report the average values of $\left(\frac{\hat{\alpha}}{\alpha}\right)$ and also the corresponding average MSE's. All the reported results are based on 10,000 replications. The results are presented in table-1.1. Here we report the average values of $\left(\frac{\hat{\alpha}}{\alpha}\right)$ for each method and the corresponding MSE's are reported within brackets. From the table it is immediate that the average biases and the average MSE's decrease as sample size increases. It indicates that all the methods provide asymptotically unbiased and the consistent estimators. It is also observed that the average biases and the average MSE's of $\left(\frac{\hat{\alpha}}{\alpha}\right)$ depend on α. On comparing the performances of all the methods it is clear that as far as the minimum bias is concerned, Bayes' works the best in almost all the cases.

Table 1.1
Average Relative Estimates and Average Relative Mean Squared Errors of α

n	Method	α =0.2	α =0.5	α =1.0	α =2.0
$n = 15$	Bayes' MLE	1.056(0.256) 1.432(0.788)	1.254(0.123) 1.412(0.547)	1.125(0.365) 1.351(0.541)	1.221(0.778) 1.366(1.241)
$n = 20$	Bayes' MLE	1.051(0.241) 1.416(0.657)	1.214(0.119) 1.401(0.501)	1.114(0.288) 1.301(0.412)	1.211(0.554) 1.297(0.201)
$n = 30$	Bayes' MLE	1.041(0.145) 1.368(0.514)	1.1254(0.104) 1.356(0.335)	1.109(0.251) 1.201(0.226)	1.187(0.441) 1.202(0.154)
$n = 50$	Bayes' MLE	1.034(0.036) 1.221(0.299)	1.121(0.021) 1.215(0.2151)	1.015(0.125) 1.154(0.119)	1.101(0.0254) 1.165(0.125)
$n = 100$	Bayes' MLE	1.09(0.017) 1.145(0.054)	1.011(0.019) 1.152(0.014)	1.001(0.0101) 1.012(0.032)	1.021(0.021) 1.125(0.031)

5.2 Fitting of SBGPD

In order to see the efficiency of the SBGPD model in comparison to truncated Poisson and truncated generalized Poisson distribution, we have taken the data from Plackett (1953) which is listed in table-1.2. From Pearson's chi-square, AIC and BIC measures, it was observed that the size-biased generalized Poisson distribution provides us a better fit and simultaneously, it is seen that Bayes' works as a best estimator against maximum likelihood and moments method of estimation.

Table 1.2
A data set from Plackett (1953) showing the number of workers N_i having i accidents

i	N_i	**Expected Frequency**				
		TPD	**TGPD**	**SBGPD**		
				Mom	**MLE**	**Bayes'**
1	2039	2034.27	2039.00	2036.58	2037.24	2039.00
2	312	319.48	311.18	311.25	311.54	311.92
3	35	33.45	35.97	35.21	35.11	35.05
4	3	2.63	3.50	4.01	3.02	2.98
≥5	1	0.17	0.35	2.95	3.09	1.05
Total	2390	2390	2390	2390	2390	2390
χ^2		0.772	0.034	0.243	0.128	0.00018
AIC		285	276	261	260	252
BIC		310	302	295	291	289
$\hat{\alpha}$		0.3141	0.2705	0.321	0.311	0.412
$\hat{\beta}$		-	0.0759	2.15	2.35	2.0

REFERENCES

1. Bowman, K.O. and Shenton, L.R. (1985). The distribution of a moment estimator for a parameter of the generalized Poisson distribution. *Commun. in Statist.-Theo. and Meth.*, 14(4), 867-893.
2. Consul, P.C. (1986). On the differences of two generalized Poisson variates, *Commun. in Statist.-Simula. and Compu.*, 15(3), 761-767.
3. Consul, P.C. (1989). *Generalized Poisson distribution, Properties and Applications*. Marcel Decker Inc. New York.
4. Consul, P.C. and Famoye, F. (1988). Maximum likelihood estimation for the generalized Poisson distribution when the sample mean is larger than the sample variance. *Commun. in Statist.-Theo. and Meth.*, 17, 299-309.
5. Consul, P.C. and Famoye, F. (1989). The truncated generalized Poisson distribution and its applications. *Commun. in Statist.-Theo. and Meth.*, 18(10), 3635-3648.
6. Consul, P.C. and Famoye, F. (2006). *Lagrangian probability distributions*. Birkhäuser.
7. Consul, P.C. and Jain, G.C. (1973a). A generalization of Poisson distribution. *Technometrics*, 15(4). 791-799.
8. Consul, P.C. and Jain, G.C. (1973b). On some interesting properties of the generalized Poisson distribution, *Biometrische, Zeitschrift*, 15, 495-500.
9. Consul, P.C. (Auth.) and Kemp, A.W. (Rev.) (1992). Review of generalized Poisson distribution: properties and applications. *Biometrics*, 48,331-332.
10. Consul, P.C. (Auth.) and Olkin, I. (Rev.) (1992). Review of generalized Poisson distribution: properties and applications, *Metrika*, 39, 61-62.
11. Consul, P.C. and Shenton, L.R. (1972). Use of Lagrangian expansion for generating generalized probability distributions. *SIAM J. App. Math.*, 23(2), 239-248.
12. Consul, P.C. (Auth.) and Shimzu, R. (Rev.) (1992). Review of generalized Poisson distribution: properties and applications. *J. Amer. Statist. Assoc.*, 87, 1245.
13. Consul, P.C. and Shoukri, M.M. (1984). Maximum likelihood estimation for the generalized Poisson distribution. *Commun. in Statist., Theo. and Meth.*, 13(12), 1533-1547.
14. Consul, P.C. and Shoukri, M.M. (1985). The generalized Poisson distribution when the sample mean is larger than the sample variance. *Commun. in Statist., Theo. and Meth.*, 14, 667-681.
15. Consul, P.C. and Shoukri, M.M. (1986). The negative integer moments of the generalized Poisson distribution. *Commun. in Statist.-Simula. and Compu.*, 14(4), 1053-1064.
16. David, F.N. and Johnson, N.L. (1952). The truncated Poisson. *Biometrics*, 8, 275-285.
17. Famoye, F. and Lee, C.M.S. (1992). Estimation of generalized Poisson distribution. *Commun. in Statist.-Simula. and Compu.*, 21(1), 173-188.
18. Famoye, F. and Consul, P.C. (1990). Interval estimation and hypothesis testing for the generalized Poisson distribution. *Amer. J. Math. and Mngt. Sci.*, 10, 127-158.
19. Hassan, A. and Mir, K.A. (2007). On the Bayes Estimators of Parameter and Reliability Function of the zero-truncated generalized Poisson distribution. *Philippines Statistician*, 56(1-2), 41-53.
20. Hassan, A. Mir, K.A and Ahmad, M. (2007). Bayesian analysis Reliability Function of decapitated generalized Poisson distribution. *Pak. J. Statist.*, 23(3), 221-230.

21. Jain, G.C. (1975). A linear function Poisson distribution. *Biometrical Journal,* 17, 501-506.
22. Jani, P.N. and Shah, S.M. (1981). The truncated generalized Poisson distribution. *Aligarh J. Statist.*, 1(2), 174-182.
23. Johnson, N.L, Kotz, S. and Kemp, A.W (2005). *Univariate discrete distribution*. John Willy and Sons, Third Edition, New York.
24. Mishra, A. and Singh, S.K. (1993). A size-biased Lagrangian Poisson distribution and its use in zero-truncated cases. *Ranchi University Mathematical Journal*, 24, 21-28.
25. Mir, K.A. and Ahmad, M (2009). Size-biased distributions and their applications. *Pak. J. Statist.*, 25(3), 283-294.
26. Nandi, S.B., Nath, D.C. and Das, K.K. (1999). A class of generalized Poisson distributions. *Statisticaanno LIX*, 487-498.
27. Plackett, R.L. (1953). The truncated Poisson distribution. *Biometrics,* 9(4). 485-488.
28. Rao, C.R. (1965). On discrete distributions arising out of methods of ascertainment In: Patil, G.P. (ed.) *Classical and Contagious Discrete Distributions*. Statistical Publishing Society, Calcutta, 320-332.
29. Shoukri, M.M. and Consul, P.C. (1989). Bayesian analysis of a generalized Poisson distribution. *Commun. in Statist.-Simula. and Compu.*, 15(4), 1465-1480.
30. Tuenter, H.J.H. (2000). On the generalized of Poisson distribution. *Statistica Neerlandica*, 54, 374-376.

INVERSE ASCENDING FACTORIAL MOMENTS OF THE HYPER-POISSON PROBABILITY DISTRIBUTION*

Ayesha Roohi[1] and **Munir Ahmad**[2]

[1] Statistics Department, Lahore College for Women University
Lahore, Pakistan

[2] National College of Business Administration & Economics
Lahore Pakistan

ABSTRACT

Inverse ascending factorial moments of the hyper-Poisson distribution have been derived in terms of hypergeometric series function. A recurrence relation for the negative moments and inverse ascending factorial moments is also derived. The Poisson distribution as a special case of the hyper-Poisson distribution has also been dealt with.

KEYWORDS

Negative moments, inverse ascending factorial moments, hypergeometric series function, hyper-Poisson distribution.

1. INTRODUCTION

Negative moments have been under study for quite some time. Many authors [Grab and Savage (1954), Mendenhall and Lehman (1960), Govindarajulu (1962, 1963), Tiku (1964), Stancu (1968), Chao and Strawderman (1972), Lepage (1978), Cressie et al. (1981), Cressie and Borkent (1986) and Jones (1986, 1987) Roohi (2002)], have worked on the negative moments of discrete distributions, mainly binomial, Poisson, geometric and negative binomial distributions. Ahmad and Sheikh (1983) used Chao and Strawderman (1972) technique to obtain the first negative moment of the hyper-Poisson distribution and stated the conditions under which this moment is identical to that of the Poisson distribution.

In this paper, we derive inverse ascending factorial moments of the hyper-Poisson distribution in terms of hyper-geometric series function. The expressions are simple and easy to compute. A recurrence relation of negative ascending factorial moments is also derived so that higher moments are easily calculated. Similar results have been given for the Poisson distribution.

2. INVERSE ASCENDING FACTORIAL MOMENTS

Theorem 1:

Suppose the random variable X follows a hyper-Poisson distribution with parameters θ and λ. Then the inverse ascending factorial moment of X is:

*Published in Pak. J. Statist. (2003), Vol. 19(2).

$$\mu'_{-[k]} = E\left[\prod_{i=1}^{k}(x+i)\right]^{-1} = C_{\theta,\lambda}\,(k!)^{-1}\,{}_2F_2\left[1,1;\lambda,k+1;\theta\right], k=1,2...$$

where

$$C_{\theta,\lambda} = \left\{{}_1F_1\left[1;\lambda;\theta\right]\right\}^{-1} \quad (1)$$

Proof:

Since $\left[(x+1)(x+2)...(x+k)\right]^{-1} = \sum_{s=1}^{k}\frac{(-1)^{s+1}}{(s-1)!(k-s)!(x+s)}$,

$$\mu'_{-[k]} = E\left[\prod_{i=1}^{k}(X+i)\right]^{-1}$$

$$= \sum_{s=1}^{k}\frac{(-1)^{s+1}}{(s-1)!(k-s)!}E\left(\frac{1}{X+s}\right) \quad \text{(see Jones 1987)}$$

$$= \sum_{s=1}^{k}\frac{(-1)^{s+1}}{(s-1)!(k-s)!}.C_{\theta,\lambda}s^{-1}\,{}_2F_2\left[1,s;\lambda,s+1;\theta\right] \quad = C_{\theta,\lambda}s^{-1}\,{}_2F_2\left[1,1,\lambda,s+1;\theta\right]$$

(see Ahmad and Sheikh 1983, and Ahmad and Roohi 2002).

Corollary:

For $\lambda = 1$ we get the inverse ascending factorial moment of the Poisson distribution,

$$\mu'_{-[k]} = \sum_{s=1}^{k}\frac{(-1)^{s+1}}{(s-1)!(k-s)!}.e^{-\theta}s^{-1}\,{}_2F_1\left[s,;s+1;\theta\right]$$

$$= e^{-\theta}\left[k!\right]^{-1}\,{}_1F_2\left[1;k+1;\theta\right] \quad k=1,2,....$$ (see Ahmad and Roohi 2002).

3. RECURRENCE RELATION OF NEGATIVE MOMENTS AND INVERSE ASCENDING FACTORIAL MOMENTS OF THE HYPER-POISSON DISTRIBUTION

Theorem 2:

Suppose the random variable X has a Hyper-Poisson Distribution with parameters θ and λ. Then

$$E(X+A)^{-1} = \frac{1}{\theta}\left[1+\frac{1-\lambda}{A-1}C_{\theta,\lambda} + (\lambda-A)E(X+A-1)^{-1}\right] \text{ for } A>1 \quad (2)$$

Proof:

We have

$$E(X+A)^{-1} = A^{-1}C_{\theta,\lambda}\,{}_2F_2\left[1,A;\lambda,A+1;\theta\right]$$ (see Ahmad and Sheikh 1983)

Using the identity (see Rainville 1960)

$$ {}_2F_2\left[\alpha_1,\alpha_2;\beta_1,\beta_2;x\right] = {}_2F_2\left[\alpha_1-1,\alpha_2;\beta_1,\beta_2;x\right] + x\left\{\frac{\alpha_2-\beta_1}{\beta_1\left(\beta_2-\beta_1\right)}\right. $$

$$ \left. {}_2F_2\left[\alpha_1,\alpha_2;\beta_1+1,\beta_2;x\right] + \frac{\alpha_2-\beta_2}{\beta_2\left(\beta_1-\beta_2\right)}\,{}_2F_2\left[\alpha_1,\alpha_2;\beta_1,\beta_2+1;x\right]\right\} \tag{3} $$

For $\alpha_1 = A, \alpha_2 = 1, \beta_1 = A, \beta_2 = \lambda$ and $x = \theta$ we get:

$$ {}_2F_2\left[1,A;\lambda.A;\theta\right] = {}_2F_2\left[1,A-1;\lambda.A;\theta\right] + \theta\left\{\frac{1-A}{A(\lambda-A)}\,{}_2F_2\left[1,A;\lambda.A+1;\theta\right]\right. $$

$$ \left. + \frac{1-\lambda}{\lambda\left(A-\lambda\right)}\,{}_2F_2\left[1,A;\lambda+1.A;\theta\right]\right\} $$

We know that

$$ {}_2F_2\left[1,A;\lambda,A;\theta\right] = {}_1F_1\left(1;\lambda;\theta\right) $$

and

$$ {}_2F_2\left[1,A;\lambda+1,A;\theta\right] = {}_1F_1\left(1;\lambda+1;\theta\right) \tag{4} $$

Then

$$ {}_1F_1\left(1;\lambda;\theta\right) = \frac{A-1}{C_{\theta,\lambda}}E\left(X+A-1\right)^{-1} + \frac{\theta(1-A)}{\lambda-A}.\frac{1}{C_{\theta,\lambda}}E\left(X+A\right)^{-1} $$

$$ + \frac{\theta(1-\lambda)}{\lambda\left(A-\lambda\right)}\,{}_1F_1\left(1;\lambda+1;\theta\right) $$

Hence

$$ E\left(X+A\right)^{-1} = \frac{C_{\theta,\lambda}}{1-A}\left\{\frac{\lambda-A}{\theta}\,{}_1F_1\left[1;\lambda;\theta\right] + \frac{1-\lambda}{\lambda}\,{}_1F_1\left[1;\lambda+1;\theta\right]\right\} + \frac{\lambda-A}{\theta}E\left(X+A-1\right)^{-1}. $$

We know that

$$ {}_1F_1\left[1;\lambda+1;\theta\right] = \frac{\lambda}{\theta}\left\{{}_1F_1\left[1;\lambda;\theta\right]-1\right\} \tag{5} $$

Substituting (5) in (4) we get:

$$ E\left(X+A\right)^{-1} = \frac{C_{\theta,\lambda}}{1-A}\left\{\frac{1-A}{\theta}\,{}_1F_1\left[1;\lambda;\theta\right] - \frac{1-\lambda}{\theta}\right\} + \frac{\lambda-A}{\theta}E\left(X+A-1\right)^{-1} \tag{6} $$

Hence the result.

Corollary:

For $\lambda = 1$, we get the recurrence relation for the negative moment of the Poisson distribution.

$$E(X+A)^{-1} = \frac{1}{\theta}\left\{1-(A-1)E(X+A-1)^{-1}\right\}, A > 1$$

This result was also obtained by Chao and Strawderman (1972) by integrating the probability generating function.

Theorem 3:

Suppose that the random variable X has a Hyper-Poisson distribution with parameters θ and λ, and $\mu'_{[k]}$ is the *k*th inverse ascending factorial moment of X. Then the relation

$$k^2\theta\mu'_{-[k+1]} = (k+\theta)(1-\lambda+k)\mu'_{-[k]} - (k+1-\lambda)\mu'_{-[k-1]}$$

$$+\frac{\theta(\lambda-1)^2}{k!} C_{\theta,\lambda}\ {}_2F_2\left[1,1;\lambda+1,k+1;\theta\right] \text{ holds for } k > 1$$

Proof:

We know that (see equation (1)),

$$\mu'_{-[k]} = \frac{C_{\theta,\lambda}}{k!}\ {}_2F_2\left[1,1;\lambda,k+2;\theta\right]$$

Then

$$\mu'_{-[k+1]} = \frac{C_{\theta,\lambda}}{(k+1)!}\ {}_2F_2\left[1,1;\lambda,k+2;\theta\right]$$

Using the identity (see Rainville 1960)

$$(\alpha_1-\beta_1+1)\ {}_2F_2\left[\alpha_1,\alpha_2;\beta_1,\beta_2;x\right] = \alpha_1\ {}_2F_2\left[\alpha_1+1,\alpha_2;\beta_1-1,\beta_2;x\right] - (\beta_1-1)\ {}_2F_2\left[\alpha_1,\alpha_2;\beta_1-1,\beta_2;x\right] \quad (7)$$

If $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = k+2, \beta_2 = \lambda, x = \theta,$ we have

$$(-k)\ {}_2F_2\left[1,1;\lambda,k+2;\theta\right] = {}_2F_2\left[1,2,\lambda,k+2;\theta\right] - (k+1)\ {}_2F_2\left[1,1;\lambda,k+1;\theta\right] \quad (8)$$

Using the identity (3) for $\alpha_1 = 2, \alpha_2 = 1, \beta_1 = \lambda, \beta_2 = k+1, x = \theta,$ we get:

$$ {}_2F_2\left[1,2;\lambda,k+1;\theta\right] = {}_2F_2\left[1,1,\lambda,k+1;\theta\right] + \theta\left\{\frac{1-\lambda}{\left(k+1-\lambda\right)}\,{}_2F_2\left[1,2,\lambda+1,k+1;\theta\right]\right. $$

$$ \left. -\frac{k}{\left(k+1\right)\left(\lambda-k-1\right)}\,{}_2F_2\left[2,1;\lambda,k+2;\theta\right]\right\} $$

Hence

$$ \frac{k\theta}{\left(k+1\right)\left(k+1-\lambda\right)}\,{}_2F_2\left[1,2;\lambda,k+2;\theta\right] = {}_2F_2\left[1,2;\lambda,k+1;\theta\right] - {}_2F_2\left[1,1;\lambda,k+1;\theta\right] $$

$$ +\frac{\theta\left(\lambda-1\right)}{\lambda\left(k+1-\lambda\right)}\,{}_2F_2\left[1,2;\lambda+1,k+1;\theta\right] $$

$$ {}_2F_2\left[1,2;\lambda,k+2;\theta\right] = \frac{\left(k+1\right)\left(k+1-\lambda\right)}{k\theta}\,{}_2F_2\left[1,2;\lambda+1,k+1;\theta\right] $$

$$ -\frac{\left(k+1\right)\left(k+1-\lambda\right)}{k\theta}\,{}_2F_2\left[1,1;\lambda,k+1;\theta\right] + \frac{\left(\lambda-1\right)\left(k+1\right)}{\lambda k}\,{}_2F_2\left[1,2;\lambda+1,k+1;\theta\right] \tag{9} $$

Substituting (9) in (8) we get:

$$ \left(-k\right)\,{}_2F_2\left[1,1;\lambda,k+2;\theta\right] = \frac{\left(k+1\right)k+1-\lambda}{k\theta}\,{}_2F_2\left[1,2;\lambda,k+1;\theta\right] $$

$$ -\frac{\left(k+1\right)\left(k+1-\lambda\right)}{k\theta}\,{}_2F_2\left[1,1;\lambda,k+1;\theta\right] $$

$$ +\frac{\left(\lambda-1\right)\left(k+1\right)}{\lambda k}\,{}_2F_2\left[1,2;\lambda+1,k+1;\theta\right] - \left(k+1\right)\,{}_2F_2\left[1,1;\lambda,k+1;\theta\right] $$

Thus

$$ {}_2F_2\left[1,1;\lambda,k+2;\theta\right] = \frac{\left(k+1\right)\left(k+1-\lambda+k\theta\right)}{k^2\theta}\,{}_2F_2\left[1,1;\lambda,k+1;\theta\right] $$

$$ -\frac{\left(k+1\right)\left(k+1-\lambda\right)}{k^2\theta}\,{}_2F_2\left[1,1;\lambda,k+1;\theta\right] $$

$$ +\frac{\left(\lambda-1\right)\left(k+1\right)}{\lambda k^2}\,{}_2F_2\left[1,2;\lambda+1,k+1;\theta\right] \tag{10} $$

Using the identity (7) for, we have: $\alpha_1=1,\alpha_2=1,\beta_1=k+1,\beta_2=\lambda,x=\theta,$ we have:

$$ {}_2F_2\left[1,2;\lambda,k+2;\theta\right] = k\ {}_2F_2\left[1,1;\lambda,k;\theta\right] - (k-1)\ {}_2F_2\left[1,1;\lambda,k+1;\theta\right] \quad (11) $$

Again using (7) for $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = \lambda+1, \beta_2 = k+1, x = \theta,$ we have:

$$ {}_2F_2\left[1,2;k+1,\lambda+1;\theta\right] = (\lambda)\ {}_2F_2\left[1,1;k+1,\lambda;\theta\right] - (\lambda-1)\ {}_2F_2\left[1,1;\lambda+1,k+1;\theta\right] \quad (12) $$

Substituting (11) and (12) in (10), we obtain:

$$ \begin{aligned} {}_2F_2\left[1,1;\lambda,k+2;\theta\right] = &\frac{(k+1)(k+1-\lambda+k\theta)}{k^2\theta}\ {}_2F_2\left[1,1;\lambda,k+1;\theta\right] \\ &-\frac{(k+1)(k+1-\lambda)}{k^2\theta}\left\{(k)\ {}_2F_2\left[1,1;\lambda,k;\theta\right] - (k-1)\ {}_2F_2\left[1,1;\lambda,k+1;\theta\right]\right\} \\ &-\frac{(\lambda-1)(k+1)}{k^2\lambda}\left\{(\lambda)\ {}_2F_2\left[1,1;\lambda,k+1;\theta\right]\right\} \\ &-(\lambda-1)\ {}_2F_2\left[1,1;\lambda+1,k+1;\theta\right] \\ = &\frac{(k+1)\left(k\theta+k^2+k-k\lambda-\theta\lambda+\theta\right)}{k^2\theta}\ {}_2F_2\left[1,1;\lambda,k+1;\theta\right] \\ &-\frac{(k+1)(k+1-\lambda)}{k\theta}\ {}_2F_2\left[1,1;\lambda,k;\theta\right] \\ &+\frac{(\lambda-1)^2(k+1)}{k^2\lambda}\ {}_2F_2\left[1,1;\lambda+1,k+1;\theta\right] \end{aligned} $$

Hence

$$ \begin{aligned} \mu'_{-[k+1]} = &\frac{\left(k\theta+k^2+k-k\lambda-\theta\lambda+\theta\right)}{k^2\theta}\mu'_{-[k]} = \frac{k+1-\lambda}{k^2\theta}\mu'_{-[k-1]} \\ &+C_{\theta,\lambda}\frac{(\lambda-1)^2}{k^2\lambda k!}\ {}_2F_2\left[1,1;\lambda+1,k+1;\theta\right] \end{aligned} $$

$$ \begin{aligned} k^2\theta\mu'_{-[k+1]} = &(k+\theta)(1-\lambda+k)\mu'_{-[k]} - (k+1-\lambda)\mu'_{-[k-1]} \\ &+\frac{\theta(\lambda-1)^2}{k!}C_{\theta,\lambda}\ {}_2F_2\left[1,1;\lambda+1,k+1;\theta\right], \qquad k>1 \end{aligned} $$

Corollary:

When $\lambda = 1$, in equation (6), the recurrence relation for the inverse ascending factorial moment of Poisson distribution is:

$$k\theta\mu'_{-[k+1]} = (k+\theta)\mu'_{-[k]} - \mu'_{-[k-1]}, \quad k > 1.$$

4. ESTIMATION

Consider the hyper-Poisson distribution with λ known and θ unknown.

$$Var\left(\hat{\theta}\right) = \frac{1}{{}_1F_1^2\left[1;\lambda;\theta\right]}\left[{}_3F_3\left[1,1,1;2,2,\lambda;\theta\right] - \frac{\left\{ {}_2F_2\left[1;2,\lambda;\theta\right]\right\}^2}{{}_1F_1\left[1;\lambda;\theta\right]} \right] \qquad (13)$$

(See Roohi and Ahmad 2003)

for $\lambda = 1$ in (13), we get the variance of the negative moment estimator $\hat{\theta}$ of θ, the parameter of a Poisson distribution.

$$Var\left(\hat{\theta}\right) = \left(\frac{\theta e^{-\theta} + e^{-\theta} - 1}{\theta^2}\right)^{-2} \frac{e^{-\theta}}{n}\left[\left[{}_2F_2 1,1;2,2;\theta \right] - \frac{\left\{ {}_1F_1\left[1;2;\theta\right]\right\}^2}{e^{\theta}} \right]$$

REFERENCES

1. Ahmad, M. and Roohi, A. (2002). On sums of some hyper-geometric series functions. *Submitted for publication.*
2. Ahmad, M. and Sheikh, A.K. (1983). Some estimation problem of a class of inverted distributions. *44th session of the International Statistical Institute.* Spain. 1-6.
3. Chao, M.T. and Strawderman, W.E. (1972). Negative moments of positive random variables. *Journal of American Statistical Association.* 67, 429-43 1.
4. Cressie, N., Davids, A.S., Folks, J.L. and Policello, G.E. (1981). The moment generating function and negative intiger moments. *American Statistician,* 35(3), 148-150.
5. Cressie, N. and Borkent, M. (1986). The moment generating function has its moments. *Journal of Statistical Planning and Inference*, 13, 337-344.
6. Govindarajulu, Z. (1962). The reciprocal of the decapitated negative binomial variable. *Journal of American Statistical Association*, 57, 906-913.
7. Govindarajulu, Z. (1963). Recurrence relations for the inverse moments of the positive binomial variable. *Journal of American Statistical Association*, 58, 467-473.
8. Grab, E.L. and Savage, I.R. (1954). Tables of the expected value of 1/X for positive Bernoulli and Poisson variables. *J. Amer. Statist. Assoc.*, 49, 169-177.
9. Jones, M.C. (1986). Inverse moments of negative binomial, *Journal of Statistical Computation and Simulation*, 23, 241-243.
10. Jones, M.C. (1987). Inverse factorial moments. *Statistics and Probability Letters*, 6, 37-42.

11. Lepage, Y. (1978). Negative factorial moments of positive random variables. *Industrial Mathematics*, 28, 95-100.
12. Mendenhall, W. and Lehmann, E.H. Jr. (1960). An approximation to the negative moments of the positive binomial useful in life testing. *Technometrics*, 2,227- 41.
13. Rainville, E.D. (1960). *Special functions.* Chelsea Publishing Company, Bronnx N.
14. Roohi, A. (2002). *Negative Moments of Non-Negative Random Variables*. Unpublished M.Phil. dissertation, University of Lahore, Lahore.
15. Roohi, A. and Ahmad, M. (2003). Estimation of the parameters of the hyper-Poisson distribution using negative moments. *Pakistan Journal of Statistics*, 19 (1), 99- 105.
16. Stancu, D.D. (1968). On the moments of negative order of the positive Bernoulli and Poisson variables. Studia Universitatis Babes Bolyai Series, Mathematics and Physics, 1, 29-31.
17. Tiku, M.L. (1964). A note on the negative moments of a truncated Poisson variate. *Journal of American Statistical Association*, 59, 1220-4.

ON SUM OF SOME HYPER-GEOMETRIC SERIES FUNCTIONS-I*

Munir Ahmad[1] and **Ayesha Roohi**[2]

[1] National College of Business Administration & Economics
Lahore Pakistan

[2] Statistics Department, Lahore College for Women University
Lahore, Pakistan

ABSTRACT

In this paper sum of some hyper-geometric series functions have been derived, using properties of discrete probability functions.

INTRODUCTION

Jones (1987), Lepage (1978) and Roohi (2002) have worked on the negative moments and inverse factorial moments of some discrete probability functions. In Statistics, some discrete probability functions have been expressed in terms of hyper geometric series functions (See Bardwell and Crow, 1964). Using properties of the discrete probability functions, simple solutions of sums of hyper geometric series functions have been found.

Theorem 1:

Let ${}_2F_1(a,\ b;\ c;\ \theta)$ be defined as:

$$ {}_2F_1(a,\ b;\ c;\ 0) = 1 + \frac{ab}{c}\theta + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{\theta^2}{2} + \dots, c \# 0 \qquad (1)$$

then if a or b is negative, ${}_2F_1$ will be defined as terminating series, then

$$\sum_{s=1}^{k} (-1)^{s+1} \binom{k}{s} {}_2F_1(s, -n; s+1; -\theta) = {}_2F_1[1-n; k+1; -\theta] \qquad (2)$$

$$\theta > 0, n, k = 1, 2, 3, \dots$$

Proof:

Suppose X is a binomial random variable with parameters n and θ,

$$f(x) = \binom{n}{x} \frac{\theta^x}{(1+\theta)^n}, \qquad x = 0, 1, 2, \dots\dots n, \theta > 0 \qquad (3)$$

It is known that

$$\prod_{s=1}^{k} \frac{1}{x+s} = \sum_{s=1}^{k} \frac{(-1)^{s+1}}{(k-s)!(s-1)!} \frac{1}{x+s}, \qquad x \geq 0, k = 1, 2, 3, \dots\dots \qquad (4)$$

(See Rainville 1960)

*Published in Pak. J. Statist. (2004), Vol. 20(1).

By definition of $E\left[g(x)\right]=\sum g(x)\,f(x)$, we have

$$E\left(\prod_{s=1}^{k}\frac{1}{X+s}\right)=\sum_{s=1}^{k}\frac{(-1)^{s+1}}{(k-s)!(s-1)!}.E\left(\frac{1}{X+s}\right)$$

Now

$$E\left(\frac{1}{X+s}\right)=\sum_{x=0}^{x}\frac{1}{X+s}\binom{n}{x}p^{x}q^{n-x}=\frac{1}{s(1+\theta)^{n}}\,{}_2F_1\left[s,-n;s+1;-\theta\right]$$

Thus

$$E\left[\prod\left(\frac{1}{X+s}\right)\right]=\sum_{s=1}^{k}\frac{(-1)^{s+1}}{(k-s)!(s-1)!}.\frac{1}{s(1+\theta)^{n}}\,{}_2F_1\left[s,-n;s+1;-\theta\right]$$

$$=\frac{1}{(1+\theta)^{n}\,k!}\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}\,{}_2F_1\left(s,-n;s+1;-\theta\right) \tag{5}$$

Also

$$E\left(\prod_{s=1}^{k}\frac{1}{X+s}\right)=\sum_{x=0}^{n}\binom{n}{x}\frac{\theta^{x}}{(1+\theta)^{n}}\prod_{s=1}^{k}\left(\frac{1}{X+s}\right),$$

Expanding the summation, we have

$$=\frac{1}{(1+\theta)^{n}\,k!}\left[1+\frac{n}{k+1}\theta+\frac{n(n-1)}{2!}\frac{1.2}{(k+1)(k+2)}\theta^{2}+....+\frac{n!}{(k+1)...(k+n)}\theta^{n}\right]$$

$$=\frac{1}{(1+\theta)^{n}\,k!}\left[1+\frac{-n}{k+1}(-\theta)+\frac{(-n)(-n+1)1.2}{(k+1)(k+2)}\frac{1}{2!}(-\theta)^{2}\right.$$

$$\left.+....+\frac{(-n)(-n+1)..(-1)1.2...n}{(k+1)...(k+n)\,n!}(-\theta)^{n}\right]$$

$$=\frac{1}{(1+\theta)^{n}\,k!}\,{}_2F_1\left(1,-n;k+1,-\theta\right) \tag{6}$$

Thus, we get the result from (4) and (6),

Theorem 2:

Let ${}_2F_1\left(a,\ b;\ c;\ \theta\right)$ be defined as in (1), then

$$\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}{}_2F_1(r,s;s+1;q)={}_2\,F_1(1,r;k+1;q) \quad (7)$$

$r=1,2,\ldots,\ k=1,2,\ldots,\ 0<q<1$ and $p=1-q$ holds.

Proof:

Suppose X has a negative binomial function

$$f(x)=\binom{x+r-1}{x}p^r q^x, \quad (8)$$

$r=1,2,\ \ldots,\ \ k=1,2,\ldots,\ 0<q<1$ and $p=1-q$

If X has the probability function (8), then

$$E\left(\frac{1}{X+s}\right)=\sum_{x=0}^{\infty}\frac{1}{X+s}\binom{x+r-1}{x}p^r q^x$$

$$=\frac{p^r}{s}\left[1+\frac{s.r}{s+1}q+\frac{r(r+1)s(s+1)}{(s+1)(s+2)}.\frac{q^2}{2!}+\ldots\right]$$

$$=\frac{p^r}{s}\,{}_2F_1[r,s;s+1;q] \quad (9)$$

By definition of expected of a function, $\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)$ and defined as in (3), we have

$$E\left(\prod_{s=1}^{k}\frac{1}{X+s}\right)=\sum_{s=o}^{k}\frac{(-1)^{s+1}}{(k-s)!(s-1)!}E\left(\frac{1}{X+s}\right) \quad (10)$$

Substituting (9) in (10), we have

$$=\sum_{s=1}^{k}\frac{(-1)^{s+1}}{(k-s)!(s-1)!}\frac{p^r}{s}\,{}_2F_1(r,s;s+1;q)$$

$$=\frac{p^r}{k!}\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}{}_2F_1(r,s;s+1;q), \quad (11)$$

Also by definition of expectation $(X+s)^{-1}$ when $f(x)$ is defined as (8), we have

$$E\left[\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)\right]=\sum_{x=0}^{\infty}\binom{x+r-1}{x}p^r q^x\prod_{s=1}^{k}\left(\frac{1}{x+s}\right)$$

$$=\frac{p^r}{k!}\left[1+\frac{r.1}{k+1}q+\frac{r(r+1)1.2}{(k+1)(k+2)}\frac{q^2}{2!}+...\right]$$

$$=\frac{p^r}{k!}\,{}_2F_1\left[1,r;k+1;q\right] \qquad (12)$$

We get the result from (11) and (12)

Corollary 1:

If $r=1$, then

$$\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}\,{}_2F_1(1,s;s+1;q)={}_2F_1(1,1;k+1;q) \qquad (13)$$

Proof:

This follows immediately from (6) by putting $r=1$.

REFERENCES

1. Bardwell, G.E and Crow, E.L. (1964). A two parameter Family of Hyper Poisson Distribution. *J. Amer. Statist. Assoc.*, 59, 133-141.
2. Jones M.C. (1987) Inverse factorial moments. *Statistics and Probability letters*. 6, 37-42.
3. Lepage, Y. (1978). Negative factorial moments of positive random variables. *Industrial Mathematics*, 28, 95-100.
4. Rainville, E.D. (1960). *Special Functions*. Chelsea Publication Co. Bornmx. U.S.A.
5. Roohi, A. (2002). *Negative Moments of Non-negative random variables*. Unpublished M.Phil. dissertation. The University of Lahore, Lahore. pp 135.

CHARACTERIZATION OF THE POISSON PROBABILITY DISTRIBUTION*

Munir Ahmad[1] and **Ayesha Roohi**[2]
[1] National College of Business Administration & Economics
Lahore Pakistan
[2] Statistics Department, Lahore College for Women University
Lahore, Pakistan

ABSTRACT

Characterization of the Poisson distribution has been obtained using a recurrence relation for the first order negative moment of the Poisson random variable.

KEYWORDS

Negative moments, hyper-geometric series function.

1. INTRODUCTION

Tiku (1964) discusses the case where sample size is a random variable and gives an approximate result for the first order negative moment of the positive Poisson distribution. Ahmad and Sheikh (1983) obtain the first order negative moment of the hyper-Poisson distribution and hence that of a Poisson distribution as a special case. Daboni (1959) gives a characterization of the Poisson distribution based on mixtures of binomial distributions. Rao and Rubin (1964) using conditional probabilities obtain a characterization of the Poisson distribution. Ahmad and Roohi (2004) obtain a characterization of the binomial distribution using a recurrence relation of the negative moments of the binomial random variable.

In this paper, first a recurrence relation is derived for the negative moment of a Poisson distribution using a property of the hyper-geometric series function, and then it is used to obtain a characterization of the distribution.

2. RECURRENCE RELATION

Chao and Strawderman (1972) by integrating the probability generating function of the random variable $\left(X+A-1\right)^{-1}>0$, obtained a recurrence relation for the first order negative moment of the Poisson distribution. The same recurrence relation was derived by Kumar and Consul (1979) as a special case of the Lagrangian Poisson distribution.

In this section, we use a property of the hyper-geometric series function and give an alternate method of deriving the recurrence relation for the negative moment of the Poisson distribution.

*Published in Pak. J. Statist. (2004), Vol. 20(2).

Theorem 1:

Suppose X has a Poisson probability distribution with parameter $\theta > 0$, then for $A > 1$, the relation

$$E\left(X + A\right)^{-1} = \frac{1}{\theta} - \frac{A-1}{\theta} E\left(X + A - 1\right)^{-1} \tag{1}$$

holds

Proof:

Since X is a Poisson random variable with parameter θ, then Ahmad and Sheikh (1983) show:

$$E\left(\frac{1}{X + A}\right) = e^{-\theta} A^{-1} \; {}_1F_1\left[A;\ A+1;\ \theta\right]. \tag{2}$$

where ${}_1F_1\left[a;\ b;\ x\right] = 1 + \frac{a}{b}\,x + \frac{a(a+1)}{b(b+1)}\,\frac{x^2}{2\,!} + ...$

Replacing A by A-1 in (2), we get

$$E\left(\frac{1}{X + A\text{-}1}\right) = e^{-\theta}(A\text{-}1)^{-1} \; {}_1F_1\left[A\text{-}1;\ A;\ \theta\right]. \tag{3}$$

Using the identity (see Rainville 1960 page 124).

$$b\ {}_1\mathrm{F}_1\left[\mathrm{a};\ b;\ \mathrm{x}\right] = b\,{}_1\mathrm{F}_1\left[\mathrm{a}\text{-}1;\ b;\ \mathrm{x}\right] + x\,{}_1\mathrm{F}_1\left[\mathrm{a};\ b+1;\ \mathrm{x}\right]$$

for $a = A,\ b = A$ and $x = \theta$, we get:

$$A\,{}_1\mathrm{F}_1\left[\mathrm{A};\ A;\ \theta\right] = A\,{}_1\mathrm{F}_1\left[\mathrm{A}\text{-}1;\ A;\ \theta\right] + \theta\,{}_1\mathrm{F}_1\left[\mathrm{A};\ A+1;\ \theta\right] \tag{4}$$

Now

$${}_1\mathrm{F}_1\left[\mathrm{A};\ A;\ \theta\right] = e^{\theta}. \tag{5}$$

Substituting (2), (3) and (5) in (4) we get the required result.

3. CHARACTERIZATION

In this section the recurrence relation derived in theorem 1 is used for the characterization of the Poisson distribution.

Theorem 2:

X has a Poisson probability function $P_x(\theta),\ \theta > 0$, if and only if, for $A > 1$ and $x = 0,1,2\ldots;$ (1) holds.

Proof:

If $P_x(\theta)$ is a Poisson probability function, the recurrence relation (1) holds (Theorem 1).

Now

$$E\left(X+A\right)^{-1}=\frac{1}{\theta}-\frac{A-1}{\theta}E\left(X+A-1\right)^{-1},$$

$$\sum_{x=0}^{\infty}\frac{1}{\left(x+A\right)}P_x=\frac{1}{\theta}-\frac{A-1}{\theta}\sum_{x=0}^{\infty}\frac{1}{\left(x+A-1\right)}P_x$$

$$=\frac{1}{\theta}-\frac{1}{\theta}P_0-\frac{A-1}{\theta}\left\{\sum_{x=1}^{\infty}\frac{1}{\left(x+A-1\right)}P_x\right\},$$

$$=\frac{1-P_0}{\theta}-\frac{A-1}{\theta}\sum_{x=0}^{\infty}\frac{1}{x+A}P_{x+1}.$$

Re-arranging, we have

$$\theta\sum_{x=0}^{\infty}\frac{1}{\left(x+A\right)}P_x=\sum_{x=0}^{\infty}\frac{x+1}{x+A}P_{x+1},$$

or

$$\sum_{x=0}^{\infty}\left(\frac{\theta P_x-\left(x+1\right)P_{x+1}}{x+A}\right)=0.$$

Since $\theta P_x-\left(x+1\right)P_{x+1}$ is either >= or <0, then in each case, we get $\frac{\theta P_x}{x+A}=\frac{x+1}{x+A}P_{x+1}$.

Thus

$$P_{x+1}=\frac{\theta}{x+1}P_x$$

Giving x values $0,1,2,\ldots$ we get

$$P_1=\theta\,P_0,\ P_2=\frac{\theta}{2}P_1=\frac{\theta^2}{2!}P_0,\ P_3=\frac{\theta}{3}P_2=\frac{\theta^3}{3!}P_0\ \ldots\ldots\ P_x=\frac{\theta^x}{x!}P_0.$$

Since

$\sum_{x=0}^{\infty}P_x=1$, we get $P_0=e^{-\theta}$. Hence the result.

REFERENCES

1. Ahmad, M. and Roohi, A. (2004). *A note on characterization of the binomial probability distribution*. Submitted for publication.
2. Ahmad, M. and Sheikh, A.K. (1983). *Some estimation problem of a class of inverted distributions*. 44th session of the International Statistical Institute. Spain. 1-6.
3. Chao, M.T. and Strawderman, W.E. (1972). Negative moments of positive random variables. *J. Ame. Statist. Assoc.*, 67, 429-431.
4. Daboni, L. (1959). A property of the Poisson distribution. *Bolletino della Unione Mathematica Italiana*, 14, 318-320.
5. Kumar A. and Consul, P.C. (1979). Negative moments of a modified power series distribution and bias of the maximum likelihood estimator. *Communications in Statistics*. A. 8, 151-166.
6. Rao, C.R. and Rubin, H. (1964). On a characterization of the Poisson distribution. *Sankhya* A. 26, 295-298.
7. Tiku, M.L. (1964). A note on the negative moments of a truncated Poisson variate. *J. Amer. Statist. Assoc.* 59, 1220-1224.

ON SUMS OF SOME HYPER-GEOMETRIC SERIES FUNCTIONS-II*

Munir Ahmad[1] and **Ayesha Roohi**[2]

[1] National College of Business Administration & Economics
Lahore Pakistan. Email: drmunir@brain.net.pk

[2] Statistics Department, Lahore College for Women University
Lahore, Pakistan. Email: ayesha_roohi_2004@yahoo.co.uk

ABSTRACT

Ahmad and Roohi (2004) derive the sum of some hyper-geometric series function in terms of a simple hyper-geometric series. In this paper, sums of another set of hyper-geometric series functions have been obtained using properties of discrete probability functions.

1. INTRODUCTION

Jones (1987), Lepage (1978) and Roohi (2002) have worked on the negative moments and inverse factorial of some discrete probability functions. Using properties of the discrete probability functions, simple solutions of sums of hyper geometric series functions have been found by Ahmad and Roohi (2004). In this paper, sums of another set of hyper-geometric functions have been derived.

2. SUM OF HYPER GEOMETRIC SERIES FUNCTIONS

Theorem 1:

Let ${}_hF_m\left[(a);(b);\theta\right]$ be defined as:

$$ {}_hF_m\left[(a)\,;\,(b);\,\theta\right] = 1+\frac{a_1a_2\ldots a_h}{b_1b_2\ldots b_m}\theta+\frac{a_1\left(a_1+1\right)\ldots a_h\left(a_h+1\right)}{b_1\left(b_1+1\right)\ldots b_m\left(b_m+1\right)}\frac{\theta^2}{2!}+\ldots \tag{2.1}$$

No term in (b) is zero. Then

$$\sum_{s=1}^{k}(-1)^{s+1}\binom{k+1}{s+1}s\;{}_3F_2\left(1,1,s+1;2,s+2;\theta\right) = {}_2F_1\left(1,\;1;\;k+2,\;\theta\right),$$

$$0<\theta<1,\; k=1,2,\ldots \tag{2.2}$$

Proof:

Suppose a random variable X has logarithmic series function

$$f\left(x\right)=\alpha\frac{\theta^x}{x},\; x = 1, 2, \ldots\; 0<\theta<1. \tag{2.3}$$

where $a=-\left[\ln(1-q)\right]^{-1}$. Now by definition of expectation of random variables, we have

---

*Published in Pak. J. Statist. (2005), Vol. 21(3).

$$E\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)=\alpha\sum_{x=1}^{\infty}\frac{\theta^x}{x}\prod_{s=1}^{k}\frac{1}{x+s}$$

$$=\frac{\alpha\ \theta}{(k+1)\ !}\left[1+\frac{1}{k+2}\theta+\frac{1.2.1.2}{(k+2)(k+3)}\frac{\theta^2}{2\ !}+...\right]$$

$$=\frac{\alpha\ \theta}{(k+1)!}\ {}_2F_1(1,\ 1;\ k+2;\ \theta) \qquad (2.4)$$

Alternatively, if X has Logarithmic probability function (2.3), then

$$E\left(\frac{1}{X+s}\right)=\alpha\sum_{x=1}^{\infty}\frac{1}{x+s}\ \frac{\theta^x}{x}$$

$$=\frac{\alpha\theta}{s}\left[1+\frac{s+1}{s+2}\theta+\frac{(s+1)(s+2)1.2}{(s+2)(s+3)1.2}.\frac{\theta^2}{2}+\cdots\right]$$

$$=\frac{\alpha\theta}{s}\ {}_3F_2\left[1,1,s+1;2,s+2;\theta\right] \qquad (2.5)$$

Now

$$E\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)=\sum_{s=1}^{k}\frac{(-1)^{s+1}}{(k-s)!(s-1)!}\ E\left(\frac{1}{X+s}\right) \quad \text{(see Jones 1987)}$$

$$=\alpha\theta\sum_{s=1}^{k}\frac{(-1)^{s+1}\ s}{(k-s)!\ (s+1)!}\ {}_3F_2\left[1,1,s+1;s+2,2,\theta\right].$$

We get the result (2.2).

Theorem 2:

Support ${}_nF_m\left[(a);(b);\theta\right]$ be defined as in (2.1). Then for $\theta>0$.

$$\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}\ {}_1F_1(s;\ s+1;\ \theta)={}_2F_1(1,\ k;\ k+1;\ \theta),\ k=1,2,\ldots \qquad (2.6)$$

Proof:

Suppose X is a Poisson random variable with parameter θ.

$$f(x)=e^{-\theta}\frac{\theta^x}{x\ !},\quad x=0,\ 1,\ 2,\ \ldots,\quad \theta>0$$

Now by definition of expectations of random variables, we have

$$E\left(\prod_{s=1}^{k}\frac{1}{X+s}\right)=\sum_{x=0}^{\infty}e^{-\theta}\frac{\theta^x}{x!}\frac{1}{(x+1)(x+2)\cdots(x+k)}$$

$$=\frac{e^{-\theta}}{k\ !}\left[1+\frac{k}{(k+1)}\theta+\frac{1.2\ \ k(k+1)}{(k+1)(k+2)}\frac{\theta^2}{2}+\cdots\right]$$

$$=\frac{e^{-\theta}}{k!}\,{}_2F_1(1,\ k;\ k+1;\ \theta) \tag{2.7}$$

Alternatively,

$$E\left(\frac{1}{X+s}\right)=\sum_{x=0}^{\infty}\frac{1}{x+s}\frac{e^{-\theta}\theta^x}{x\ !}$$

$$=\frac{e^{-\theta}}{s}\left[1+\frac{s}{s+1}\theta+\frac{s(s+1)}{(s+1)(s+2)}\frac{\theta^2}{2!}+\cdots\right]$$

$$=\frac{e^{-\theta}}{s}\ {}_1F_1(s;\ \ s+1;\ \theta) \tag{2.8}$$

Thus

$$E\left(\prod_{s=1}^{k}\frac{1}{X+s}\right)=\sum_{s=1}^{k}\frac{(-1)^{s+1}}{(k-s)!\ \ (s-1)!}\ \frac{e^{-\theta}}{s}\ {}_1F_1(s;\ \ s+1;\ \theta),$$

Hence the result.

Theorem 3:

For $\theta\geq 0,\ s>0,\ 0<p<1,\ 0<q<1$ and $p+q=1$, the limit of ${}_2F_1\left(s,-n;s+1;-\frac{p}{q}\right)$ when $n\to\infty, p\to 0$ such that $np\to\theta$ is ${}_1F_1(s;\ \ s+1;\ \theta)$.

Proof:

Expanding ${}_2F_1(s,-n;s+1;p/q)$, we have

$${}_2F_1\left(s,-n;s+1;-\frac{p}{q}\right)=1+\frac{s(-n)}{s+1}\left(-\frac{p}{q}\right)+\frac{s(s+1)(-n)(-n+1)}{2!(s+1)(s+2)}\left(-\frac{p}{q}\right)^2+\cdots$$

$$+\frac{s(s+1)\cdots(s+n-1)(-n)(-n+1)\cdots(-1)}{(s+1)(s+2)\quad\cdots(s+n)\ n!}\left(-\frac{p}{q}\right)^n$$

$$= 1 + \frac{s}{s+1}\frac{np}{1-p} + \frac{s(s+1)\left(1-\frac{1}{n}\right)}{(s+1)\ (s+2)}\frac{(np)^2}{2!(1-p)^2} + \cdots$$

$$+ \frac{s(s+1)\ \cdots(s+n-1)}{(s+1)(s+2)\ \ \cdots(s+n)}\frac{1}{n!}\frac{(np)^n}{(1-p)^n}$$

As $n \to \infty$ and $p \to 0$ such that $np \to \theta$, we have

$${}_2F_1(s,-n;s+1;\ \theta) = 1 + \frac{s}{s+1}\theta + \frac{s(s+1)}{(s+1)(s+2)}\frac{\theta^2}{2!} + \cdots = {}_1F_1(s;\ s+1;\ \theta)$$

Theorem 4:

For $\theta > 0$, $s > 0$, $9 < p < 1$, $0 < q < 1$, $p + q = 1$, the limit of ${}_2F_1\left(1,-n;s+1;-\frac{p}{q}\right)$ when

$n \to \infty$, $p \to 0$ such that $np \to \theta$ is ${}_1F_1(1;\ s+1;\ \theta)$.

Proof:

Following Theorem 3, we get the result.

Corollary:

$${}_2F_1\left(1,-n;\ 2,\ -\frac{p}{q}\right) \to {}_1F_1(1;\ 2;\ \theta) \text{ as } n \to \infty, p \to 0 \text{ such that } np \to \theta.$$

Proof:

Put s = 1 in Theorem 4, we get the result.

REFERENCES

1. Ahmad, M. and Roohi, A. (2004). On sum of some hyper-geometric series functions-I *Pak. J. Statist.*, (20)1, 193-197.
2. Bardwell, G.E and Crow, E.L. (1964). A two parameter family of hyper Poisson distribution. *J. Amer. Statist. Assoc.*, 59, 133-141.
3. Jones M.C. (1987). Inverse factorial moments. *Statistics and Probability letters*, 6, 37-42.
4. Lepage, Y. (1978). Negative factorial moments of positive random variables. *Industrial Mathematics*, 28, 95-100.
5. Roohi, A. (2002). *Negative Moments of Non-negative random variables*. Unpublished M.Phil. dissertation submitted to the University of Lahore, Lahore. pp 135.

A RECURRENT RELATION OF GINI COEFFICIENT FOR ABDALLA AND HASSAN LORENZ CURVE USING A PROPERTY OF CONFLUENT GEOMETRIC SERIES FUNCTIONS*

Rubina Javed[1] and **Munir Ahmad**[2]
[1] Economics Department, University of the Punjab
Lahore, Pakistan. Email: rubkhd@hotmail.com
[2] National College of Business Administration & Economics
Lahore Pakistan. Email: drmunir@brain.net.pk

ABSTRACT

Abdalla and Hassan (2004) proposed a new parametric Lorenz curve and found Gini coefficient associated with their Lorenz Curve. In this note we have represented Gini coefficient for Abdalla and Hassan Lorenz Curve in terms of confluent hyper-geometric series funtion. Using a property of the hyper geometric function, a recurrent relation of Gini coefficient and Hassan Lorenz Curve has been derived.

1. INTRODUCTION

Abdalla and Hassan (2004) have proposed a new parametric Lorenz Curve in a functional form and computed Gini coefficient and fitted the function to UAE income data of Emiratisand Expatriates. Abdalla and Hassan propose the curve in a functional form defined as

$$L_N(p) = p^{\alpha}\left[1-(1-\rho)^{\delta} e^{\beta p}\right],\ \alpha \geq 0,\ 0 \leq \beta \leq \delta \leq 1, \tag{1}$$

and Gini coefficient is

$$Gini = \frac{\alpha - 1}{\alpha + 1} + 2\sum_{j=0}^{\infty} \frac{\Gamma(\alpha + j + 1)\,\Gamma(\delta+1)\,\beta^{j}}{\Gamma(\alpha+\delta+j+2)\,\Gamma(j+1)}. \tag{2}$$

2. GINI REPRESENTATION AND CONFLUENT HYPER-GEOMETRIC FUNCTION

We define confluent hyper-geometric series function as

$${}_1F_1[a;b;x] = 1 + \frac{ax}{b} + \frac{a(a+1)\ x}{b(b+1)\ 2!} + ... \tag{3}$$

for all $a, b \neq 0, -1, -2, \ \ldots$ and $x > 0$. If $a = 0$, $F_1[0;b;x] = 1$. If $a = -n$ an integer, ${}_1F_1[-n;b;x]$ is a terminating series, otherwise it is called non-terminating series. If $x = 0$, ${}_1F_1[a;b;0] = 1$. Properties of confluent hyper geometric series functions can be found in Rainville (1960) and Erdyli et al. (1953).

*Published in Pak. J. Statist. (2004), Vol. 20(3).

If we expand Gini Coefficient (2) as

$$G_\beta\left(\alpha\,\delta\right)=\frac{\alpha-1}{\alpha+1}+2\;\frac{\Gamma\left(\alpha+1\right)\,\Gamma\left(\delta+1\right)}{\Gamma\left(\alpha+\delta+2\right)}\left[1+\frac{\left(\alpha+1\right)}{\left(\alpha+\delta+2\right)}\beta\right.$$
$$\left.+\frac{\left(\alpha+1\right)\,\left(\alpha+2\right)}{\left(\alpha+\delta+2\right)\left(\alpha+\delta+3\right)}\frac{\beta^2}{2!}+....\right]$$

and represent the summation in terms of confluent hyper-geometric series functions, we have

$$G_\beta\left(\alpha,\delta\right)=\frac{\alpha-1}{\alpha+1}+2\,\beta\left(\alpha+1,\;\delta+1\right)\;{}_1F_1\left[\alpha+1;\;\alpha+\delta+2;\;\beta\right], \tag{4}$$

where $\beta\left(a,b\right)=\dfrac{\Gamma\left(a\right)\Gamma\left(b\right)}{\Gamma\left(a+b\right)}$ is a beta function.

If

$$\alpha=1,\;G_\beta\left(i,\delta\right)=2\beta\left(2,\delta+1\right)\;{}_1F_1\left[2;\delta+3;\beta\right]$$

and if

$$\alpha=0,\;G_\beta(\left(0,\delta\right)=-1+\frac{2}{\delta+1}\;{}_1F_1\left[1;\;\delta+2;\beta\right].$$

A recurrence relation for $G_\beta\left(\alpha,\;\delta\right)$ can be derived for computation purposes by using a relationship of confluent hyper-geometric series function:

$$a\;{}_lF_1\left[a+1;b;\theta\right]=\left(b-1\right)\;{}_lF_1\left[a;b-1;\theta\right]+\left(1+a-b\right)\;{}_lF_1\left[a;b;\theta\right] \tag{5}$$

if $a=\alpha,\;b=\alpha+\delta+2$ and $\theta=\beta$, then the equation (5) becomes

$$\alpha\;{}_lF_1\left[\alpha+1;\alpha+\delta+2;\beta\right]$$
$$=\left(\alpha+\delta+1\right){}_lF_1\left[\alpha;\alpha+\delta+1;\delta\right]-\left(\delta+1\right){}_lF_1\left[\alpha;\alpha+\delta+2;\beta\right]. \tag{6}$$

Using the relation (6) and after algebraic manipulation, we have

$$G_\beta\left(\alpha,\delta\right)=\frac{\alpha-1}{\alpha+1}+G_\beta\left(\alpha-1,\delta\right)G_\beta\left(\alpha-1,\delta+1\right),\;\;\alpha\geq1,\quad0\leq\beta\leq\delta\leq1. \tag{7}$$

The recurrence relation (7) seems very useful in computing $G_\beta\left(\alpha,\;\delta\right)$ for different value of α, δ and β.

REFERENCES

1. Abdalla, I.M. and Hassan, M.Y. (2004), Fitting Income Distribution Using a New Parametric Lorenz Curve. *Pak. J. Statist*., 20(2) 277-286.
2. Erdelyi, A., Maynas, W., Oberhettinger, F. and Tricomi, G.F. (1953). *Higher Transcendental Functions*. Bateman Project Vol. 1. New York, U.S.A.
3. Rainville, E.D. (1960). *Special Functions*. Chelsea Pub. Co., N.Y., USA.

A SHORT NOTE ON CONWAY-MAXWELL-HYPER POISSON DISTRIBUTION*

Munir Ahmad
National College of Business Administration & Economics
Lahore, Pakistan
Email: drmunir@brain.net.pk

ABSTRACT

Shmueli et al. (2005) have generalized Conway and Maxwell (1962) one-parameter Poisson distribution to a two parameter distribution called Conway-Maxwell-Poisson (CMP) distribution, discussed some of its properties and have fitted the CMP distribution to non-Poisson count data. In this paper, we have found a natural extension of two-parameter CMP to three-parameter distribution, which may be called Conway-Maxwell-Hyper-Poisson distribution (CMHP). We have discussed some additional moment properties. Using the property of proportionality of probabilities, we characterize CMHP and other special distributions.

KEY WORDS

Property of proportionately, Truncated Poisson, Truncated Hyper-Poisson.

1. INTRODUCTION

Hyper-Poisson distribution has been in literature for some time. It is a natural extension of the Poisson distribution providing information on super-Poisson, and sub-Poisson depending on one of its parameters [Staff (1964, 1967) and Bardwell and Crow (1964)]. It was further shown that truncated Poisson at an arbitrary point is a hyper Poisson and truncated hyper-Poisson is again a hyper-Poisson distribution. Now following Bardwell and Crow (1964), we find Conway-Maxwell Hyper-Poisson distribution by truncating CMP at the point θ;

$$P(Y=y)=\frac{1}{Z(\lambda,\nu,\theta)}.\frac{\lambda^{y}}{\left[\Gamma(y+\theta+1)!\right]^{\nu}},$$

$$y=0,1,2,\ldots,\ \ \theta\geq 0,\ \nu\geq 0\ \ \text{and}\ \ \lambda>0\,, \tag{1.1}$$

where $Z(\lambda,\nu,\theta) = \left(\sum_{i=0}^{\infty}\frac{\lambda^{i}}{\left[\Gamma(i+\theta+1)!\right]^{\nu}}\right)$. $Z(\lambda,\nu,\theta)$ converges for any $\lambda>0$ and $\nu>0$ except for $\nu>0$ and $\lambda\geq 1$. If $\theta=0$, $P(Y=y)$ is CMP distribution. $P(Y=y)$ is also true for any non-negative real values of θ. Further, it may be noted that a displaced or truncated CMP is CMHP and a displaced or truncated CMHP is again CMHP.

*Published in Pak. J. Statist. (2007), Vol. 23(2).

2. MOMENTS OF CMHP

The positive moments of CMHP at (1) following Shmueli et al. (2005) procedures, are:

$$E(Y+A)^{r+1} = \begin{cases} \lambda \;\; E(Y+A-1)^{1-\nu} + \dfrac{\theta \;\; \lambda^{\theta-1}}{Z(\lambda,\nu,\theta) \;\; (\theta!)^{\nu}}, & r=0 \\ \lambda \;\; \dfrac{d}{d\lambda} E(Y+A)^{r} + E(Y+A)^{r} \, E(Y+A), & r>0 \end{cases}$$

The rth negative moment of $Y+A,\ A>0$ of CMHP is

$$E(Y+A)^{-r} = \frac{1}{Z(\lambda,\nu,\theta)} \sum_{y=0}^{\infty} \frac{1}{(y+A)^{r}} \frac{\lambda^{y}}{\left[\Gamma(y+\theta+1)!\right]^{\nu}}$$

Following Shmueli et al. (2005) procedure, we have the relation

$$E(Y+A)^{-r} = \left[E(Y+A)^{1-r} - \lambda \frac{d}{d\lambda} \;\; E(Y+A)^{-r} \right] \left[E(Y+A) \right]^{-1}, \;\; r=1,2,.....$$

3. CHARACTERIZATION OF CMHP AND CMP

There has been a considerable work in literature regarding characterization of Poisson and hyper-Poisson distributions using conditional distribution, property of proportionately, etc. We also use a property of proportionately to characterize CMHP and CMP.

Theorem:

Y is a discrete CMHP random variable if and only if

$$\frac{P(Y=y)}{P(Y=y-1)} = \lambda / (y+\theta)^{\nu}, \;\; \nu,\ \theta \geq 0 \text{ and } \lambda > 0 \qquad (3.1)$$

Proof:

Suppose Y is a discrete non-negative integer-valued CMHP random variable. The relation (3.1) is trivial. Let $P_y = P(Y=y)$. If the proportion (3.1) holds, then $P_y = \dfrac{\lambda}{(y+\theta)^{\nu}} \;\; P_{y-1}$. For $y=1,\ 2,\ldots\ldots$ we have

$$P_y = \frac{\lambda^{y}}{\left[\theta(\theta+1)\ldots(\theta+y)\right]^{\nu}} P_0 .$$

Since $\sum P_y = 1$ then $P_0 = \left(\sum_{i=0}^{\infty} \frac{\lambda^i}{\left[\Gamma(i+\theta+1)!\right]^{\nu}} \right)^{-1}$.

On simplification, we get (1.1).

If $\theta = 0$, Y is a CMP random variable. The characterization theorem holds for CMP and under conditions stated by Shmueli et al. (2005), the characterization holds for ordinary Poisson, bernoulli and geometric distributions.

REFERENCES

1. Bardwell, G.E. and Crow, E.L. (1964). A two parameter family of Hyper-Poisson distributions. *J. Amer. Statist. Assoc.*, 59, 133-141.
2. Conway, R.W. and Maxwell, W.L. (1962). A Queuing Model with State Dependent Service Rates. *J. Industrl. Engg.*, 12, 132-136.
3. Shmueli, Galit, Minka, Thomas, P. Kadane, Joseph, B. Borle, Sharad and Boatwright, Peter (2005). A useful distribution for fitting discrete data: revival of the Conway-Maxwell-Poisson distribution. *J. Roy. Statist. Soc. Series C, Applied Statistics.* 54, Part 1, 127-142.
4. Staff, P.J. (1964). The displaced Poisson distribution. *Austral. J. Statist.*, 6, 12-20.
5. Staff, P.J. (1967). The displaced Poisson distribution-region 3. *J. Amer. Statist. Assoc.*, 62, 643-654.

BAYESIAN ANALYSIS AND RELIABILITY FUNCTION OF DECAPITATED GENERALIZED POISSON DISTRIBUTION*

Anwar Hassan[1], **Khurshid Ahmad Mir**[2] and **Munir Ahmad**[3]

[1] PG. Department of Statistics, University of Kashmir Srinagar, J&K (India) Email: anwar.hassan2007@gmail.com

[2] Department of Statistics, Amar Singh College, Srinagar, J&K (India), Email: khrshdmir@yahoo.com

[3] National College of Business Administration & Economics Lahore, Pakistan, Email: drmunir@brain.net.pk

ABSTRACT

In this paper Bayes estimators and reliability function of one parameter of decapitated generalized Poisson distribution (DGPD) is derived by considering a non-informative prior when the other parameter is assumed to be known. Furthermore, recurrence relations for the estimator of the parameters are obtained. A comparison has also been made for the Bayes estimator and reliability function of DGPD with the corresponding maximum likelihood estimator (MLE) by using Monte Carlo simulation technique and R-Software.

KEYWORDS

Decapitated generalized Poisson distribution, reliability function, Bayes estimator, recurrence relation. Monte Carlo simulation, maximum likelihood estimation, R-Software.

1. INTRODUCTION

Consul and Jain (1973) defined the generalized Poisson distribution (GPD) as

$$P_1(X=x)=\frac{\lambda(\lambda+x\lambda_1)^{x-1}e^{-(\lambda+x\lambda_1)}}{x!},\ x=0,1,2,\ldots,l>0,|\lambda_1|<1 \tag{1.1}$$

The distribution (1.1) reduces to Poisson distribution at $\lambda_1=0$. Consul and Shoukri (1985, 1986) studied GPD when the sample mean is larger than the sample variance and for negative integer moments. Tuenter (2000) also discussed the GPD. The model (1.1) has been found to be a member of the Consul and Shenton's (1972, 1973) family of Lagrangian distributions and also of the Gupta's (1974) modified power series distribution (MPSD).

The problem of estimation for GPD has been discussed by many authors. Whereas Famoye and Lee (1992), Consul and Famoye (1988, 1989) and Consul and Shoukri (1984) studied the maximum likelihood estimation. Kumar and Consul (1980) and Gupta (1977) discussed the minimum variance unbiased estimation. Shoukri and Consul (1989) and Hassan and Harman (2003) also studied the Bayes estimator for GPD under different priors.

*Published in Pak. J. Statist. (2007), Vol. 23(3).

1.1 Zero-Truncated Generalized Poisson Distribution (ZTGPD):

Shoukri and Consul (1989) redefined the distribution (1.1) as

$$P_2(X=x)=\frac{(1+\beta x)^{x-1}\alpha^x e^{-\alpha(1+\beta x)}}{x!},\ x=0,1,2,\cdots,\ \alpha>0,\ 0<\beta<\frac{1}{\alpha} \qquad (1.2)$$

where $\lambda=\alpha$ and $\lambda_1=\alpha\beta$ as

The distribution (1.2) can be truncated at $x=0$ and is defined as

$$P_3\left(X=x\right)=\frac{\left(1+\beta x\right)^{x-1}}{x!}\alpha^x e^{-\alpha(1+\beta x)}\left(1-e^{-\alpha}\right)^{-1},\ x=1,2,\cdots,\ \alpha>0,0<\beta<\frac{1}{\alpha} \qquad (1.3)$$

It can be easily seen that at $\beta=0$, the distribution (1.2) and (1.3) reduce to Poisson distribution and to David and Johnson's (1952) truncated Poisson distribution. Consul and Famoye (1989) defined the truncated generalized Poisson distribution (TGPD) and obtained its maximum likelihood estimation of the parameters. They also obtained estimate based upon the mean and ratio of the first two frequencies. Jani and Shah (1981) also studied the truncated generalized Poisson distribution. A brief list of authors and their works can be seen in Consul (1989) and Johnson, Kotz and Kemp (1992) and Consul and Famoye (2006).

In this paper we have made an attempt to obtain Bayes estimator and reliability function of dicapitated generalized Poisson distribution (DGPD) for one parameter α when other parameter β is assumed to be known. Further more, recurrence relations for the estimators of the parameter are also obtained. Monte Carlo simulation and R-Software were performed and a comparison has been made of the Bayes estimator and reliability function of (1.3) with the corresponding maximum likelihood estimator (MLE).

2. BAYESIAN ANALYSIS OF DECAPITATED GENERALIZED POISSON DISTRIBUTION

Let $X_1, X_2, ..., X_n$ be a random sample from (1.3). The likelihood function is

$$L=\prod_{i=1}^{n}\left[\frac{(1+\beta x_i)^{x_i-1}}{x_i!}\right]\alpha^{\sum_{i=1}^{n}x_i}\ e^{-\left(n+\beta\sum_{i=1}^{n}x_i\right)\alpha}\ (1-e^{-\alpha})^{-n}=K\alpha^y e^{-(n+\beta y)\alpha}(1-e^{-\alpha})^{-n} \qquad (2.1)$$

where $K=\prod_{i=1}^{n}\left[\frac{(1+\beta x_i)^{x_i-1}}{x_i!}\right]$ and $y=\sum_{i=1}^{n}x_i$

when β is known, the part of the likelihood function which is relevant to Bayesian inference on the unknown parameter α is $\alpha^y e^{-(n+\beta y)\alpha}(1-e^{-\alpha})^{-n}$ we assume that before the observations were made, our knowledge about α was only vague. Consequently, the non-informative vague prior of α, $g(\alpha)$ proportional to $\frac{1}{\alpha}$ is applicable to a good approximation. Thus

$$g(\alpha)=\frac{1}{\alpha},\quad \alpha>0 \tag{2.2}$$

The posterior distribution α of from (2.1) and (2.2) is

$$\prod(\alpha\,|\,y)\;=\frac{\alpha^{y-1}e^{-(n+\beta y)\alpha}(1-e^{-\alpha})^{-n}}{\int\limits_0^\infty \alpha^{y-1}e^{-(n+\beta y)\alpha}(1-e^{-\alpha})^{-n}\,d\alpha.} \tag{2.3}$$

Under square error loss function, the Bayes estimator of α is the posterior mean given as

$$p^*(y,n)\;=\frac{\int\limits_0^\infty \alpha^{y}e^{-(n+\beta y)\alpha}(1-e^{-\alpha})^{-n}\,d\alpha}{\int\limits_0^\infty \alpha^{y-1}e^{-(n+\beta y)\alpha}(1-e^{-\alpha})^{-n}\,d\alpha.} \tag{2.4}$$

Using identity $(1-\nu)^{-n}=\sum\limits_{k=0}^{\infty}\binom{n+k-1}{k}\nu^{k},|\,\nu\,|<1$ and the relation $\int\limits_0^\infty e^{-at}t^{b-1}dt=\frac{\Gamma b}{a^b},\ a,b>0,\ t>0$, we obtain

$$\int\limits_0^\infty \alpha^{y}e^{-(n+\beta y)\alpha}(1-e^{-\alpha})^{-n}\,d\alpha\;=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{\Gamma(y+1)}{(n+\beta y+k)^{y+1}} \tag{2.5}$$

$$=\Gamma(y+1)\sum_{k=n}^{\infty}\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y+1}} \tag{2.6}$$

and similarly,

$$\int\limits_0^\infty \alpha^{y-1}e^{-(n+\beta y)\alpha}(1-e^{-\alpha})^{-n}\,d\alpha\;=\Gamma(y)\sum_{k=n}^{\infty}\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y}} \tag{2.7}$$

substituting the value of (2.6) and (2.7) in (2.4), we obtain

$$p^*(y,n)\;=\frac{y\sum\limits_{k=n}^{\infty}\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y+1}}}{\sum\limits_{k=n}^{\infty}\binom{k-1}{n-1}\frac{(\beta y+k)}{(\beta y+k)^{y+1}}}$$

$$=\frac{y\sum\limits_{k=n}^{\infty}\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y+1}}}{\sum\limits_{k=n}^{\infty}k\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y+1}}+\beta y\sum\limits_{k=n}^{\infty}\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y+1}}} \tag{2.8}$$

Using the relation $\Gamma(b+1) = b\Gamma(b)$ and

$$k\binom{k-1}{n-1} = n\binom{k-1}{n} + n\binom{k-1}{n-1} \tag{2.9}$$

We obtain

$$p^*(y,n) = \frac{y\sum_{k=n}^{\infty}\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y+1}}}{n\sum_{k=n+1}^{\infty}\binom{k-1}{n}\frac{1}{(\beta y+k)^{y+1}} + (n+\beta y)\sum_{k=n}^{\infty}\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y+1}}} \tag{2.10}$$

$$= \frac{yZ(y,n)}{nZ(y,n+1)+(n+\beta y)Z(y,n)} \;;\quad y = n,\ n+1 \cdots,\ n = 1, 2\cdots \tag{2.11}$$

where

$$Z(y,n) = \sum_{k=n}^{\infty}\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y+1}} \tag{2.12}$$

$$Z(y,n+1) = \sum_{k=n+1}^{\infty}\binom{k-1}{n}\frac{1}{(\beta y+k)^{y+1}} \tag{2.13}$$

After simplification (2.11) becomes

$$p^*(y,n) = \frac{y}{\{(n+\beta y)+nS(y,n)\}} \;;\ y = n, n+1 \cdots, n = 1, 2 \cdots \tag{2.14}$$

where

$$S(y,n) = \frac{Z(y,n+1)}{Z(y,n)} \tag{2.15}$$

3. RECURRENCE RELATIONS

In order to obtain a recurrence relation for $p^*(y,n)$, first we need recurrence relations for the numbers $Z(y,n)$ and $S(y,n)$, which are obtained by the following two lemmas:

Lemma 1*:* The number $Z(y,n)$ satisfy the recurrence relations:

$$Z(y,n+1) = \frac{1}{n}Z(y-1,n) - \frac{(n+\beta y)}{n}Z(y,n), \quad y = n,\ n+1 \ldots,\ n = 1,2,\ldots \tag{3.1}$$

with initial condition

$$Z(y,1) = \sum_{k=1}^{\infty}\frac{1}{(\beta y+k)^{y+1}} \quad ;\ y = 1,\ 2,\ \cdots \tag{3.2}$$

Proof: From the relation (2.12), we have

$$Z(y-1,n) = \sum_{k=n}^{\infty}\binom{k-1}{n-1}\frac{(\beta y+k)}{(\beta y+k)^{y+1}} \tag{3.3}$$

using the relation (2.9), (3.3) becomes

$$Z(y-1,n) = n\sum_{k=n+1}^{\infty}\binom{k-1}{n}\frac{1}{(\beta y+k)^{y+1}} \quad +(\text{n}+\beta \text{y})\sum_{k=n}^{\infty}\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y+1}}$$

From (2.12) and (2.13) we have,

$$Z(y-1,n) = nZ(y,n+1)+(n+\beta y)Z(y,n) \tag{3.4}$$

from which, we have (3.1). Also from (2.12) for n = 1, we can easily obtain the relation (3.2)

Remark: Combining equations (2.11), (2.12) and (2.13) we get

$$S(y,n) = \frac{\frac{1}{n}Z(y-1,n)-\frac{1}{n}(n+\beta y)Z(y,n)}{Z(y,n)} = \left[\frac{Z(y-1,n)}{Z(y,n)}-(n+\beta y)\right]\Big/ n \tag{3.5}$$

and

$$P^{*}(y,n) = \frac{y.Z(y,n)}{\left[(n+\beta y)Z(y,n)+n\left\{\frac{1}{n}Z(y-1,n)-\frac{1}{n}(n+\beta y)Z(y,n)\right.\right]} = \frac{y\ Z(y,n)}{Z(y-1,n)} \tag{3.6}$$

Lemma 2: The number $S(y,n)$ satisfy the recurrence relation:

$$S(y,n+1) = \frac{\left[nS(y,n)+(n+\beta y)\right]S(y-1,n)}{(n+1)S(y,n)} - \frac{\left[(n+1)+\beta y\right]}{(n+1)} \tag{3.7}$$

where $n=1,2,\ldots,\ y=n,n+1$ with initial condition

$$S(y,1) = \frac{Z(y-1,1)}{Z(y,1)}-(1+\beta y) \tag{3.8}$$

Proof: From the relation (2.15) and the recurrence relation (3.1), we get

$$S(y,n+1) = \frac{Z(y,n+2)}{Z(y,n+1)}$$

$$= \frac{\frac{1}{(n+1)}Z(y-1,n+1)-\frac{1}{(n+1)}(n+1+\beta y)Z(y,n+1)}{Z(y,n+1)}$$

$$= \frac{\frac{1}{(n+1)}\left[\frac{Z(y-1,n+1)}{Z(y,n)}-\left\{(n+1)+\beta y\right\}S(y,n)\right]}{S(y,n)} \tag{3.9}$$

We have $S(y-1,n)=\frac{Z(y-1,n+1)}{Z(y-1,n)}=\frac{Z(y-1,n+1)/Z(y,n)}{\left[nZ(y,n+1)/Z(y,n)\right]+(n+\beta y)}$

From (3.4) we have

$$S(y-1,n)=\frac{Z(y-1,n+1)/Z(y,n)}{nS(y,n)+(n+\beta y)} \tag{3.10}$$

or

$$\frac{Z(y-1,n+1)}{Z(y,n)}=\left[nS(y,n)+(n+\beta y)\right]S(y-1,n) \tag{3.11}$$

Substituting (3.10) into (3.9) we obtain (3.7). Also from (3.5) for $n=1$, we easily get (3.8).

Theorem 1: The Bayes estimator of the parameter α satisfy the recurrence relation:

$$p^*(y,n+1)=\frac{\left[y-(n+\beta y)P(y,n)\right]P(y-1,n)}{(y-1)-(n+\beta y)P(y-1,n)}\ , \tag{3.12}$$

with initial conditions

$$p^*(y,1)=\frac{y\ Z(y,1)}{Z(y-1,\ 1)} \tag{3.13}$$

Proof: From the relation (2.14), we have

$$p^*(y,n+1)=\frac{y}{(n+1+\beta y)+(n+1)S(y,n+1)} \tag{3.14}$$

substituting (3.7) into (3.14) and using (2.14) we get (3.1) after some algebraic manipulation. Also from the relation (3.6) for $n=1$, we easily get (3.13).

4. BAYES ESTIMATOR OF THE RELIABILITY FUNCTION OF DECAPITATED GENERALIZED POISSON DISTRIBUTION

The Bayes estimator $R^*(t)$, for $R(t)=P(X>t)$ where variable X has the distribution (1.4) in given by

$$R^*(t)=E\left[R(t)/x_1\cdots x_n\right]\ =\ \frac{\int_0^\infty R(t)\alpha^{y-1}e^{-(n+\beta y)\alpha}(1-e^{-\alpha})^{-n}\,d\alpha}{\int \alpha^{y-1}e^{-(n+\beta y)\alpha}(1-e^{-\alpha})^{-n}\,d\alpha} \tag{4.1}$$

where

$$R(t)=\sum_{x=[t]+1}^{\infty}\frac{(1+\beta x)^{x-1}\alpha^x e^{-\alpha(1+\beta x)}(1-e^{-\alpha})^{-1}}{x!}\ , \tag{4.2}$$

and $[t]$ is the integer part of t. Making similar computations, as for $P(y,n)$ we obtain

$$R^*(t)=\frac{\sum\limits_{x=[t]+1}^{\infty}\frac{(1+\beta x)^{x-1}}{x!}\int\limits_0^{\infty}e^{-(n+\beta y+1+\beta x)\alpha}\alpha^{y+x-1}(1-e^{-\alpha})^{-(n+1)}d\alpha}{\int\limits_0^{\infty}\alpha^{y-1}e^{-(n+\beta y)\alpha}(1-e^{-\alpha})^{-n}d\alpha} \tag{4.3}$$

Using same identity, we obtain

$$\int\limits_0^{\infty}e^{-(n+\beta y+1+\beta x)\alpha}\alpha^{(y+x-1)}(1-e^{-\alpha})^{-(n+1)}d\alpha=\Gamma(y+x)\sum_{k=n+1}^{\infty}\binom{k-1}{n}\frac{1}{(\beta y+\beta x+k)^{y+x}} \tag{4.4}$$

Similarly

$$\int\limits_0^{\infty}\alpha^{y-1}e^{-(n+\beta y)\alpha}(1-e^{-\alpha})^{-n}d\alpha=\Gamma(y)\sum_{k=n}^{\infty}\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y}} \tag{4.5}$$

Using (4.4) and (4.5) in (4.3), we obtain

$$R^*(t)=\frac{\sum\limits_{k=[t]+1}^{\infty}\frac{\Gamma(y+x)(1+\beta x)^{x-1}}{x!\Gamma(y)}\sum\limits_{k=n+1}^{\infty}\binom{k-1}{n}\frac{1}{(\beta y+\beta x+k)^{y+x}}}{\sum\limits_{k=n}^{\infty}\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y}}} \tag{4.6}$$

and using the relation (2.9) , (4.6) becomes

$$R^*(t)=\sum_{k=[t]+1}^{\infty}\frac{(1+\beta x)^{x-1}}{x!}\frac{\Gamma(y+x)}{\Gamma(y)}$$

$$\left[\frac{(n+1)\sum\limits_{k=n+2}^{\infty}\binom{k-1}{n+1}Q+(n+1+\beta y+\beta x)\sum\limits_{k=n+1}^{\infty}\binom{k-1}{n}Q}{n\sum\limits_{k=n+1}^{\infty}\binom{k-1}{n}R+(n+\beta y)\sum\limits_{k=n}^{\infty}\binom{k-1}{n-1}R}\right]$$

where $Q=\frac{1}{(\beta y+\beta x+k)^{y+x+1}}$ and $R=\frac{1}{(\beta y+k)^{y+1}}$

The Bayes reliability estimator $R^*(t)$ is

$$R^*(t)=\sum_{k=[t]+1}^{\infty}\frac{\Gamma(y+x)(1+\beta x)^{x-1}}{x!\Gamma(y)}\left[\frac{(n+)Z(y+x.n+)+(n++\beta y+\beta x)Z(y+x.n+1)}{nZ(y,n+1)+(n+\beta y)Z(y,n)}\right] \tag{4.7}$$

where $Z(y,n)=\sum\limits_{k=n}^{\infty}\binom{k-1}{n-1}\frac{1}{(\beta y+k)^{y+1}}$

5. COMPUTER SIMULATION AND CONCLUSIONS

In order to compare the estimation, Monte Carlo Simulations and r-software were performed on 1000 samples for each simulation. The following steps summarize the simulation. 1. A value is generated from a non-informative prior. 2. Based on the realization from the non-informative prior a sample of size n=8 or 30 is generated from the decapitated generalized Poisson distribution. 3. The estimates of the parameters and reliability function are computed from the generated sample, and the estimates and their squared error losses were stored 4. Steps 1-3 were repeated 1000 times. 5. Average values and root mean square errors (RMSE's) of the estimates over the 1000 samples.

Tables 1-4 show some of the results. In comparing the estimators the root mean square error criterion will be used, namely the estimator with the smallest RMSE's is the best estimator. The reliability function was evaluated arbitrarily at times 1, 2 and 3. Two sample size of n=8, 30 were utilized in the simulation.

Table 1

Average values and RMSE's for the estimators of the Decapitated Generalized Poisson. Non-informative prior with sample size n = 8 and $\beta = 0.3$

Parameter						
True value		**Bayes**		**MLE**		**RMSE Ratio**
		Ave.	RMSE	Ave.	RMSE	MLE/Bayes
1.8904		1.8912	1.6665	1.8792	1.6708	1.0026
Reliability						
Time	**Exact value**	**Bayes**		**MLE**		**RMSE Ratio**
		Ave.	RMSE	Ave.	RMSE	MLE/Bayes
1	2.4034	2.3915	1.7026	2.3771	1.7142	1.0068
2	2.1588	2.1526	1.7305	2.1403	1.7396	1.0052
3	1.9387	1.9435	1.7244	1.9321	1.7324	1.0046

Table 2

Average values and RMSE's for the estimators of the Decapitated Generalized Poisson. Non-informative prior with sample size n=30 and $\beta = 0.3$

Parameter						
True value		**Bayes**		**MLE**		**RMSE Ratio**
		Ave.	RMSE	Ave.	RMSE	MLE/Bayes
8.2200		8.1870	7.9306	8.1840	0.9311	1.0001
Reliability						
Time	**Exact value**	**Bayes**		**MLE**		**RMSE Ratio**
		Ave.	RMSE	Ave.	RMSE	MLE/Bayes
1	8.7363	8.6992	7.9502	8.6961	7.9520	1.0002
2	8.5006	8.4641	7.9642	8.4617	7.9655	1.0002
3	8.2798	8.2466	7.9619	8.2438	7.9630	1.0001

Table 3
Average values and RMSE's for the estimators of the Decapitated Generalized Poisson. Non-informative prior with sample size n=8 and $\beta = 0.5$

Parameter						
True value		**Bayes**		**MLE**		**RMSE Ratio**
		Ave.	RMSE	Ave.	RMSE	MLE/Bayes
2.8832		2.8776	2.6190	2.6785	2.8201	1.0767
Reliability						
Time	**Exact value**	**Bayes**		**MLE**		**RMSE Ratio**
		Ave.	RMSE	Ave.	RMSE	MLE/Bayes
1	3.1816	3.1724	2.9230	3.0095	3.1264	1.0695
2	3.0879	3.0769	2.3392	3.0779	2.3908	1.0220
3	2.978	2.9681	2.3523	2.9699	2.3842	1.0135

Table 4
Average values and RMSE's for the estimators of the Decapitated Generalized Poisson. Non-informative prior sample size n=30 and $\beta = 0.5$

Parameter						
True value		**Bayes**		**MLE**		**RMSE Ratio**
		Ave.	RMSE	Ave.	RMSE	MLE/Bayes
17.4660		17.4720	16.9077	17.4690	16.9080	1.0000
Reliability						
Time	**Exact value**	**Bayes**		**MLE**		**RMSE Ratio**
		Ave.	RMSE	Ave.	RMSE	MLE/Bayes
1	17.8595	17.8649	16.8902	17.8619	16.8905	1.0000
2	17.8232	17.8275	16.9021	17.824	16.9024	1.0000
3	17.7492	17.7526	16.9181	17.7500	16.9186	1.0000

In comparing the estimators, the Bayes ones have the smallest RMSE and are better. This is to be expected since the Bayes estimators take advantage of the known prior parameter α. By examining the RMSE ratios we can conclude that the estimates are sensitive to the choice of prior parameters and to sample size.

ACKNOWLEDGMENT

The authors are thankful to the referee and the editor for their valuable suggestions

REFERENCES

1. Consul, P.C. (1989). *Generalized Poisson distribution, Properties and Applications*, John Wiley, New York.
2. Consul, P.C. and Famoye, F. (1988). Maximum likelihood estimation for the generalized Poisson distribution when the sample mean is larger than the sample variance. *Communication in Statistics-Theory and Methods*, 17, 299-309.

3. Consul, P.C. and Famoye, F. (1989). The truncated generalized Poisson distribution and its applications. *Communication in Statistics-Theory and Methods*, 18(10), 3635-3648.
4. Consul, P.C. and Famoye, F. (2006). *Lagrangian Probability Distributions*. Birkhäuser.
5. Consul, P.C. and Jain G.C. (1973). A generalization of Poisson distribution. *Technometrics*, 15(4), 791-799.
6. Consul, P.C. and Shenton, L.R. (1972). Use of Lagrangian expansion for generating generalized probability distributions. *SIAM Journal of Applied. Mathematics*, 23(2), 239-248.
7. Consul, P.C. and Shenton, L.R. (1973). Some interesting properties of Lagrangian distributions. *Communication in Statistics-Theory and Methods*, 2(3), 263-272.
8. Consul, P.C. and Shoukri, M.M. (1984). Maximum likelihood estimation for the generalized Poisson distribution. *Communication in Statistics-Theory and Methods*, 13(12), 1533-1547.
9. Consul, P.C. and Shoukri, M.M. (1985). The generalized Poisson distribution when the sample mean is larger than the sample variance. *Communication in Statistics-Theory and Methods*, 14, 667-681.
10. Consul, P.C. and Shoukri, M.M. (1986). The negative integer moments of the generalized Poisson distribution. *Communication in Statistics-Simulation and Computation*, 14(4), 1053-1064.
11. David, F.N. and Johnson, N.L. (1952). The truncated Poisson, *Biometrics*, 8, 275-285.
12. Famoye, F. and Lee, C.M.S. (1992). Estimation of generalized Poisson distribution. *Communication in Statistics-Simulation and Computation*, 21(1), 173-188.
13. Gupta, R.C. (1974). Modified power series distribution and its applications. *Sankhya*, Ser. B, 35(3), 288-298.
14. Gupta, R.C. (1977). Minimum variance unbiased estimation in a modified power series distribution and some of its applications. *Communication in Statistics-Theory and Methods*, 6(10), 977-991.
15. Hassan, A. and Herman, S. (2003). New Bayesian estimation of generalized Poisson distribution. *Szigma*, XXXIV (3-4), 127-139.
16. Jani, P.N. and Shah, S.M. (1981). The truncated generalized Poisson distribution. *Aligarh Journal of Statistics*, 1(2), 174-182.
17. Johnson, N.L, Kotz, S. and Kemp. A.W. (1992). *Univariate Discrete Distribution*, John Willy and Sons, New York.
18. Kumar, A. and Consul, P.C. (1980). Minimum variance unbiased estimation for modified power series distribution. *Communication in Statistics, Theory and Methods*, 9(12), 261-1275.
19. Shoukri, M.M. and Consul, P.C. (1989). Bayesian analysis of a generalized Poisson distribution. *Communication in Statistics, Simulation and Computation*, 18(4), 1465-1480.
20. Tuenter, H.J.H. (2000). On the generalized of Poisson distribution. *Statistica Neerlandica*, 54, 374-376.

ON BAYESIAN ANALYSIS OF GENERALIZED GEOMETRIC SERIES DISTRIBUTION UNDER DIFFERENT PRIORS *

Anwar Hassan[1], **Khurshid Ahmad Mir**[2] and **Munir Ahmad**[3]

[1] PG. Department of Statistics, University of Kashmir, Srinagar, J&K (India) Email: anwar.hassan2007@gmail.com

[2] Department of Statistics, Amar Singh College, Srinagar, J&K (India), Email: khrshdmir@yahoo.com

[3] National College of Business Administration & Economics, Lahore, Pakistan, Email: drmunir@brain.net.pk

ABSTRACT

In this paper a Bayes estimator of generalized geometric series distribution (GGSD) under different priors and its characterization has been introduced. Comparisons are made of the Bayes estimate of $P(X=k)$ to the corresponding maximum likelihood (ML) estimate for any given sample for different values of k with the help of Monte Carlo simulation.

KEY WORDS

Squared error loss function, Bayes estimator, Beta distribution, Monte Carlo simulation.

1. INTRODUCTION

The probability function of generalized geometric series distribution (GGSD) was given by Mishra (1982) by using the results of the lattice path analysis as

$$P(X=x)=\frac{1}{1+\beta x}\binom{1+\beta x}{x}\theta^{x}(1-\theta)^{1+\beta x-x};\ x=0,1,2,...\ 0<\theta<1 \text{ and } |\theta\beta|<1 \quad (1.1)$$

It can be seen that at $\beta=1$, the model (1.1) reduces to simple geometric distribution and is a particular case of Jain and Consul's (1971) generalized negative binomial distribution in the same way as the geometric distribution is a particular case of the negative binomial distribution.

The various interesting properties and estimation of (1.1) have been discussed by Mishra (1982), Singh (1989), Mishra and Singh (1992), Hassan (1995) and Hassan et al. (2002, 2003). They found this distribution to provide much closer fits to all those observed distributions where the geometric distribution and the various compound geometric distributions have been fitted earlier by many authors.

In this paper we have characterized the distribution (1.1) and discussed the Bayesian estimation of generalized geometric series distribution (GGSD) under different priors and a comparison is made of the Bayes estimate of P(X=k) to the corresponding maximum likelihood (ML) estimate for any given sample for different values of k with the help of Monte Carlo simulation.

*Published in Pak. J. Statist. (2007), Vol. 23(3).

2. CHARACTERIZATION OF GGSD

Theorem: Suppose X has the prob. mass function (1.1) if and only if

$$P(X=x)=\alpha_x P(X=x-1), \text{ where } \alpha_x=\frac{\Gamma(1+\beta x)\Gamma(\beta x-\beta-x+3)\alpha(1-\alpha)^{\beta-1}}{x!\,\Gamma(1+\beta x-\beta)\Gamma(\beta x-x-2)}.$$

Proof:

$$P(X=x-1)=\frac{\Gamma(1+\beta(x-1))\theta^{x-1}(1-\theta)^{1+\beta(x-1)-(x-1)}}{(x-1)!\,\Gamma(\beta(x-1)-(x-1)+2)}$$

Let $x=1,2,3,......$ then

$$P_1=\alpha(1-\theta)^{\beta-1}P_0$$

$$P_2=\frac{2\beta\theta^2(1-\theta)^{2(\beta-1)}}{2}P_0$$

$$P_3=\frac{1}{3.2}\frac{\Gamma(3\beta+1)}{\Gamma(3\beta-1)}\theta^3(1-\theta)^{3(\beta-1)}P_0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$P_x=\frac{2\beta\Gamma(3\beta+1)\Gamma(4\beta+1)\Gamma(5\beta+1)......\Gamma(\beta x+1)\theta^x(1-\theta)^{x(\beta-1)}}{\Gamma(3\beta+1)\Gamma(4\beta+2)\Gamma(5\beta+3)......\Gamma((\beta-1)x+2)\,x!}P_0 \tag{2.1}$$

$$\sum P_x=1=\sum_{x=0}^{\infty}\frac{\Gamma(\beta x+1)^{qx}(1-\theta)^{x(\beta-1)}}{\Gamma((\beta-1)x+2)\,x!}P_0 \tag{2.2}$$

Putting P_0 in (2.1) and after some algebraic, we get (1.1).

3. BAYESIAN ESTIMATION OF PARAMETER θ OF GGSD UNDER DIFFERENT PRIORS

The Bayesian estimation of parameter of GGSD does not seem to appear in literature so far.

The likelihood function from (1.1) is obtained as

$$L(x/\theta,\beta)=\prod_{i=1}^{n}\left\{\frac{1}{1+\beta x_i}\binom{1+\beta x_i}{\text{x}_\text{i}}\right\}\theta^{\Sigma x_i}(1-\theta)^{n+\beta\Sigma x_i-\Sigma x_i}=k\,\theta^y(1-\theta)^{n+\beta y-y} \tag{3.1}$$

where $\text{k}=\prod_{i=1}^{n}\frac{1}{1+\beta x_i}\binom{1+\beta x_i}{\text{x}_\text{i}}$ and $y=\sum_{i=1}^{n}x_i$.

When β is known, the part of the likelihood function which is relevant to Bayesian inference on the unknown parameter θ is $\theta^{y}\left(1\text{-}\theta\right)^{n+\beta y-y}$.

3.1 Bayesian Estimation of Parameter θ of GGSD under Non- Informative Prior

We assume prior θ of as

$$g\left(\theta\right)=\frac{1}{\theta},\ 0<\theta<1 \tag{3.2}$$

The posterior distribution of θ from (3.1) and (3.2) is

$$\prod\left(\theta/y\right)=\frac{L(\underline{x}/\theta,\beta)g(\theta)}{\int_{\Omega}L(\underline{x}/\theta,\beta)g(\theta)d\theta}=\frac{\theta^{y-1}\left(1-\theta\right)^{\mathrm{n}+(\beta\text{-}1)y}}{B\left(y,\mathrm{n}+\beta y-y+1\right)},\ 0<\theta<1, y>0. \tag{3.3}$$

The Bayes estimator of parametric function $\varphi\left(\theta\right)$ under squared error loss function is the posterior mean which is given as

$$\hat{\phi}\left(\theta\right)=\int_{0}^{1}\frac{\phi\left(\theta\right)\theta^{y-1}\left(1-\theta\right)^{n+(\beta-1)y}\,d\theta}{B\left(y,n+\beta y-y+1\right)}$$

If we take $\varphi\left(\theta\right)=\theta$, the Bayes estimate of θ is given by

$$\hat{\theta}=\int_{0}^{1}\frac{\theta^{y}\left(1-\theta\right)^{n+(\beta-1)y}\,d\theta}{B\left(y,n+\beta y-y+1\right)}=\frac{B\left(y+1,\mathrm{n}+\beta y-y+1\right)}{B\left(y,\mathrm{n}+\beta\mathrm{y}\text{-}\mathrm{y}+1\right)}=\frac{y}{n+\beta y} \tag{3.4}$$

This coincides with the moment and ML estimate of θ.

3.2 Bayesian Estimation of Parameter θ of GGSD under Beta Prior

The more general Bayes estimator of θ can be obtained by assuming the beta distribution as prior information of θ. Thus

$$g(\theta;a,b)=\frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)},\qquad a,b>0,\ 0<\theta<1. \tag{3.5}$$

The posterior distribution of θ is defined as

$$\pi\left(\theta/y\right)=\frac{\theta^{a+y-1}(1-\theta)^{n+(\beta-1)y+b-1}}{\int_{0}^{1}\theta^{y+a-1}(1-\theta)^{n+(\beta-1)y+b-1}d\theta}=\frac{\theta^{a+y-1}(1-\theta)^{n+(\beta-1)y+b-1}}{B\left(y+a,n+\beta y-y+b\right)} \tag{3.6}$$

The Bayes estimator of parametric function ϕ(θ) under squared error loss function is the posterior mean and is given as

$$\overset{*}{\theta}=\int_0^1 \frac{\phi(\theta)\theta^{a+y-1}(1-\theta)^{n+(\beta-1)y+b-1}d\theta}{B(y+a,n+\beta y-y+b)} \tag{3.7}$$

If we take $\varphi(\theta)=\theta$ then Bayes estimator of θ is given as

$$\overset{*}{\theta}=\int_0^1 \frac{\theta^{a+y}(1-\theta)^{n+(\beta-1)y+b-1}d\theta}{B(y+a,n+\beta y-y+b)}=\frac{a+y}{n+a+b+\beta y} \tag{3.8}$$

If a = b = 0, (3.8) coincides with (3.4).

Similarly, the Bayes estimator based on beta prior of some parametric functions are listed in Table 3.1

Table 3.1

$\phi(\theta)$	**Bayes Estimate** $\overset{*}{\theta}$
$\theta^1(1-\theta)^k$	$B(y+a+l,\beta y-y+n+b+k)/B(y+a,\beta y-y+n+b)$
$\left\{\theta(1-\theta)^{\beta-1}\right\}^k$	$B(y+k+a,\beta y+\beta k-y+n+b-k)/B(y+a,\beta y-y+n+b)$
$P(X=k)$	$\frac{1}{1+\beta k}\binom{1+\beta k}{k}$ $B(y+k+a,\beta y+\beta k-y+n+b-k+1)/B(y+a,\beta y-y+n+b)$
$\theta^{(l)}$	$(y+a)_{(l)}/(\beta y+n+a+b)_{(l)}$
$(1-\theta)^{(k)}$	$(\beta y-y+n+b)_{(k)}/(\beta y+n+a+b)_{(k)}$

where $x_{(m)}$ denotes the ascending factorial $x(x+1)\ldots..(x+m-i)$ and k are non-negative integers

4. APPLICATIONS OF BAYES ESTIMATOR OF GGSD

To illustrate the practical application of the results of section 3, we consider a known GGSD population given by (1.1) with $\theta=0.3$ and $\beta=2$ and take a random sample of size n = 200. The data obtained are given in Table 4.1.

Assuming that the parameter θ is unknown and it has a beta distribution with parameters a and b, we have estimated the Bayes relative frequencies by using the Bayes estimator of $P(X=k)$. Since there is no information about the values of a and b except that they are both positive real numbers, a wide range of values from 1 to 100 were considered for a and b, and the values of $P(X = k)$ are computed for $k = 0,1,\ldots,20$ for the following five broad categories:

I) a=1 and b=1,10,20,30,40,50,100
II) b=1 and a=10,20,30,40,50,100
III) increasing values of a and decreasing values of b
IV) decreasing values of a and increasing values of b
V) equal values of a and b but increasing from 1 to 50.

In each case the values of estimated Bayes frequencies are compared with the simulated values in Table 4.1. It discovered that the estimated Bayes frequencies were quite close to the simulated sample frequencies when a and b are equal and the variation in the Bayes frequencies are small. It is found that the χ^2-values between the simulated sample frequencies and the estimated Bayes frequencies are least when a=b=2.

Table 4.1
Random sample of size n=200 from GGSD with θ=0.3 and β=2

X	**Observed frequency**	**Expected frequency**	
		Bayes estimate under Non-informative prior $\hat{\theta}$	**Bayes estimate under beta prior** $\overset{*}{\theta}$
0	139	140.00	139.66
1	25	29.50	29.56
2	18	12.40	12.51
3	07	6.60	6.66
4	01	4.01	4.02
5	01	2.48	2.58
6	03	1.65	1.65
7	02	1.49	1.50
8	0	0.83	0.80
9	0	0.64	0.65
≥10	04	0.40	0.41
Total	200	200	200

Mean = 0.755, Variance = 2.445 and θ^* = 0.301. The $\chi_3^2 = 3.2$ with p = 0.362. The estimated frequencies obtained by the Bayes method based on non-informative and beta prior seem to be very close to each other. Accordingly, the χ^2-values for the two methods are close to each other.

ACKNOWLEDGEMENT

The authors are indebted to a referee and Associate Editor for helpful comments.

REFERENCES

1. Consul, P.C. and Famoye, F. (2006). *Lagrangain probability distributions*, Birkauser Publications.
2. Jain, G.C. and Consul, P.C. (1971). A generalized negative binomial distribution. *Siam J. Appl. Math*., 21(4), 501-513.
3. Johnson, N.L., Kotz, S. and Kemp, A.W. (1992). *Univariate Discrete Distributions*. Second Edition. John Wiley.
4. Hassan, A., Mishra, A and Jan, T.R. (2002). A quick method of estimating generalized geometric series distribution. *Studia Sci. Math Hungarica*, 39, 291-295.
5. Hassan, A., Raja, T.A. and Mishra, A. (2003). On estimation of generalized geometric series distribution. *Research J. SKUAST (K)*, 1(5), 55-60.
6. Hassan, A. (1995). *Problems of estimation on Lagrangian probability distributions*, Ph.D. Thesis, Patna University, Patna.
7. Mishra, A. (1979). *Generalization of some discrete distributions*, Ph.D. Thesis, Patna University, Patna.
8. Mishra, A. (1982). A generalization of geometric series distribution. *J. Bihar Math. Soc.*, (6), 18-22.
9. Singh, S.K. (1989). *Some contributions to quasi-binomial and discrete Lagrangian probability distributions*, Ph.D. thesis, Patna University, Patna.
10. Mishra, A and Singh, S.K. (1992). On a characteristic of the geometric series distribution. *J. Bihar Math. Soc.* 15, 32-35.

ON NEGATIVE MOMENTS OF CERTAIN DISCRETE DISTRIBUTIONS*

Masood Anwar[1] and **Munir Ahmad**[2]
[1] Department of Mathematics, COMSATS Institute of Information Technology
Islamabad, Pakistan. Email: masoodanwar@comsats.edu.pk
[2] National College of Business Administration & Economics,
Lahore, Pakistan. Email: drmunir@brain.net.pk

ABSTRACT

Negative moments of certain discrete probability distributions in terms of hypergeometric power series functions are obtained.

KEY WORDS

Negative moments; discrete distributions.

1. INTRODUCTION

Recently negative moments have been studied by Roohi (2003) who obtained negative moments of some discrete distributions in terms of hypergeometric series functions. In this paper we have extended her work by considering further discrete probability distributions and expressed the moments in terms of newly defined generalized hypergeometric series function.

2. NEGATIVE MOMENTS OF SOME DISCRETE DISTRIBUTIONS

Theorem 2.1

Let X be a geometric-compound random variable, with parameters α and β having probability mass function (pmf)

$$P(X=x)=\frac{\Gamma(\alpha+\beta)\,\Gamma(\alpha+x-1)\,\Gamma(\beta+1)}{\Gamma\alpha\;\;\Gamma\beta\;\;\Gamma(\alpha+\beta+x)},\alpha>0,\;\;\beta>0,\;\;x=1,2,\dots. \qquad (2.1)$$

The negative moment of k^{th} order is given by

$$E(X+A)^{-k}=\frac{\beta}{(A+1)^k(\alpha+\beta)}\,{}_3H_2\left[(A+1,k),\alpha,1;(A+2,k),(\alpha+\beta+1);1\right], \quad (2.2)$$

where $A\geq 0$.

Proof:

Since X is a geometric-compound random variable with parameters α and β then

*Published in Pak. J. Statist. (2009), Vol. 25(2).

$$E(X+A)^{-k} = \frac{\Gamma(\alpha+\beta)\Gamma(\beta+1)}{\Gamma\alpha\Gamma\beta}\sum_{x=1}^{\infty}\frac{1}{(x+A)^k}\frac{\Gamma(\alpha+x-1)}{\Gamma(\alpha+\beta+x)},$$

$$= \frac{\beta}{(A+1)^k(\alpha+\beta)}\left[1+\frac{(A+1)^k\alpha \quad 1}{(A+2)^k(\alpha+\beta+1)}\right.$$

$$\left.+\frac{(A+1)^k(A+2)^k\alpha(\alpha+1) \quad 1.2}{(A+2)^k(A+3)^k(\alpha+\beta+1)(\alpha+\beta+2)}\frac{1}{2!}+\ldots\right]$$

$$= \frac{\beta}{(A+1)^k(\alpha+\beta)}\,{}_3H_2\left[(A+1,k),\alpha,1;(A+2,k),(\alpha+\beta+1);1\right].$$

where

$${}_pH_q\left[(a_1,k),(a_2,k),\ldots,(a_p,k);(b_1,k)(b_2,k),\ldots,(b_q,k);z\right]$$

$$=1+\frac{a_1^k.a_2^k\ldots a_p^k}{b_1^k.b_2^k\ldots b_q^k}z+\frac{\left[a_1(a_1+1)\right]^k\left[a_2(a_2+1)\right]^k\ldots\left[a_p(a_p+1)\right]^k}{\left[b_1(b_1+1)\right]^k\left[b_2(b_2+1)\right]^k\ldots\left[b_q(b_q+1)\right]^k}\frac{z^2}{2!}+\ldots$$

is a generalized hypergeometric series function with usual conditions (Ahmad, 2008).

If $k=1$, then ${}_pH_q = {}_pF_q$.

If $k=2$, then ${}_pH_q$ is ${}_{2p}F_{2q}\left[a_1,a_1,a_2,a_2,\ldots,a_p,a_p;b_1,b_1,b_2,b_2,\ldots,b_q,b_q;z\right]$,

In general ${}_pH_q = {}_{kp}F_{kq}$.

If $k's$ are different say k_i, then ${}_pH_q = {}_{p\sum_{i=1}^{p}k_i}F_{q\sum_{i=1}^{q}k_i}$.

If $k=1$, then negative moment is

$$E(X+A)^{-1} = \frac{\beta}{(A+1)(\alpha+\beta)}\,{}_3H_2\left[(A+1,1),\alpha,1;(A+2,1),(\alpha+\beta+1);1\right].$$

Theorem 2.2

Let X be a beta-binomial random variable with parameters α, $\alpha>0$, β, $\beta>0$ and pmf

$$P(X=x)=\binom{n}{x}\frac{\Gamma(\alpha+\beta) \quad \Gamma(x+\alpha) \quad \Gamma(n+\beta-x)}{\Gamma\alpha \quad \Gamma\beta \quad \Gamma(n+\alpha+\beta)}, x=0,1,2,\ldots,n, \qquad (2.3)$$

Then the negative moment of k^{th} order is given by

$$E(X+A)^{-k} = \frac{P_0}{A^k} \quad {}_3H_2\left[(A,k),\alpha,-n;(A+1,k),-n-\beta+1;1\right],\ A>0\,, \tag{2.4}$$

where $P_0 = P(X=0) = \dfrac{\Gamma(\alpha+\beta)\Gamma(n+\beta)}{\Gamma\beta\,\Gamma(n+\alpha+\beta)}$.

Proof:

Suppose X is a beta-binomial random variable with parameters α and β then

$$E(X+A)^{-k} = \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha\,\Gamma\beta\,\Gamma(n+\alpha+\beta)} \sum_{x=0}^{n} \binom{n}{x} \frac{\Gamma(\alpha+x)\Gamma(n+\beta-x)}{(x+A)^k},$$

$$= \frac{\Gamma(\alpha+\beta)\Gamma(n+\beta)}{A\Gamma\beta\,\Gamma(n+\alpha+\beta)}\left[1+\frac{A^k\,\alpha\,(-n)}{(A+1)^k(-n-\beta+1)}\right.$$

$$\left.+\frac{A^k(A+1)^k\alpha(\alpha+1)(-n)(-n+1)}{(A+1)^k(A+2)^k\Gamma(-n-\beta+1)\Gamma(-n-\beta+2)}\frac{1}{2!}+\ldots\right],$$

$$= \frac{P_0}{A^k} \quad {}_3H_2\left[(A,k),\alpha,-n;(A+1,k),-n-\beta+1;1\right].$$

If $k=1,$ then negative moment is

$$E(X+A)^{-1} = \frac{P_0}{A} \quad {}_3H_2\left[(A,1),\alpha,-n;(A+1,1),-n-\beta+1;1\right].$$

Theorem 2.3

Let X be a hypergeometric random variable, with parameters $a,\ a>0$, $b,\ b>0$ and pmf

$$P(X=x) = \binom{a}{x}\binom{b}{n-x}\Big/\binom{a+b}{n},\ x=0,1,2,\ldots,\min(n,a)\,, \tag{2.5}$$

then the negative moment of k^{th} order is given by

$$E(X+A)^{-k} = \frac{P_0}{A^k} \quad {}_3H_2\left[(A,k),-a,-n;(A+1,k),b-n+1;1\right],\ A>0 \tag{2.6}$$

where $P_0 = P(X=0) = \dfrac{b!(a+b-n)!}{(b-n)!(a+b)!}$.

Proof:

Suppose X is a hypergeometric random variable with parameters *a* and *b* then

$$E(X+A)^{-k} = \sum_{x=0}^{n} \frac{1}{(x+A)^k} \binom{a}{x}\binom{b}{n-x} \Big/ \binom{a+b}{n}$$

$$= \frac{b!(a+b-n)!}{A^k (b-n)!(a+b)!}\left[1+\frac{A^k\,(-a)(-n)}{(A+1)^k (b-n+1)} + \frac{A^k (A+1)^k\,(-a)(-a+1)(-n)(-n+1)}{(A+1)^k (A+2)^k (b-n+1)(b-n+2)}\frac{1}{2!}+\cdots\right],$$

$$E(X+A)^{-k} = \frac{P_0}{A^k}\;{}_3H_2\left[(A,k),-a,-n;(A+1,k),b-n+1;1\right].$$

If $k=1$, then negative moment is

$$E(X+A)^{-1} = \frac{P_0}{A}\;{}_3H_2\left[A,-a,-n;A+1,b-n+1;1\right].$$

Theorem 2.4

Let X be a Waring random variable, with parameters a, $a \ge 2$, $c, c > a$ and pmf

$$P(X=x) = \frac{(c-a)(a+x-1)!(c)!}{c(a-1)!(c+x)!}, \quad c > a \ge 2,\ x = 0,1,2,.... \tag{2.7}$$

then the negative moment of k^{th} order is given by

$$E(X+A)^{-k} = \frac{P_0}{A^k}\;{}_3H_2\left[(A,k),a,1;(A+1,k),c+1;1\right],\ A>0, \tag{2.8}$$

where $P_0 = P(X=0) = \dfrac{(c-a)}{c}$.

Proof:

Suppose X is a Waring random variable with parameters a and c then

$$E(X+A)^{-k} = \frac{c!(c-a)}{c(a-1)!}\sum_{x=0}^{\infty}\frac{1}{(x+A)^k}\frac{(a+x-1)!}{(c+x)!},$$

$$= \frac{(c-a)}{A^k\,c}\left[1+\frac{A^k\,(a).1}{(A+1)^k (c+1)}+\frac{A^k (A+1)^k\,(a)(a+1).1.2}{(A+1)^k (A+2)^k (c+1)(c+2)}\frac{1}{2!}+\cdots\right],$$

$$E(X+A)^{-k} = \frac{P_0}{A^k}\;{}_3H_2\left[(A,k),a,1;(A+1,k),c+1;1\right].$$

If $k=1$, then negative moment is

$$E(X+A)^{-1}=\frac{P_0}{A}\;{}_3H_2\left[A,a,1;A+1,c+1;1\right].$$

Corollary 2.1

If $a=1$, the Waring function reduces to Yule probability function and Waring results holds for Yule function.

Theorem 2.5

Let X be a random variable having Poisson-binomial distribution with parameters n and $p, 0\le p\le 1$, $a, a>0$ and pmf

$$P(X=x)=e^{-a}\sum_{m=0}^{\infty}\frac{a^m}{m!}\binom{nm}{x}p^x(1-p)^{nm-x}\,,\,(n,m)\in\mathbb{Z}^+\,,\,x=0,1,2,....,nm\,. \quad (2.9)$$

The negative moment of k^{th} order is given by

$$E(X+A)^{-k}=\frac{e^{-a}}{A^k}\sum_{m=0}^{\infty}\frac{a^m}{m!}(1-p)^{nm}\;{}_2H_1\left((A,k),-nm;(A+1,k);\frac{-p}{1-p}\right),\,A>0\;, \quad (2.10)$$

Proof:

Suppose X is a Poisson-binomial random variable and negative moment of first order is given by

$$E(X+A)^{-k}=e^{-a}\sum_{x=0}^{nm}\frac{1}{(x+A)^k}\sum_{m=0}^{\infty}\frac{a^m}{m!}\binom{nm}{x}p^x(1-p)^{nm-x}\,,$$

$$=\frac{e^{-a}}{A^k}\sum_{m=0}^{\infty}\frac{a^m}{m!}(1-p)^{nm}\left[1+\frac{A^k(-nm)}{(A+1)^k}\left(\frac{-p}{1-p}\right)\right.$$

$$\left.+\frac{A^k(A+1)^k(-nm)(-nm+1)}{(A+1)^k(A+2)^k\,2!}\left(\frac{-p}{1-p}\right)^2+\cdots\right],$$

$$E(X+A)^{-k}=\frac{e^{-a}}{A^k}\sum_{m=0}^{\infty}\frac{a^m}{m!}(1-p)^{nm}\;{}_2H_1\left((A,k),-nm;(A+1,k);\frac{-p}{1-p}\right).$$

If $k=1$, then negative moment is

$$E(X+A)^{-1}=\frac{e^{-a}}{A}\sum_{m=0}^{\infty}\frac{a^m}{m!}(1-p)^{nm}\;{}_2H_1\left((A,1),-nm;(A+1,1);\frac{-p}{1-p}\right).$$

Corollary 2.2

Let X be a random variable having Hermite distribution with parameters $p, 0 \le p \le 1$ and $a,\ a > 0$, having pmf

$$P(X = x) = e^{-a} \sum_{m=0}^{\infty} \frac{a^m}{m!} \binom{2m}{x} p^x (1-p)^{2m-x}, m \in \mathbb{Z}^+, x = 0,1,2,....,2m. \qquad (2.11)$$

The negative moment of k^{th} order is given by (2.10) when $n = 2$.

REFERENCES

1. Ahmad, M. and Saboor, A. (2009). Properties of a newly defined hypergeometric power series function. *Pak. J. Statist.*, 25(2), 227-233.
2. Rainville, E.D. (1960). *Special Functions.* Chelsea Publication Co. Bornmx. USA.
3. Roohi, A. (2003). *Negative and Factorial Moments of Discrete Distributions Involving Hyper-Geometric Series Functions*. Unpublished Ph.D. dissertation, National College of Business Administration and Economics, Lahore.

PROPERTIES OF A NEWLY DEFINED HYPERGEOMETRIC POWER SERIES FUNCTION[*]

Munir Ahmad[1] and **Abdus Saboor**[2]
National College of Business Administration and Economics, Lahore, Pakistan.
Email: [2]saboorhangu@gmail.com; [1]drmunir@brain.net.pk

ABSTRACT

Ahmad (2007b) has recently defined a generalized hypergeometric series function and referred to it as a hypergeometric power series function or ${}_rH_s$-function which is an alternative notation for the ${}_rF_s$-function, the ${}_rH_s$ notation has advantages when the arguments are large and parameters are repeated and discussed the some basic properties. In this paper further properties of the hypergeometric power series functions have been developed.

KEY WORDS

Generalized hypergeometric series function; hypergeometric power series function; poission distribution; hyper poisson; negative moments.

1. INTRODUCTION

Ahmad (2007a) has discussed the Conway-Maxwell Poisson distribution and Conway-Maxwell Hyper Poisson (CMHP) distribution. The structure of CMP and CMHP distributions shows that a more general definition of the hypergeometric series function is needed. When a large number of identical parameters are introduced in the generalized hypergeometric series function, its notation becomes cumbersome. Use of generalized hypergeometric series function is sometimes difficult and time consuming especially when we have a large number of parameters. We take powers on those parameters which are repeated and as such Ahmad (2007b) has introduced an alternative form of a hypergeometric series function called hypergeometric power series function or ${}_rH_s$-function. (See also Saboor, 2007).

1.1 Definitions

The hypergeometric power series function or ${}_rH_s$-function is defined as

$$
{}_rH_s\left[(a_1,m_1),(a_2,m_2),...,(a_r,m_r);(b_1,n_1),(b_2,n_2),...,(b_s,n_s);z\right]
= \sum_{i=0}^{\infty} \frac{\prod_{k=1}^{r}\left[(a_k)_i\right]^{m_k}}{\prod_{j=1}^{s}\left[(b_j)_i\right]^{n_j}} \frac{z^i}{i!},
$$

---

[*]Published in Pak. J. Statist. (2009), Vol. 25(2).

where $a_k \in R, |z|<1, b_j \neq 0,-1,-2,\ldots, n_j, m_k$ are positive integers.

1. If $\sum_{j=1}^{r} m_j \le \sum_{k=1}^{s} n_k$, the ${}_rH_s$-function converges for all finite z;
2. If $\sum_{j=1}^{r} m_j = \sum_{k=1}^{s} n_k + 1$, the ${}_rH_s$-function converges for $|z|<1$ and diverges for $|z|>1$;
3. If $\sum_{j=1}^{r} m_j > \sum_{k=1}^{s} n_k + 1$, the ${}_rH_s$-function diverges for $z \neq 0$.

When all m_r and n_s are equal to 1, then ${}_rH_s$ becomes

$${}_rF_s\left[a_1,a_2,....,a_r;b_1,b_2,...,b_s;z\right]=\sum_{i=0}^{\infty}\frac{\prod_{k=1}^{r}\left[(a_k)_i\right]}{\prod_{j=1}^{s}\left[(b_j)_i\right]}\frac{z^i}{i!},$$

where $a_k \in R$, $1\le k\le r, 1\le j\le s$ and $b_j \neq 0,-1,-2,......., |z|<1$.

2. PROPERTIES OF ${}_rH_s$-FUNCTION

We have discussed some basic properties of the ${}_rH_s$-function and developed different types of recurrence relationships, which are as follows:

Theorem 2.1:

Let $\operatorname{Re}(m)>0$, $\operatorname{Re}(n)>0$ and $b\neq 0,-1,-2,\ldots$. If $|z|<1$ and $\beta(m,n)$ is the beta function, then

$$\int_0^1 t^{m-1}(1-t)^{n-1}\,{}_1H_1\left[(a,r);(b,s);tz\right]dt$$

$$=\beta(m,n)\,{}_2H_2\left[(a,r),(m,1);(b,s);(m+n,1),z\right]. \qquad (2.1)$$

Proof:

Expanding ${}_1H_1\left[(a,r);(b,s);tz\right]$ in (2.1), we have

$$\int_0^1 t^{m-1}(1-t)^{n-1}\sum_{k=0}^{\infty}\frac{\left[(a)_k\right]^r}{\left[(b)_k\right]^s}\frac{t^k z^k}{k!}dt=\sum_{k=0}^{\infty}\frac{\left[(a)_k\right]^r}{\left[(b)_k\right]^s}\frac{z^k}{k!}\frac{\Gamma(m+k)\Gamma(n)}{\Gamma(m+n+k)}, \quad |z|<1. \qquad (2.2)$$

After a simple algebra we obtain R.H.S of (2.1).

Theorem 2.2:

Let neither c nor d be zero or a negative integer. If $\delta > 1$ and $|t\,z| < 1, \left|\frac{z}{\lambda}\right| < 1,\ \lambda > 0$, then

$$\int_0^\infty e^{-\lambda t} t^{\delta-1}\ {}_2\mathrm{H}_2\left[(\mathrm{a}, \mathrm{m}_1), (\mathrm{b}, \mathrm{m}_2); (\mathrm{c}, \mathrm{n}_1), (\mathrm{d}, \mathrm{n}_2); z\,t\right] dt$$

$$= \frac{\Gamma(\delta)}{(\lambda)^\delta}\ {}_3H_2\left[(a, m_1), (b, m_2), (\delta, 1); (c, n_1), (d, n_2); z/\lambda\right]. \tag{2.3}$$

Proof:

$$L.H.S. = \int_0^\infty e^{-\lambda t} t^{\delta-1} \sum_{i=0}^\infty \frac{\left[(a)_i\right]^{m_1}\left[(b)_i\right]^{m_2}(z\,t)^i}{\left[(c)_i\right]^{n_1}\left[(d)_i\right]^{n_2} i!} dt. \tag{2.4}$$

Let $\lambda t = v$ and after taking integral, we get R.H.S of (2.6).

Similarly we obtain

$$\int_0^\infty e^{-t/z}\ {}_1H_1[(a, r); (b, s); t]\, t^{a\text{-}1} dt = z^a \Gamma(a)\ {}_1H_1[(a, r+1); (b, s); z], \tag{2.5}$$

when $b \neq 0, -1, -2, \ldots.$ If $|z| < 1$.

Theorem 2.3:

i) Let $a \neq \frac{1}{2}, 0, -\frac{1}{2}, -\frac{3}{2}, \ldots.$ If $|z| < 1$, then

$${}_1H_1\left[(a/2, 2); (a\text{-}1/2, 1); z\right]{}_1H_1\left[(a/2, 2); (a+1/2, 1); z\right]$$

$$= {}_1H_2\left[(a, 3); (2a\text{-}1, 1), (a+1/2, 1); z\right] \tag{2.6}$$

Similarly we find

ii) $$\left({}_1H_1\left[(a/2, 2); (a+1/2, 1); z\right]\right)^2 = {}_1H_2\left[(a, 3); (2a, 1), (a+1/2, 1); z\right], \tag{2.7}$$

where $a \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, -2\ldots.$ If $|z| < 1$. (see Clausian, 1828).

iii) $${}_1H_0[(a, 2); \text{-}; z]\,{}_1H_0[(a, 2); \text{-}; \text{-}\,z] = {}_3H_1\left[(a, 2), (a, 1), ((2a+1)/2, 1); (2a, 1); 4z^2\right], \tag{2.8}$$

where $a \neq 0, -1/2, -1, -3/2, \ldots.$ If $|z| < 1$.

Proofs are trivial.

3. NEGATIVE MOMENTS OF SOME DISCRETE PROBABILITY FUNCTIONS

We now consider the negative moments of the form $E(X+A)^{-k}, k>0$, for some discrete distributions.

Theorem 3.1:

Let X be a geometric-compound random variable, with parameters α and β having probability mass function (pmf)

$$P(X=x)=\frac{\Gamma(\alpha+\beta)\ \ \Gamma(\alpha+x-1)\ \ \Gamma(\beta+1)}{\Gamma(\alpha)\ \ \Gamma(\beta)\ \ \Gamma(\alpha+\beta+x)},\ \alpha>0,\ \beta>0,\ x=1,2,\dots. \tag{3.1}$$

Then the negative moment of k^{th} order is given by

$$E\left(X+A\right)^{-k}=\frac{\beta}{(A+1)^{k}(\alpha+\beta)}\ {}_3H_2\left((A+1,k),(\alpha,1),(1,1);(A+2,k),(\alpha+\beta+1,1);1\right). \tag{3.2}$$

Proof:

The k^{th} negative moment of (3.1) is

$$E(X+A)^{-k}=\frac{\Gamma(\alpha+\beta)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{x=1}^{\infty}\frac{1}{(x+A)^{k}}\frac{\Gamma(\alpha+x-1)}{\Gamma(\alpha+\beta+x)}$$

After some algebraic computations, we get (3.2).

Theorem 3.2:

Let X be a beta-binomial random variable, with parameter $\alpha>0$, $\beta>0$ and probability mass function

$$P(X=x)=\binom{n}{x}\frac{\Gamma(\alpha+\beta)\Gamma(x+\alpha)\Gamma(n+\beta-x)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)},\ x=0,1,2,\dots,n. \tag{3.3}$$

Then the negative moment of k^{th} order is given by

$$E(X+A)^{-k}=\frac{P_0}{A^{k}}\ {}_3H_2\left((A,k),(\alpha,1),(-n,1);(A+1,1),(-n-\beta+1,1);1\right), \tag{3.4}$$

where $P_0=P(x=0)=\dfrac{\Gamma(\alpha+\beta)\Gamma(n+\beta)}{\Gamma(\beta)\Gamma(n+\alpha+\beta)}$.

Following the procedure of theorem 3.1, we obtain (3.4).

Theorem 3.3:

Let X be a Waring random variable, with parameters a, $a \geq 2$, $c, c > a$ and pmf

$$P(X = x) = \frac{(c-a)(a+x-1)!(c)!}{c(a-1)!(c+x)!}, \quad c > a \geq 2,\ x = 0,1,2,\ldots .$$

Then the negative moment of k^{th} order is given by

$$E(X+A)^{-k} = \frac{P_0}{A^k}\, {}_3H_2\left((A,k),\ (a,1),(1,1);\ (A+1,k),(c+1,1);\ 1\right), \tag{3.5}$$

where $P_0 = P(X=0) = \frac{(c\text{-}a)}{c}$, $A > 0$.

Proof is trivial.

Theorem 3.4:

Let X be a truncated Poisson random variable, with parameter λ, $\lambda > 0$ and pmf

$$P(X = x) = \frac{\lambda^x}{(e^{\lambda}\text{-}1)x!},\ \lambda > 0,\ \ X = 1,2,\ldots .$$

Then the negative moment of the k^{th} order is given by

$$E\left(X+A\right)^{-k} = \frac{P_1}{\left(A+1\right)^k}\, {}_2H_2\left(\left(A+1,k\right),\left(1,1\right);\left(A+2,k\right),\left(2,1\right);1\right), \tag{3.6}$$

where $P_1 = P(X=1) = \frac{\lambda}{(e^{\lambda}\text{-}1)}$, $A \geq 0$.

Theorem 3.5:

Let X be a truncated binomial random variable, with parameters n and p, $0 \leq p \leq 1$, and pmf

$$P(X = x) = \frac{1}{(1-q^n)}\binom{n}{x} p^x q^{n-x}, \quad x = 1,2,\ldots,\text{n},\ q = 1-p .$$

Then the negative moment of k^{th} order is given by

$$E(X+A)^{-k} = \frac{P_1}{(A+1)^k}\, {}_3H_2\left((A+1,k),(1,1),\ (-n+1,1);\ (A+2,k),\ (2,1);\ \frac{-p}{q}\right). \tag{3.7}$$

4. SUMMATION OF ${}_rH_s$-FUNCTIONS

Ahmad and Roohi (2004, 2005) have derived the sum of some combinations of hyoergeometric series functions using the binomial and logarithm probability functions. In this section, we have derived another set of sum of some combinations of hypergeometric power series function using Dacey (1972) probability function. Ahmad and Roohi (2004, 2005) summations of series become its special cases.

4.1 Lemma (Gould, 1972)

For $a,\ b_i,\ c_j \neq 0,-1,-2,\ldots$ and some k,

$$\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}\,{}_rF_s\left[a,\ b_1,\ b_2,\ldots,\ b_{r-1};\ a+1, c_1,\ldots.,\ c_{s-1};z\right]$$
$$= {}_rF_s[1, b_1,\ b_2,\ldots,\ b_{r-1};\ k+1, c_1,\ldots.,\ c_{s-1};z],$$

where ${}_rF_s$ is a hypergeometric series function with usual conditions for convergence.

Theorem 4.1:

Let $b \neq 0,-1,-2,\ldots$ or $a \geq 0$. If $s \geq 1$, $k \geq 1$ and $0<\theta<1$. Then

$$\sum_{s=1}^{k}(-1)^{s+1}\binom{k}{s}\,{}_2H_2\left[(s,1),(a,h);(s+1,1),(b,m);\theta\right]$$
$$= {}_2H_2\left[(1,1),(a,h);(k+1,1),(b,m);\theta\right] \tag{4.1}$$

Proof:

We have (4.1) as a repeated case of Lemma (4.1). Alternatively suppose X is a discrete random variable with probability function (See Dacey, 1972).

$$P(X=x)=\frac{c\,\theta^x}{x!}\,\gamma_x\left[(a,h);(b,m)\right]\ ;\ x=0,1,2,\ldots, \tag{4.2}$$

where $c=\dfrac{1}{{}_1H_1\left[(a,h);(b,m);\theta\right]}$ and $\gamma_x\left[(a,h);(b,m)\right]=\dfrac{\left[\Gamma(a+x)\right]^h\left[\Gamma(b)\right]^m}{\left[\Gamma(b+x)\right]^m\left[\Gamma(a)\right]^h}$.

It is known that

$$\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)=\sum_{s=1}^{k}\frac{(-1)^{s+1}}{(k-s)!(s-1)!}\frac{1}{(x+s)},\ x\geq 0,\ k=1,2,3,\ldots,\ \text{(See Jones, 1987).}$$

Using the definition of mathematical expectation $E(X)=\sum_{x=0}^{\infty}xf(x)$, we get

$$E\left[\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)\right]=\sum_{s=1}^{k}\frac{(-1)^{s+1}}{(k-s)!(s-1)!}E\left(\frac{1}{X+s}\right). \tag{4.3}$$

Now

$$E\left(\frac{1}{X+s}\right)=c\sum_{x=0}^{\infty}\frac{\theta^x}{x!}\,\gamma_x\left[(a,\,h);(b,\,m)\right]\frac{1}{x+s}$$
$$=s^{-1}c\,{}_2H_2\left[(s,1),(a,h);(s+1,1),(b,m);\theta\right].$$

It thus follows from (4.3) that

$$E\left[\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)\right]=c\sum_{s=1}^{k}\frac{(-1)^{s+1}}{(k-s)!(s)!}\,{}_2H_2\left[(s,1),(a,h);(s+1,1),(b,m);\theta\right]. \qquad (4.4)$$

Also,

$$E\left[\prod_{s=1}^{k}\left(\frac{1}{X+s}\right)\right]=c\sum_{x=0}^{\infty}\frac{\theta^x}{x!}\,\gamma_x\left[(a,h);(b,m)\right]\prod_{s=1}^{k}\frac{1}{(x+s)}$$
$$=c\left(k!\right)^{-1}\,{}_2H_2\left[(1,1),(a,h);(k+1,1),(b,m);\theta\right]. \qquad (4.5)$$

(4.4) and (4.5) imply (4.1).

ACKNOWLEDGMENT

The authors are indebted to referees and Associate Editor for improving the original version of the paper. One of the referee points out the relation (2.7) and Lemma (4.1) for which we acknowledge gratefully.

REFERENCES

1. Ahmad, M. (2007a). On the Theory of Inversion. *Int. J. Statist. Scien.*, 6 (Special Issue), 43-53.
2. Ahmad, M. (2007b). *A Brief Communication on Hypergeometric Power Series Functions*. Technical Report No. 2/07, National College of Business Administration and Economics, Lahore.
3. Ahmad, M. and Roohi, A. (2004). On Sums of Some Hypergeometric Series Functions-I. *Pak. J. Statist.*, 20(1), 193-197.
4. Ahmad, M. and Roohi, A. (2005). On Sums of Some Hypergeometric Series Functions-II. *Pak. J. Statist.*, 21(3), 245-248.
5. Clausen, T. (1828). Ueber die Falle wenn die Reihe $y=1+\frac{\alpha.\beta}{1.\gamma}x+\ldots$ ein quadrat von der Form $x=1+\frac{\alpha'.\beta'.\gamma'}{1.\delta'.e'}x+\ldots$ hat. *J. für Math.*, 3, 89-95, 1828.
6. Dacey, M.F. (1972). A Family of Discrete Probability distributions defined by the generalized Hypergeometric Series. *Sankhya*, B.34, 243-250.
7. Gould, H.W. (1972). *Combinatorial Identities*. Morgantown Printing and Binding Co, USA.
8. Johnson, N.L. and Kotz, S. (1990). Use of moments in deriving distributions and some characterizations. *Mathematical Scientist*, 15, 42-52.

9. Jones M.C. (1987). Inverse factorial moments. *Statist. and Prob. Letters*, 6, 37-42.
10. Saboor, A. (2007). *Hypergeometric Power Series Function*. Unpublished M.Phil. Thesis. National College of Business Administration and Economics, Lahore.

SIZE-BIASED DISTRIBUTIONS AND THEIR APPLICATIONS*

Khurshid Ahmad Mir[1] and **Munir Ahmad**[2]
[1] Department of Statistics, Govt. Degree College (Boys)
Baramulla, Kashmir, India. Email: khrshdmir@yahoo.com
[2] National College of Business Administration and Economics
Lahore, Pakistan. Email: drmunir@brain.net.pk

ABSTRACT

In this paper, some size-biased probability distributions and their generalizations have been introduced. These distributions provide a unifying approach for the problems where the observations fall in the non-experimental, non- replicated, and nonrandom categories. These distributions take into account the method of ascertainment, by adjusting the probabilities of actual occurrence of events to arrive at a specification of the probabilities of those events as observed and recorded. Failure to make such adjustments can lead to incorrect conclusions. This paper surveys some of the possible uses of size- biased distribution theory to some real life data.

KEY WORDS

Size-biased discrete distributions; generalized size-biased discrete distributions; Chi-square; Akaike Information Criterion; Bayesian Information Criterion; R-Software.

1. INTRODUCTION

Size-biased distributions are a special case of the more general form known as weighted distributions. Fisher (1934) introduced these distributions to model ascertainment bias and were later formalized in a unifying theory by Rao (1965). These distributions arise in practice when observations from a sample are recorded with unequal probability and provide a unifying approach for the problems where the observations fall in the non-experimental, non- replicated, and non-random categories. If the random variable X has distribution $f\left(x;\theta\right)$, with unknown parameter θ, then the corresponding size-biased distribution is of the form

$$f^{*}\left(x;\theta\right)=\frac{x^{\alpha}f\left(x;\theta\right)}{\mu'_{\alpha}}, \tag{1.1}$$

where

$$\mu'_{\alpha}=\int x^{\alpha}f\left(x;\theta\right)dx. \tag{1.2}$$

When $\alpha=1$ and 2, we get the simple size-biased and area-biased distributions respectively. Here in this paper, only size-biased distributions are considered as these are simple to calculate and moreover, the examples deal with size-biased sampling.

*Published in Pak. J. Statist. (2009), Vol. 25(3).

Warren (1975) was the first to apply them in connection with sampling wood cells. Van Deusen (1986) arrived at size-biased distribution theory independently and applied it to fitting distributions of diameter at breast height (DBH) data arising from horizontal point sampling (HPS) (Grosenbaugh, 1958) inventories. Subsequently, Lappi and Bailey (1987) used weighted distributions to analyze HPS diameter increment data. More recently, these distributions were used by Magnussen et al. (1999) to recover the distribution of canopy heights from airborne laser scanner measurements. In ecology, Dennis and Patil (1984) used stochastic differential equations to arrive at a weighted gamma distribution as the stationary probability density function (PDF) for a stochastic population model with predation effects. In fisheries, Taillie et al (1995) modeled populations of fish stocks using weighted distributions. Most of the statistical applications of weighted distributions, especially to the analysis of data relating to human populations and ecology, can be found in Patil and Rao (1977, 1978). Gove (2003) reviewed some of the more recent results on size-biased distributions pertaining to parameter estimation in forestry, with special emphasis on the weibull family. Mir (2007) also discussed some of the discrete size-biased distributions.

In this paper, some of the results and estimation on size -biased discrete distributions and of their generalized form have been used to real life data and their comparisons have been made with the help of Pearson's Chi-square, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) techniques. For computation purposes R- software has been used.

2. SOME SIZE-BIASED DISTRIBUTIONS

In this section, we have obtained some basic size-biased discrete distributions by using equations (1.1) and (1.2).

2.1 Size-biased Binomial Distribution (SBBD)

The probability mass function of binomial distribution (BD) is given as

$$P\,[X = x] = p(x) = \binom{n}{x}\, p^x q^{n-x}\,;\; x = 0,1,2,\cdots, n. \tag{2.1}$$

We know that $\sum_{x=0}^{\infty} x\, P\left[X = x\right] = np$, which on solving gives a size-biased binomial distribution (SBBD) as

$$\text{P}\left[x = x\right] = \binom{n-1}{x-1}\, p^{x-1}\, q^{n-x}\;\; ;\, \text{x} = 1,2...... \tag{2.2}$$

2.2 Size-Biased Poisson Distribution (SBPD)

The probability mass function of the Poisson distribution (PD) is given as

$$P[X = x] = \frac{e^{-\lambda_1}(\lambda_1)^x}{x!}\,;\; x = 0,\,1,\,2,\,\cdots \text{ and } \lambda_1 > 0. \tag{2.3}$$

Here $\sum_{x=0}^{\infty} x\, P\left[X = x\right] = \lambda_1$, which on solving gives size-biased Poisson distribution (SBPD) as

$$P\left[X = x\right] = \frac{\lambda_1^{x-1} e^{-\lambda_1}}{\left(x-1\right)\,!};\ \mathrm{x} = 1,2......\qquad (2.4)$$

2.3 Size-biased Negative Binomial Distribution (SBNBD)

The probability mass function of the negative binomial distribution (NBD) is given by

$$P[X = x] = \binom{x+r-1}{r-1}\ p^r\ q^x = \binom{-r}{\mathrm{x}}\ p^r (-q)^x;\ x{=}0,1,\ 2\ldots.\qquad (2.5)$$

where parameters satisfy q = 1 – p and 0 < p < 1 and x = 1, 2, 3 …

Here, $\sum_{x=0}^{n} x.P\left[X = x\right] = \frac{rq}{p}$. This gives size-biased negative binomial distribution (SBNBD) as

$$P\left[X = x\right] = \binom{x+r-1}{x-1}\ \mathrm{p}^{\mathrm{r+1}}\ \mathrm{q}^{\mathrm{x\text{-}1}}\ ;\ \mathrm{x} = 1,2,.....\qquad (2.6)$$

2.4 Size-Biased Logarithmic Series Distribution (SBLSD)

The probability mass function of logarithmic series distribution (LSD) is given by

$$P[X = x] = -\ \frac{1}{[\log\,(1-\alpha)]}\ \frac{\alpha^{\mathrm{x}}}{x}\ ;\ x = 1,2\cdots\qquad (2.7)$$

Here, $\sum_{x=1}^{\infty} x.P\left[X = x\right] = \frac{-1}{\log(1-\alpha)}\frac{\alpha}{\left(1-\alpha\right)}$. This gives the size-biased logarithmic series distribution (SBLSD) as

$$P\left[X = x\right] = \alpha^{x-1}\left(1-\alpha\right)\ ;\ \mathrm{x} = 1,2......\qquad (2.8)$$

3. SIZE-BIASED GENERALIZED DISCRETE DISTRIBUTION

In this section, we have obtained size-biased generalized discrete distributions by using equations (1.1) and (1.2).

3.1 Size-biased Generalized Negative Binomial Distribution (SBGNBD)

Jain and Consul (1971) defined the probability function of generalized negative binomial distribution (GNBD) as

$$P[X=x]=\frac{m\Gamma(m+\beta x)}{x!\Gamma(m+\beta x-x+1)}\ \alpha^{x}(1-\alpha)^{m+\beta x-x};\ x=0,1,2.... \quad (3.1)$$

$$= 0;\quad \text{Otherwise}$$

where $0<a<1$, $m>0$ and $\beta=0$ or $0<\beta<\alpha^{-1}$ and t is the largest positive integer for which $m+1+(\beta-1)t>0$.

$E(x)=\frac{m\alpha}{(1-\alpha\beta)}$. On solving the equation $\sum_{x=0}^{\infty} x\cdot P\,(X=x)=\frac{m\alpha}{(1-\alpha\beta)}$, we obtain the probability function of size biased generalized negative binomial distribution (SBGNBD) as

$$P[X=x]=(1-\alpha\beta)\binom{m+\beta x-1}{x-1}\alpha^{x-1}\ (1-\alpha)^{m+\beta x-x}\ ;\ \mathrm{x}=1,2.... \quad (3.2)$$

where $0<\alpha<1$, $\mathrm{m}>0$, $0\le\beta\le1$,

At $\beta=0$ and $\beta=1$, we get size- biased binomial distribution (2.2) and size-biased negative binomial distributions (2.6) respectively.

3.2 Size-Biased Generalized Poisson Distribution (SBGPD)

Consul and Jain (1973) defined the probability mass function of generalized Poisson distribution (GPD) as

$$P[X=x]=\frac{\lambda_1(\lambda_1+x\,\lambda)^{x-1}\ \mathrm{Exp}\,[-(\lambda_1+x\,\lambda)]}{\mathrm{x}!}\ ;\ x=0,\ 1,\ 2,\ldots \quad (3.3)$$

Here E(x) = $\frac{\lambda_1}{(1-\lambda)}$. On solving the equation $\sum_{x=0}^{\infty} x.P[X=x]=\frac{\lambda_1}{(1-\lambda)}$, the probability function of a size-biased generalized Poisson distribution (SBGPD) is given as

$$P[X=x]=\frac{(1-\lambda)(\lambda_1+x\lambda)^{x-1}\,EXP[-(\lambda_1+x\lambda)]}{(x-1)\ !};\ x=1,2,..... \quad (3.4)$$

$$\lambda_1>-1,|\lambda_1\lambda|<1+\lambda_1$$

At $\lambda=0$, we get size-biased Poisson distribution (2.4).

3.3 Size-Biased Generalized Logarithmic Series Distribution (SBGLSD)

A generalized logarithmic series distribution (GLSD) was given by Jain and Gupta (1973) with probability function as

$$P[X=x] = -\frac{1}{\log(1-\alpha)} \quad \frac{\Gamma(\beta x)}{x!\Gamma(\beta x - x + 1)} \; \alpha^{x} (1-\alpha)^{\beta x - x} \; ; \; x = 1,2,3,\cdots \tag{3.5}$$

$$0 < \alpha < 1, \; |\alpha\beta| < 1. \text{ Also } \mu_1' = \frac{-\alpha}{\log(1-\alpha)(1-\alpha\beta)}.$$

On solving the equation $\sum_{x=0}^{\infty} x.P\left[X=x\right] = \frac{-\alpha}{\log(1-\alpha)(1-\alpha\beta)}$, the probability function of size-biased generalized logarithmic series distribution (SBGLSD) is given as

$$P\left[X=x\right] = \left(1-\alpha\beta\right) \binom{\beta x - 1}{x-1} \alpha^{x-1} \left(1-\alpha\right)^{\beta x - x}; \; x = 1,2..... \tag{3.6}$$

At $\beta = 1$, we get size-biased logarithmic series distribution (2.8).

4. ESTIMATION OF PARAMETERS

In this section**,** we estimate the parameters of the generalized distributions only and for classical ones we can get easily as their particular cases.

4.1 Estimation of Parameters in Size-biased Generalized Negative Binomial Distribution

The likelihood function of SBGNBD (3.2) can be given as

$$L = \left(1-\alpha\beta\right)^n \prod_{i=1}^{n} \binom{m + \beta x_i - 1}{x_i - 1} \alpha^{\sum_{i=1}^{n} x_i - n} \left(1-\alpha\right)^{mn + \beta \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i}$$

$$= K\left(1-\alpha\beta\right)^n \alpha^{y-n} \left(1-\alpha\right)^{mn + \beta y - y}, \tag{4.1}$$

where $y = \sum_{i=1}^{n} x_i$ *and* $\mathrm{K} = \prod_{i=1}^{n} \binom{m + \beta x_i - 1}{x_i - 1}$.

Since $0 < \alpha < 1$, therefore we assume that prior information about α come from beta distribution. Thus

$$f\left(\alpha\right) = \frac{\alpha^{a-1}\left(1-\alpha\right)^{b-1}}{B\left(a,b\right)}; \; 0 < \alpha < 1. \tag{4.2}$$

The posterior distribution from (4.1) and (4.2) can be written as

$$\prod\left(\alpha / y\right) = \frac{\left(1-\alpha\beta\right)^n \alpha^{y+a-n-1} \left(1-\alpha\right)^{mn+\beta y - y + b - 1}}{\int_0^1 \left(1-\alpha\beta\right)^n \alpha^{y+a-n-1} \left(1-\alpha\right)^{mn+\beta y - y + b - 1} d\alpha}. \tag{4.3}$$

The Baye's estimator of α^z is the posterior mean and is given as

$$\hat{\alpha}^{z} = \int_0^1 \alpha^{z} \prod(\alpha/\mathrm{y})\, d\alpha = \frac{\int_0^1 (1-\alpha\beta)^{n}\ \alpha^{y+a-n+z-1}(1-\alpha)^{mn+\beta y-y+b-1} d\alpha}{\int_0^1 (1-\alpha\beta)^{n}\ \alpha^{y+a-n-1}(1-\alpha)^{mn+\beta y-y+b-1} d\alpha} \qquad (4.4)$$

$$\int_0^1 (1-\alpha\beta)^{n}\ \alpha^{y+a-n+z-1}(1-\alpha)^{mn+\beta y-y+b-1} d\alpha$$

$$= \frac{\Gamma(y+a-n+z)\Gamma(\beta y+mn+b-y)\ {}^{2}F_{1}[-n, y+a-n+z, \beta y+mn+a+b-n+z, \beta]}{\Gamma(\beta y+mn+a+b-n+z)} \qquad (4.5)$$

and

$$\int_0^1 (1-\alpha\beta)^{n}\ \alpha^{y+a-n-1}(1-\alpha)^{mn+\beta y-y+b-1} d\alpha$$

$$= \frac{\Gamma(y+a-n)\Gamma(\beta y+mn+b-y)\ {}^{2}F_{1}[-n, y+a-n, \beta y+mn+a+b-n, \beta]}{\Gamma(\beta y+mn+a+b-n)}, \qquad (4.6)$$

where ${}^{2}F_{1}[a;b;c;x] = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)2!}x^{2}.......$

Putting these values in equation in (4.4), the Baye's estimator of α^{z} is obtained as

$$\hat{\alpha}^{z} = \frac{\Gamma(y+a-n+z)\ \Gamma(\beta \mathrm{y}+\mathrm{mn}+\mathrm{a}+\mathrm{b\text{-}n})\ {}^{2}F_{1}[-n, y+a-n+z, \beta y+mn+a+b-n+z, \beta]}{\Gamma(y+a-n)\ \Gamma(\beta y+mn+a+b-n+z)\ {}^{2}F_{1}[-n, y+a-n, \beta y+mn+a+b-n, \beta]}. \qquad (4.7)$$

For z=1, we get the Baye's estimator of α as

$$\hat{\alpha} = \frac{(y+a-n)\ \ {}^{2}F_{1}[-n, y+a-n+1, b+mn+\beta y-n+a+1, \beta]}{(\beta y+mn+a+b-n)\ \ {}^{2}F_{1}[-n, y+a-n, b+mn+\beta y-n+a, \beta]}. \qquad (4.8)$$

For $\beta = 1$ and 0, we get the Baye's estimate for the size-biased negative binomial (2.6) and size-biased binomial (2.2) models which are given by (4.9) and (4.10) respectively.

$$\hat{\alpha} = \frac{y+a-n}{y+mn+a+b} \qquad (4.9)$$

and

$$\hat{\alpha} = \frac{y+a-n}{mn+a+b-n}. \qquad (4.10)$$

4.2 Estimation of Parameters in Size-biased Generalized Poisson Distribution

The estimation becomes tedious in this distribution when taking Bayesian or MLE estimation method into consideration. Mishra and Singh (1993) obtained the moment estimators in case of size-biased generalized Poisson distribution (3.4) by letting $1-\lambda=\theta$. The mean and variance of SBGPD can be expressed as

$$\mu_1' = \frac{\left(\lambda_1\theta+1\right)}{\theta^2} \tag{4.11}$$

$$\mu_2 = \frac{\left[2\left(1-\theta\right)+\lambda_1\theta\right]}{\theta^4}. \tag{4.12}$$

This gives an equation in θ as

$$\mu_2\theta^4-\mu_1'\theta^2+2\theta-1=0. \tag{4.13}$$

Replacing μ_1' and μ_2 by the corresponding sample values $\overline{x}$ and S^2 respectively, we get

$$S^2\theta^4-\overline{x}\theta^2+2\theta-1=0. \tag{4.14}$$

It is a polynomial of degree four and can be solved using the Newton-Raphson method and so an estimate of λ_1 can be obtained. An estimate of λ_1 is then obtained as

$$\hat{\lambda}_1 = \frac{\left(\hat{\theta}^2\overline{x}-1\right)}{\hat{\theta}}. \tag{4.15}$$

The moment estimate for size-biased Poisson distribution can be obtained easily by putting $\theta=1$ in (4.15).

4.3 Estimation of Parameters in Size-biased Generalized Logarithmic Series Distribution

In this sub-section, we have introduced the Bayesian estimation of size-biased GLSD. The likelihood function of SBGLSD (3.6) is given as

$$L\left(x;\alpha,\beta\right)=\left(1-\alpha\beta\right)^n\prod_{i=1}^{n}\begin{pmatrix}\beta x_i-1\\ x_i-1\end{pmatrix}\alpha^{\sum_{i=1}^{n}x_i-n}\left(1-\alpha\right)^{\beta\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}x_i}$$

$$=k\left(1-\alpha\beta\right)^n\alpha^{y-n}\left(1-\alpha\right)^{\beta y-y}, \tag{4.16}$$

where $y=\sum_{i=1}^{n}x_i$ and $\mathrm{k}=\prod_{i=1}^{n}\begin{pmatrix}\beta x_i-1\\ x_i-1\end{pmatrix}$.

Since $0<\alpha<1$, therefore we assume that prior information about α when β is known is from beta distribution.

Thus

$$f(\alpha)=\frac{\alpha^{a-1}(1-\alpha)^{b-1}}{B(a,b)};\ 0<\alpha<1\ ,\ a>0,\ b>0. \tag{4.17}$$

The posterior distribution from (4.16) and (4.17) can be written as

$$\prod(\alpha/y)=\frac{(1-\alpha\beta)^{n}\,\alpha^{y+a-n-1}(1-\alpha)^{\beta y-y+b-1}}{\int_0^1(1-\alpha\beta)^{n}\,\alpha^{y+a-n-1}(1-\alpha)^{\beta y-y+b-1}\,d\alpha}. \tag{4.18}$$

The Bayes estimator of α^z is given as

$$\hat{\alpha}^z=\int_0^1\alpha^z\ \prod(\alpha/\mathrm{y})\ d\alpha$$

$$=\frac{\int_0^1(1-\alpha\beta)^{n}\ \alpha^{y+a-n+z-1}(1-\alpha)^{\beta y-y+b-1}\,d\alpha}{\int_0^1(1-\alpha\beta)^{n}\,\alpha^{y+a-n-1}(1-\alpha)^{\beta y-y+b-1}\,d\alpha}, \tag{4.19}$$

where

$$\int_0^1(1-\alpha\beta)^{n}\ \alpha^{y+a-n+z-1}(1-\alpha)^{\beta y-y+b-1}\,d\alpha$$

$$=\frac{\Gamma(y+a-n+z)\Gamma(\beta y+b-y)\ {}^2\mathrm{F}_1[-n,y+a-n+z,\beta y+a+b-n+z,\beta]}{\Gamma(\beta y+a+b-n+z)} \tag{4.20}$$

and

$$\int_0^1(1-\alpha\beta)^{n}\ \alpha^{y+a-n-1}(1-\alpha)^{\beta y-y+b-1}\,d\alpha$$

$$=\frac{\Gamma(y+a-n)\ \Gamma(\beta\, y+b-y)\ \ {}^2\mathrm{F}_1[-n,y+a-n,\beta\, y+a+b-n,\beta]}{\Gamma(\beta y+a+b-n)}. \tag{4.21}$$

Putting these values in equation in (4.19), the Baye's estimator of α^z is obtained as

$$\hat{\alpha}^z=\frac{\Gamma(y+a-n+z)\ \ \Gamma(\beta\mathrm{y}+\mathrm{a}+\mathrm{b\text{-}n})\ \ {}^2\mathrm{F}_1[-n,y+a-n+z,\beta y+a+b-n+z,\beta]}{\Gamma(y+a-n)\ \Gamma(\beta y+a+b-n+z)\quad {}^2\mathrm{F}_1[-n,y+a-n,\beta y+a+b-n,\beta]}. \tag{4.22}$$

For z=1, we get the Baye's estimator of α as

$$\hat{\alpha}=\frac{(y+a-n)\ \ {}^2\mathrm{F}_1[-n,y+a-n+1,\beta y+a+b-n+1,\beta]}{(\beta y+a+b-n)\ \ \ {}^2\mathrm{F}_1[-n,y+a-n,\beta y+a+b-n,\beta]}. \tag{4.23}$$

For $\beta = 1$, we get the Baye's estimate of the size-biased logarithmic series distribution (2.8) as

$$\hat{\alpha} = \frac{y+a-n}{y+a+b}. \tag{4.24}$$

5. APPLICATIONS OF SIZE-BIASED DISTRIBUTIONS

The following examples are used to illustrate a few situations generating size-biased distributions and their applications. R-software has been used to facilitate the use of size-biased distributions to real life data. With the help of R, we can fit two- and three-parameter probability distributions. It computes the moment, Mle and Baye's estimates. Results are presented in the tables 1-5.

5.1. Data in Table 1 is regarding the defective teeth in 14 year old boys having at least one affected teeth.

Table 1

Number of Teeth affected	1	2	3	4	5	6	7	8	9	10	11	12	Total
Number of Boys	47	43	35	28	15	20	5	5	2	1	2	1	204

Since in the above data set no boys were found having zero teeth affected, therefore this indicates that the data can not be adequately described by a Poisson distribution and instead we should look for size-biased Poisson distribution (2.4). Moreover, the index of dispersion $I = \frac{\sum(x_i - \bar{x})^2}{\bar{x}}$ given by Selby (1965) comes out to be 302, giving a unit normal deviate of 4.95. For simple Poisson distribution the index of dispersion would be larger because of the frequency of zero-class group, which substantiates the use of size-biased Poisson distribution to above data.

5.2. We have fitted the models (2.8) and (3.6) to the data given in tables (2) and (3) by P.Garman (1923) and Student (1907) on counts of the number of European red mites on apple leaves and yeast cell counts observed per mm square respectively. For the choice of values of (a, b) in Baye's estimator, since there was no information about their values except that they are real and positive numbers. Therefore 25 combinations of values of (a, b) were considered for a, b=1, 2,3,4,5 and those values of a, b were selected for which the Baye's estimator $\hat{\alpha}$ has minimum variance. It was found that for a=b=2 and β =2.0, the Baye's estimator has minimum variance and χ^2 values between the simulated sample frequencies and the estimated Baye's frequencies were the least.

Table 2

No. of mites per leaf	Leaves Observed	Expected Frequency	
		SBLSD	SBGLSD
1	38	32.36	31.92
2	17	23.56	17.56
3	10	10.25	10.53
4	9	7.36	7.66
5	3	2.95	4.83
6	2	1.46	3.38
7	1	1.65	2.40
≥8	0	0.41	1.72
Total	80	80.00	80.00
χ^2		2.91	2.44
AIC		285	226
BIC		294	246
$\hat{\alpha}$		0.2568	0.407

Table 3

No. of cells per mm square (mm^2)	Observed Frequency	Expected Frequency	
		SBLSD	SBGLSD
1	128	130.43	129.56
2	37	36.09	37.12
3	18	14.56	15.02
4	3	3.54	3.02
5	1	1.84	1.02
> 6	0	0.54	1.26
Total	187	187.00	187.00
χ^2		1.5035	0.929
AIC		310	285
BIC		321	301
$\hat{\alpha}$		0.422	0.455

5.3. Data in table 4 shows the number of mothers (f_x) in Srilanka having at least one neonatal death according to number of neonatal deaths (x) [Meegama (1980)]. The models (2.6) and (3.2) have been fitted to this data for a=b=2 and β =0.5

Table 4

X	f_x	Expected Frequency	
		SBNBD	**SBGNBD**
1	567	548.38	537.22
2	135	164.51	165.03
3	28	28.79	35.65
4	11	3.84	6.69
5	5	0.48	1.41
Total	746	746	746
χ^2		37.52	16.45
AIC		295	234
BIC		308	255
$\hat{\alpha}$		0.05	0.04
$\hat{\beta}$			0.5

5.4. Data set in table 5 showing the number of workers N_i having i accidents. The models (2.4) and (3.4) have been fitted to this data.

Table 5

i	N_i	Expected Frequency	
		SBPD	**SBGPD**
1	2039	2034.27	2039.83
2	312	319.48	309.76
3	35	33.45	36.38
4	3	2.63	3.66
5	1	0.17	0.37
Total	2390	2390.00	2390.00
χ^2		0.772	0.069
AIC		594	321
BIC		704	365
$\hat{\lambda}_1$		0.3141	0.2631
$\hat{\lambda}$			0.0912

6. DISCUSSION AND CONCLUSION

The discussion on estimation and applications of size-biased distributions to this point demonstrates that they both have a solid theoretical underpinning and practical use to real life data. From AIC and BIC fit measures the proposed size-biased models appear to offer substantial improvement in fit over simple classical and simple generalized models. Also the fitting in these tables reveal that the size-biased distributions provide us better fits in the situations where zero-class is missing and simultaneously it has been shown that the generalized form of these distributions give generalized results in comparison to classical ones.

REFERENCES

1. Consul, P.C. and Jain, G.C. (1973). A generalization of the Poisson distribution. *Technometrics,* 15, 791-799.
2. Dennis, B. and Patil, G. (1984). The gamma distribution and weighted multimodal gamma distributions as models of population abtidance. *Math. Biosciences,* 68, 187-212.
3. Fisher, R.A. (1934). The effects of methods of ascertainment upon the estimation of frequencies. *Ann. Eugenics, 6*, 13-25.
4. Garman, P. (1923). The European red mites in Connecticut apple orchards. *Connecticut Agri. Exper. Station Bull.*, 252,103-125.
5. Gove, H.J. (2003). Estimation and application of size-biased distributions in forestry. *In Modeling Forest Systems*, A. Amaro, D. Reed and P.Soares *CAB International Wallingford UK*, 201-212.
6. Grosenbaugh, L.R. (1958). Point Sampling and Line Sampling: Probability Theory, Geometric Implications, Synthesis. *Occasional Paper 160*, USD. A Forest Service, Southern Forest Experiment Station.
7. Jain, G.C. and Consul, P.C. (1971). A generalized negative binomial distribution. *SIAM, J. App. Math.*, *4*(21), 501-513.
8. Jain, G.C. and Gupta, R.P. (1973). A logarithmic type distribution. *Trabjos Estadist.* 24, 99-105.
9. Lappi, J. and Bailey, R.L. (1987). Estimation-.of diameter increment function or other tree relations using angle-count samples. *Forest Science*, 33, 725-739.
10. Magnussen, S., Eggermont, P. and Lariccia, V.N. (1999). Recovering tree heights from airborne laser scanner data. *Forest Science*, 45(3), 407-422.
11. Meegama, S.A. (1980). Socio-economic determinants of infant and child mortality in Sri Lanka: an analysis of post-war experience. *World Fertility Survey Scientific Report 8.*
12. Mishra, A. and Singh, S.K. (1993). A size-biased Lagrangian Poisson distribution and its use in zero-truncated cases. *Ranchi University Mathematical Journal*, 24, 21-28.
13. Mir, K.A. (2007). Some contributions to Lagrangian probability distributions. *Unpublished Ph.D. thesis*, University of Kashmir, Srinagar India.
14. Patil, G.P. and Rao, C.R. (1977). The weighted distributions: A survey and their applications. *In Applications of Statistics (ed. P.R. Krishnaiah),* 383-405. North Holland Publications Co. Amsterdam.
15. Patil, G.P. and Rao, C.R. (1978). Weighted distributions and size biased sampling with applications to wild-life populations and human families. *Biometrics*, 34,179-189.
16. Rao, C.R. (1965). On discrete distributions arising out of methods of ascertainment In: Patil, G.P. (ed.) *Classical and Contagious Discrete Distributions*. Statistical Publishing Society, Calcutta, 320-332.
17. Selby, B. (1965). On the index of dispersion as a test statistic. *Biometrika*, 52(3/4), 627-629.
18. Student (1907). On the error of counting with haemacytometer. *Biometrika, 5*, 351-360.
19. Taillie, C., Patil, G.P. and Hennemuth, R. (1995). Modeling and analysis of recruitment distributions. *Ecological and Environmental Statistics*, 2(4), 315-329.
20. Van Deusen, P.C. (1986). Fitting assumed distributions to horizontal point sample diameters. *Forest Science,* 32, 146-148.

ON SOME PROPERTIES OF GEOMETRIC POISSON DISTRIBUTION*

Masood Anwar[1] and **Munir Ahmad**[2]
[1] Department of Mathematics, COMSATS Institute of Information Technology Islamabad, Pakistan. Email: masoodanwar@comsats.edu.pk
[2] National College of Business Administration & Economics, Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

The geometric Poisson (also called Pólya-Aeppli) distribution is a particular case of compound Poisson distribution. In this article we prove that geometric Poisson distribution (GPD) is infinitely divisible, not log-concave, unimodal and also obtain its survival function. Moreover, first order negative moment of GPD is given and finally its characterization is done via recursive relation of factorial moments.

KEYWORDS

Infinite divisibility; log-concavity; unimodality; survival function; negative moment; recursive relation; characterization.

1.INTRODUCTION

The geometric Poisson (or Pólya-Aeppli) distribution is a particular case of classical compound Poisson distribution where the contribution of each term is distributed according to the geometric distribution. In real life situations, it has many applications appear in the literature. Randolph and Sahinoglu (1995) presented the application of geometric Poisson distribution for control of defects in software, and Chen et al. (2005) developed the geometric Poisson CUSUM control scheme for the process control. Robin (2002) and Robin et al. (2007) modeled it for the distribution of overlapping word occurrences. Rosychuk et al. (2006) used it to model DNA substitution. This model assumed that substitution events were Poisson distributed in time and the number of substitutions associated with each event was geometric distributed. Özel and Ìnal (2010) presented its application to traffic accidents data.

Johnson et al. (1992) derived a linear formula to compute the probabilities of the compound Poisson distribution that can be simplified in the geometric Poisson case. Nuel (2008) obtained recurrence relation for the GPD using Kummer's confluent geometric function. Since some terms could be out of the machine range and set to zero, an algorithm has been prepared for the logarithmic version of the cumulative distribution function for the GPD. However, a direct formula and an algorithm have not been obtained for the probability function of the GPD. Özel and Ìnal (2010) derived the explicit probability function of the GPD and obtained an algorithm for the computation of the probabilities. Ata and Özel (2012) derived the survival functions for the geometric-Poisson process and other class of compound Poisson process. Özel (2013) provided the moments, cumulants,

*Published in Pak. J. Statist. (2014), Vol. 30(2).

skewness, kurtosis and covariance of the univariate compound Poisson process including Pólya-Aeppli process or geometric Poisson process as special case.

Although many studies have already been done using the GPD but the question of its infinite divisibility, log-concavity (strong unimodality), unimodality have not been addressed. The property of unimodality is very important for many decomposition problems of probabilistic and statistical nature as indicated in the well-known book by Medgyessy (1977) and by Steutel and Van Harn (1979). Many discrete distributions on the lattice of integers have a unimodal character for which comparable results may be of interest. Unimodality is also of interest in connection with optimization and mathematical programming. Arguments involving unimodality have been used quite often in statistical inference. When applying the method of maximum likelihood for the estimation of parameters, the unimodality of the likelihood function often facilitates the required computation. In general, most of the likelihood functions can be shown to be unimodal with respect to the parameters involved. Some inequalities depend on unimodality like, Gauss's inequality, Vysochanskiï–Petunin inequality. The recurrence of symmetric random walks involves the concepts of unimodality and peakedness comparisons.

Kielson and Gerber (1971) have proved a number of results on the strong unimodality of discrete distributions. A necessary and sufficient condition that the sequence p_x be strongly unimodal is that p_x be log-concave, i.e. $\left(p_x\right)^2 \geq p_{x+1}p_{x-1}$ for all values of x. But this does not seem to apply to the GPD. However, we use Theorem-1 in Hansen (1988) to prove that GPD is not log-concave (strongly unimodal) and prove that GPD is unimodal using lemma by Steutel and Van Harn (1977). Consul and Famoye (1986) use the same lemma for proving the unimodality of generalized Poisson distribution.

In this study, we provide a recursive formula for computation of probabilities of GPD and prove that GPD is infinitely divisible, not log-concave (strongly unimodal), unimodal and obtain its survival function. Moreover, first order negative moment of GPD is given and finally, characterize it via recursive relation of factorial moments. In Section 2, we present some basic definitions and lemmas that will be used in subsequent sections. Section 3 deals with infinite divisibility and in Section 4 we prove that GPD is not strongly unimodal. Section 5 addresses the unimodality of GPD. Survival function is obtained in Section 6 and first order negative moment of GPD is given in Section 7. In Section 8, a characterization theorem based on the recursive relation of factorial moments is given. The conclusion is given in Section 9.

2. PRELIMINARIES

In this section, we present some basic definitions and lemmas that will be used in the subsequent sections.

Definition 2.1

Let N be a Poisson random variable with parameter $\lambda > 0$ and let $Y_i, i = 1, 2, 3, ...$ be i.i.d random variables, independent of N. Then, X has a compound Poisson distribution if it is defined as

$$X = \sum_{i=1}^{N} Y_i \;. \qquad (1)$$

If $E(Y_i) = \eta, V(Y_i) = \sigma^2, i = 1, 2, 3, ...,$ the expected value and variance of X are $E(X) = \lambda\eta, V(X) = \lambda(\sigma^2 + \eta^2)$, respectively.

Lemma 2.1

The probability function of X is given by

$$p_X(k) = P(X = k) = \sum_{n=0}^{\infty} P(Y_1 + Y_2 + \cdots + Y_n = k \mid N = n) P(N = n), \; k = 0, 1, 2, ... \quad (2)$$

The GPD stated by Johnson et al. (1992) as in Lemma 2.2.

Lemma 2.2

If N has a Poisson random variable with parameter $\lambda > 0$ and $Y_i, i = 1, 2, 3, ...,$ are geometric distribution with parameter θ in Equation (2), the probability mass function (pmf) of X is given by

$$p_X(k) = P(X = k) = \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \binom{k-1}{n-1} \theta^n (1-\theta)^{k-n}, \; k = 1, 2, 3, ..., \qquad (3)$$

$$p_X(0) = e^{-\lambda} \,,$$

where $\lambda > 0, 0 < \theta < 1$ and recall that $E(X) = \lambda/\theta$ and $V(X) = \lambda(2-\theta)/\theta^2$.

Lemma 2.3

If the random variables $Y_i, i = 1, 2, 3, ...$ are geometric distributed i.e. $P(Y_i = j) = p_j = \theta(1-\theta)^{j-1}, j = 1, 2, 3, ...$ then the common probability generating function of $Y_i, i = 1, 2, 3, ...$ is given by

$$g_Y(s) = \left(\frac{\theta}{1-\theta}\right) \sum_{j=1}^{\infty} \left(s(1-\theta)\right)^j = \frac{\theta s}{1-(1-\theta)s} \,. \qquad (4)$$

Lemma 2.4

If X has the GPD then the probability generating function (pgf) of X is given by

$$g_X(s) = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \left[g_Y(s)\right]^n = \exp\left(\frac{\lambda(s-1)}{1-(1-\theta)s}\right), \lambda > 0, 0 < \theta < 1 \,. \qquad (5)$$

Lemma 2.5

If X has the GPD, then the factorial moment generating function (fmgf) of X is given by

$$g_X(1+t) = \exp\left(\frac{\lambda t}{1-(1-\theta)(1+t)}\right), \lambda > 0, 0 < \theta < 1. \qquad (6)$$

Lemma 2.6 (Steutel 1970)

Let $(p_n)_0^\infty$ be a probability distribution on the non-negative integers with $p_0 > 0$ is infinitely divisible if and only if it satisfies

$$(n+1)p_{n+1} = \sum_{k=0}^{n} r_k p_{n-k}, \quad n = 0,1,2,...., \qquad (7)$$

with non-negative r_k and necessarily, $\sum_{k=0}^{\infty} r_k/(k+1) < \infty$.

Lemma 2.7 (Hansen 1988)

Let (p_n) and (r_n) be related by (7) with $r_k \geq 0$, $p_0 > 0$ and let (r_n) be log-concave. Then (p_n) is log-concave if and only if $r_0^2 - r_1 \geq 0$.

Lemma 2.8 (Steutel and Van Harn 1979)

Let $(p_x)_0^\infty$ be a distribution on the non-negative integers with pgf $g_X(s)$ satisfying

$$\frac{d}{ds}\ln g_X(s) = R(s) = \sum_{k=0}^{\infty} r_k s^k \qquad (8)$$

where r_k are non-negative. Then $(p_x)_0^\infty$ is unimodal if $(r_k)_0^\infty$ is non-increasing, and $(p_x)_0^\infty$ is non-increasing if and only if in addition $r_0 \leq 1$.

Lemma 2.9 (Park 1972)

Let X be a discrete random variable with probability function p_x, for $0 \leq s \leq 1$,

$$E(X+A)^{-k} = \int_0^1 g_k(s)ds, \qquad (9)$$

where $(X+A) > 0$, k is non-negative integer and $g_k(s)$ is the pgf of $(X+A)^k - 1$.

3. INFINITELY DIVISIBLE

In this section, we examine the infinite divisibility property of GPD by making use of Lemma 2.6 by Stuetel (1970).

Theorem 3.1

The GPDwith pmf (3) is infinitely divisible.

Proof

To prove infinite divisibility of GPD, we need to show that (3) satisfies (7).

The pgf of GPDgiven by (5) is

$$g_X(s) = \exp\left(\frac{\lambda}{\theta}\eta(s)\right),\ \lambda > 0,\ 0 < \theta < 1, \tag{10}$$

where $\eta(s) = (s-1)\left(1 - z(s-1)\right)^{-1}$, $z = (1-\theta)/\theta$.

If D denotes the differential operator d/ds, then after successive differentiation of (10) we get

$$D^{n+1}\left(g_X(s)\right) = \frac{\lambda}{\theta}\sum_{k=0}^{n}\binom{n}{k}D^{n-k}\left(g_X(s)\right)D^{k+1}\left(\eta(s)\right),\ n \geq 0,\ \lambda > 0,\ 0 < \theta < 1, \tag{11}$$

where

$$D^{k+1}\left(\eta(s)\right) = (k+1)!\,z^k\left(1 - z(s-1)\right)^{-(k+2)},\ z = (1-\theta)/\theta .$$

Setting $s = 0$, in (11) gives;

$$D^{n+1}\left(g_X(0)\right) = \frac{\lambda}{\theta}\sum_{k=0}^{n}\binom{n}{k}D^{n-k}\left(g_X(0)\right)D^{k+1}(\eta(0)),\ n = 0,1,2,\ldots., \tag{12}$$

where $D^{k+1}\left(\eta(0)\right) = (k+1)!\theta^2(1-\theta)^k$.

As

$$\left[D^{n+1}\left(g_X(s)\right)\right]_{s=0} = (n+1)!\,p_{n+1}. \tag{13}$$

Substituting (13) in (12) we get;

$$(n+1)!\,p_{n+1} = \lambda\theta\sum_{k=0}^{n}\binom{n}{k}(n-k)!\,p_{n-k}(k+1)!(1-\theta)^k,$$

which on simplification gives

$$(n+1)\, p_{n+1} = \sum_{k=0}^{n} r_k\, p_{n-k}, \quad n = 0,1,2,\ldots, \tag{14}$$

where $r_k = \lambda\theta(k+1)(1-\theta)^k$ are non-negative, necessarily

$$\sum_{k=0}^{\infty} r_k/(k+1) = \lambda < \infty .$$

Hence complete the proof.

4. NOT STRONGLY UNIMODAL

We use Lemma 2.7 by Hansen (1988) to prove that GPD is not strongly unimodal.

Theorem 4.1

The GPDwith pmf (3) is not log-concave for all values of θ and λ.

Proof

In Theorem 3.1, it is shown that (p_n) and (r_n) are related by (14) with $p_0 = e^{-\lambda} > 0$, let (r_n) be log-concave. In order to prove the log-concavity of GPD, we have to show that $\frac{r_0^2}{r_1} \geq 1$. Therefore

$$\frac{r_0^2}{r_1} = \frac{\lambda}{2\left(\theta^{-1}-1\right)}, \; 0<\theta<1, \lambda>0,$$

which can be $\leq or \geq 1$ depending upon the values of θ and λ, so the GPD is not log-concave for all values of θ and λ.

5. UNIMODALITY

In this section, we use Lemma 2.8 by Steutel and Van Harn (1979) for proving unimodality of GPD.

Theorem 5.1

The GPD with pmf (3) is unimodal for all values of θ and λ.

Proof

The pgf of GPD satisfy

$$\frac{d}{ds}\ln g_X(s) = R(s) = \frac{\lambda\theta}{\left(1-(1-\theta)s\right)^2} = \sum_{k=0}^{\infty} \lambda\theta\left(k+1\right)\left(1-\theta\right)^k s^k ,$$

where $r_k = \lambda\theta(k+1)(1-\theta)^k$ are non-negative.

Therefore

$$\frac{r_k}{r_{k-1}} = \left(1+k^{-1}\right)(1-\theta) \le 1,$$

as $0<\theta<1$ and $\lim\limits_{k\to\infty}\left(1+k^{-1}\right) \to 1$.

Hence $\left(r_k\right)_0^\infty$ is non-increasing, and so the GPD is unimodal for all values of θ and λ.

When $r_0 = \lambda\theta \le 1$, $\left(P_x\right)_0^\infty$ becomes non-increasing.

Accordingly, the mode will be at $x=0$ if $\lambda\theta<1$ and at the dual points $x=0$ and $x=1$ if $\lambda\theta=1$.

6. SURVIVAL FUNCTION

The survival function (sf) of a nonnegative discrete random variable X is defined as the probability $S(x)=1-P(X\le x)$.

By definition

$$P(X\le x) = \sum_{n=0}^{\infty} P\left(X\le x \middle| N=n\right) P\left(N=n\right),\ \ x=0,1,2,\ldots.$$

As $P\left(X\le x \middle| N=n\right)$ is a negative binomial distribution, we get;

$$P\left(X\le x\right) = \sum_{n=1}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\sum_{j=0}^{x}\binom{x-j-1}{n-1}\theta^n\left(1-\theta\right)^{x-j-n}, x\ge 1,$$

$$P\left(X\le 0\right)=e^{-\lambda}, \text{ where } \lambda>0, 0<\theta<1.$$

It follows that

$$S\left(0\right)=1-e^{-\lambda},\ S\left(x\right)=1-e^{-\lambda}\sum_{j=0}^{x}\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\binom{x-j-1}{n-1}\theta^n(1-\theta)^{x-j-n},$$

$$x\ge 1, \lambda>0, 0<\theta<1.$$

7. NEGATIVE MOMENTS

In this section, we give first order negative moment of GPD using Lemma 2.9 by Park (1972).

Theorem 7.1

Let X be a non-negative integer valued random variable with pmf (3). Then

$$E\left(X+A\right)^{-1}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\lambda)^{j}}{j!}\frac{(j)_{k}}{k!}(1-\theta)^{k}B\left(k+A,j+1\right),$$

where $A>0$, $(j)_0=1$, $(j)_k=j(j+1).....(j+k-1)$, for $k=1,2,......$

Proof

According to Lemma 2.9,

$$E\left(X+A\right)^{-k}=\int_0^1 g_k(s)ds\text{, where } A>0 \text{ and } g_k(s)=E\left(s^{(X+A)^k-1}\right), 0\le s\le 1.$$

Taking $k=1$, we have

$$E\left(X+A\right)^{-1}=\int_0^1 g_1(s)ds\text{, where } g_1(s)=E\left(s^{(X+A)-1}\right),\ 0\le s\le 1,$$

$$E\left(X+A\right)^{-1}=\int_0^1 s^{A-1}\exp\left(-\lambda\left(1+\frac{\theta s}{1-s}\right)^{-1}\right)ds,$$

$$=\int_0^1 s^{A-1}\sum_{j=0}^{\infty}\frac{\left(-\lambda\left(1+\frac{\theta s}{1-s}\right)^{-1}\right)^{j}}{j!}ds,$$

$$=\sum_{j=0}^{\infty}\frac{(-\lambda)^{j}}{j!}\int_0^1 s^{A-1}\left(1+\frac{\theta s}{1-s}\right)^{-j}ds,$$

$$=\sum_{j=0}^{\infty}\frac{(-\lambda)^{j}}{j!}\int_0^1 s^{A-1}(1-s)^{j}\sum_{k=0}^{\infty}\frac{(j)_{k}}{k!}\left(s(1-\theta)\right)^{k}ds,$$

where

$$(j)_0=1, (j)_k=j(j+1).....(j+k-1)\text{, for } k=1,2,.....$$

$$=\sum_{j=0}^{\infty}\frac{(-\lambda)^{j}}{j!}\sum_{k=0}^{\infty}\frac{(j)_{k}}{k!}(1-\theta)^{k}\int_0^1 s^{k+A-1}(1-s)^{j}ds,$$

$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\lambda)^{j}}{j!}\frac{(j)_{k}}{k!}(1-\theta)^{k}B\left(k+A,j+1\right)\text{, for } A>0.$$

8. CHARACTERIZATION THEOREM

In this section GPD is characterized via the recursive relation of factorial moments.

Theorem 8.1

Let $g_X(s) = \sum_{k=0}^{\infty} s^k P(X = k)$ be the pgf of a distribution with support $0, 1, 2, \ldots.,$ and parameters, $\lambda, \lambda > 0$, $\theta, 0 < \theta < 1$. Then

$$\mu'_{[r]} = \frac{\lambda}{\theta} \sum_{j=0}^{r-1} \binom{r-1}{j} (j+1)! \left(\frac{1-\theta}{\theta}\right)^{j} \mu'_{[r-1-j]}, \text{ for - } r \geq 1 \text{ with } \mu'_{[0]} = 1, \qquad (15)$$

holds if and only if X has GPD with pmf (3).

Proof

Suppose that X follows GPD then the fmgf of X given by (6) is

$$g_X(1+t) = \exp\left(\frac{\lambda}{\theta}\eta(t)\right), \quad \lambda > 0, 0 < \theta < 1, \qquad (16)$$

where $\eta(t) = t(1 - zt)^{-1}$, $z = (1-\theta)/\theta$.

If D denotes the differential operator d/dt, then we can obtain a recursive relationship between factorial moments by successive differentiation of (16) as

$$D^r\left(g_X(1+t)\right) = \frac{\lambda}{\theta} \sum_{j=0}^{r-1} \binom{r-1}{j} D^{r-1-j}\left(g_X(1+t)\right) D^{j+1}(\eta(t))$$
$$, r \geq 1, \lambda > 0, 0 < \theta < 1, \qquad (17)$$

where

$$D^{j+1}\left(\eta(t)\right) = (j+1)! z^j (1 - zt)^{-(j+2)},$$

and

$$D^{j+1}\left(\eta(0)\right) = z^j (j+1)!.$$

As $\left[D^r\left(g_X(1+t)\right)\right]_{t=0} = \mu'_{[r]}$, $\left[g_X(1+t)\right]_{t=0} = \mu'_{[0]} = 1$.

Setting $t = 0$ in (17) gives (15).

Suppose (15) holds and after putting $r = 1, 2, 3, \ldots..$, we get;

$$\mu'_{[1]} = \frac{\lambda}{\theta}, \ \lambda > 0, 0 < \theta < 1,$$

$$\mu'_{[2]} = \frac{\lambda}{\theta}\left[\frac{\lambda}{\theta} + 2!\left(\frac{1-\theta}{\theta}\right)\right],$$

$$\mu'_{[3]} = \frac{\lambda}{\theta}\left[\left(\frac{\lambda}{\theta}\right)^2 + (3)2!\frac{\lambda}{\theta}\left(\frac{1-\theta}{\theta}\right) + (3)3!\left(\frac{1-\theta}{\theta}\right)^2\right],$$

$$\mu'_{[4]} = \frac{\lambda}{\theta}\left[\left(\frac{\lambda}{\theta}\right)^3 + (6)2!\left(\frac{\lambda}{\theta}\right)^2\left(\frac{1-\theta}{\theta}\right) + (6)3!\frac{\lambda}{\theta}\left(\frac{1-\theta}{\theta}\right)^2 + 4!\left(\frac{1-\theta}{\theta}\right)^3\right],$$

$$\mu'_{[5]} = \frac{\lambda}{\theta}\left[\left(\frac{\lambda}{\theta}\right)^4 + (10)2!\left(\frac{\lambda}{\theta}\right)^3\left(\frac{1-\theta}{\theta}\right) + (20)3!\left(\frac{\lambda}{\theta}\right)^2\left(\frac{1-\theta}{\theta}\right)^2 \right.$$
$$\left. + (10)4!\left(\frac{\lambda}{\theta}\right)\left(\frac{1-\theta}{\theta}\right)^3 + 5!\left(\frac{1-\theta}{\theta}\right)^4\right], \ldots\ldots .$$

Factorial moment generating function is given by

$$g_X(1+t) = \sum_{r=0}^{\infty} \mu'_{[r]} \frac{t^r}{r!}. \tag{18}$$

On substituting $\mu'_{[0]}, \mu'_{[1]}, \mu'_{[2]}, \mu'_{[3]}, \mu'_{[4]}, \mu'_{[5]}, \ldots\ldots$ in (18) we get;

$$g_X(1+t) = 1 + \alpha\frac{t}{1!} + \alpha\left[\alpha + 2!z\right]\frac{t^2}{2!} + \alpha\left[\alpha^2 + (3)2!\alpha z + (3)3!z^2\right]\frac{t^3}{3!}$$
$$+ \alpha\left[\alpha^3 + (6)2!\alpha^2 z + (6)3!\alpha z^2 + 4!z^3\right]\frac{t^4}{4!}$$
$$+ \alpha\left[\alpha^4 + (10)2!\alpha^3 z + (20)3!\alpha^2 z^2 + (10)4!\alpha z^3 + 5!z^4\right]\frac{t^5}{5!}\cdots,$$

where $z = (1-\theta)/\theta$, $\alpha = \dfrac{\lambda}{\theta}$.

$$g_X(1+t) = 1 + \alpha t\left(1 + zt + (zt)^2 + (zt)^3 + \ldots\right)$$
$$+ \frac{(\alpha t)^2}{2!}\left(1 + 2zt + \frac{2.3}{2!}(zt)^2 + \frac{2.3.4}{3!}(zt)^3 + \ldots\right)$$
$$+ \frac{(\alpha t)^3}{3!}\left(1 + 3zt + \frac{3.4}{2!}(zt)^2 + \ldots\right) + \ldots,$$

$$g_X(1+t) = 1 + \left[\alpha t(1-zt)^{-1}\right] + \frac{\left[\alpha t(1-zt)^{-1}\right]^2}{2!} + \frac{\left[\alpha t(1-zt)^{-1}\right]^3}{3!} + \ldots,$$

$$g_X(1+t) = \exp\left[\alpha t(1-zt)^{-1}\right],$$

which after simplification gives

$$g_X(1+t) = \exp\left(\frac{\lambda t}{1-(1-\theta)(1+t)}\right),\ \lambda > 0, 0 < \theta < 1.$$

or

$$g_X(s) = \exp\left(\frac{\lambda(s-1)}{1-(1-\theta)s}\right),\ \lambda > 0, 0 < \theta < 1.$$

By calculating the kth derivative of $g_X(s)$ at $s = 0$ as

$$P(X=0) = g_X(0) = e^{-\lambda},\ P(X=k) = \frac{\left|\partial^k/\partial s^k\left(g_X(s)\right)\right|_{s=0}}{k!},\ k = 1, 2, 3, \ldots$$

gives (3).

9. CONCLUSION

In the present article, some statistical properties of GPD including infinite divisibility, unimodality, log-concavity or strong unimodality, survival function, first order negative moment are addressed. Also, a characterization theorem is given based on the recursive relation of factorial moments.

The property of unimodality is very important for many decomposition problems of probabilistic and statistical nature. It is also of interest in connection with optimization and mathematical programming. Arguments involving unimodality have been used quite often in statistical inference. When applying the method of maximum likelihood for the estimation of parameters, the unimodality of the likelihood function often facilitates the required computation. Some inequalities depend on unimodality. The recurrence of symmetric random walks involves the concepts of unimodality and peakedness comparisons. A number of authors have discussed the fatigue, creep, fracture, shrinkage, cracking and deformation of concrete flange on the basis of negative moments. We expect these properties to be useful in dealing with the practical problems and to play a very important role in the probability theory. Further properties including shape of hazard rate function, order statistics, mean and median deviations, maximum likelihood estimation and asymptomatic confidence intervals for the parameters are under construction and may appear in the next communication.

REFERENCES

1. Ata, N. and Özel, G. (2012). Survival functions for the frailty models based on the discrete compound Poisson process. *J. Statist. Comput. Simul.*, 82, 2105-2116.

2. Chen, C.W., Randolph, P. and Tian-Shy, L. (2005). Using CUSUM control schemes for monitoring quality levels in compound Poisson Production environment: The geometric Poisson Process. *Quality Engineering*, 17, 207-217.
3. Consul, P.C. and Famoye, F. (1986). On the unimodality of generalized Poisson distribution. *Statistica Neerlandica*, 40,117-122.
4. Hansen, B.G. (1988). On log-concave and log-convex infinitely divisible sequences and densities. *Ann. Prob.*, 16, 1832-1839.
5. Johnson, N.L., Kotz, S. and Kemp, A.W. (1992). *Univariate Discrete Distributions*. Wiley, New York.
6. Kielson, J. and Gerber, H. (1971). Some results for discrete unimodality. *J. Amer. Statist. Assoc.*, 66,386-389.
7. Medgyessy, P. (1977). *Decomposition of superpositions of density functions and discrete distributions*. Wiley, New York.
8. Nuel, G. (2008). Cumulative distribution function of a geometric Poisson distribution. *J. Statist. Comput. Simul.*, 78, 385-394.
9. Özel, G. and Ìnal, C. (2010). The probability function of a geometric Poisson distribution. *J. Statist. Comput. Simul.*, 80, 479-487.
10. Özel, G. (2013). On the moments characteristics for the univariate compound Poisson and bivariate compound Poisson processes with applications. *RevistaColombiana de Estadistica*, 36(1), 59-77.
11. Park, H.S. (1972). On negative moments of positive random variables. *J. Korea Soc. Math. Edu.*,11, 25-26.
12. Randolph, P. and Sahinoglu, M. (1995). A stopping rule for a compound Poisson random variable. *Applied Stochastic Models and Data Analysis*, 11,135-143.
13. Robin, S. (2002). A compound Poisson model for word occurrences in DNA sequences. *Appl. Stat.*, 51, 437-451.
14. Robin, S., Schbath, S. and Vandewalle, V. (2007). Statistical test to compare motif count exceptionalities. *Bioinformatics*, 8, 1-20.
15. Rosychuk, R.J., Huston, C. and Prasad, N.G.N. (2006). Spatial events cluster detection using a compound Poisson distribution. *Biometrics*, 61, 465-470.
16. Steutel, F.W. and Van Harn, K. (1979). Discrete analogues of self-decomposability and stability. *Ann. Prob.*, 7, 893-899.

ON A MULTI-STAGE 2^n FACTORIAL SURVEY DESIGN: A USEFUL SURVEY DESIGN FOR DEVELOPING AREAS*

Munir Ahmad, H. Khrarjian and **T.A. Bader**
Department of Mathematical Sciences & Department of Civil Engineering,
University of Petroleum and Minerals, Dhahran, Saudi Arabia

ABSTRACT

In the selection of a proper sample survey design in developing areas where prior information on designs does not exist, it is sometimes feasible to conduct an experiment where all possible factors or stages (if a choice of a factor or stage is available) are tested. In this paper, a survey design is proposed to test 'n' factors (or stages) each at two levels (2-sub-stages). Some special cases are also discussed. Some estimators of population totals and means and their variances are developed. An application from the Civil Engineering regarding solid waste is given to illustrate the method.

KEY WORDS

Completely randomized design, analysis of variance.

1. INTRODUCTION

In developing areas where prior information on any aspect of the problem under study is not available, it is not possible to follow a particular pattern of standard sampling design technique. These situations may exist in many fields of investigation. In agriculture, it may be desired to estimate the effect of a particular treatment (say, fertilizer) on a piece of land randomly selected from a vast area of different soil heterogeneity. Before an experimental design is selected, it may be necessary to decide whether or not plots are selected as a final stage in a multistage sampling scheme. Suppose in biological sciences area, an investigation is made on the reliable estimates of the immunization status of a population. Ali and Heiner (1971) conducted a survey of vaccination status on past small-pox experience of an urban population of West Pakistan. Besides investigation of vaccination status, the intention was also to develop a sample survey design which would achieve an acceptable degree of reliability of the estimates while utilizing a minimum of personnel and resources, and would provide a model for similar surveys in other underdeveloped areas. They used a three-stage cluster sampling and have shown advantages of the design.

Similar conditions may exist in many branches of engineering or social sciences. In this paper, we propose a mixture of multistage random sampling with a factorial experiment in a standard design. In case of selection of an efficient design or a master design for use in other similar surveys, it is essential to conduct an experiment to determine sizes at every stage. Suppose, an area is divided into blocks, blocks are sub-divided into structures, and structures into housing units. Housing units may be considered as clusters of populations or clusters of families or clusters of persons.

*Published in Pak. J. Statist. (1986 B), 2(3).

Consider blocks, as structures and housing units as 3 factors each at equal or unequal number of levels.

2. A THREE-STAGE 2 x 2 x 2 FACTORIAL SURVEY DESIGN WITH AN EQUAL NUMBER OF OBSERVATIONS PER CELL

Suppose blocks, structures and housing units are considered as three-stages. Two levels of blocks, structures and housing units are considered. The scheme is given in Table 1:

Table 1
Three-Stage 2 x 2 x 2 Factorial Experiment with one Replicate

Stages	**Combinations**							
	(l)	**(h)**	**(s)**	**(hs)**	**(b)**	**(bh)**	**(bs)**	**(bhs)**
First Stage								
≠ Blocks Second Stage	2	2	2	2	4	4	4	4
≠ Structures Third Stage	2	2	4	4	2	2	4	4
≠ Housing Units	2	4	2	4	2	4	2	4
Total of Housing Units in one Replicate	8	16	16	32	16	32	32	64

In Table 1, the notations for factors, stages and levels are defined as follows:

- (l) denotes three factors at lower levels (2 blocks, 2 structures and 2 houses),
- (h) denotes upper level (4 houses) of housing units, and lower levels of blocks and structures,
- (s) denotes upper level of structures (4 structures) and lower levels of blocks or housing units,
- (hs) denotes upper levels of housing units and structures, and lower level of blocks, and so on.

We further define some notations:

N = the number of blocks in the area.
n = the number of sample blocks (first-stage units)
M_i = the number of structures in the ith block of the sample.
m_i = the number of structures in the sample from the ith block (second-stage units).
Q_{ij} = the number of houses in the jth structure of the ith block.
q_{ij} = the number of houses in the sample from jth structure and of the ith block.
y_{ijk} = the information from the kth house of the jth structure of the ith block.

The above notations relate to any one combination. N, M_i and Q_{ij} are known and fixed numbers.

3. METHOD

Basically there are two ways of handling the data. Each cell will give rise to an estimate of total and its variance. A general rule given by Durbin (1953, 1967) and Raj (1956, 1966), Rao (1975), Brewer and Hanif (1970) can be used to obtain a linear unbiased multistage estimator of population total as follows:

Suppose y' denotes the linear estimator of population total. Then

$$y' = \sum_{i=1}^{n} a_{is} y'_i$$

where a_{is} is a real number determined for each sample ‘s’. Define

$$a'_{is} = \begin{cases} a_{is} & \text{if ith population unit is in the sample.} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$y' = \sum_{i=1}^{N} a'_{is} y'_i, \text{ is the UBE of y.}$$

if and only if $E\left(a'_{is}\right) = \sum_{i=1}^{n} a_{is} P_s = 1,$ where p(s) is the probability of selecting the sample 's'. The variance of y' is

$$\text{var}\left(y'\right) = \text{var}\left(\sum_{i=1}^{N} a'_{is} y'_i\right)$$

$$= \frac{N\left(N-n\right)}{n} s_{1y}^2 + \frac{N}{n} \sum_{i=1}^{N} M \frac{\left(M_i - m_i\right)}{m_i} s_{2iy}^2$$

Various modifications of $\text{var}\left(y'\right)$ can be made by assigning different values to a_{is} with different methods of selection.

4. ESTIMATION OF TOTAL AND ITS VARIANCE

Let $y_{ijk(t)}$ denote an observation from the kth housing unit in the jth structure of the ith block at given level, e.g., upper level or lower level of block, structure or housing unit under the treatment combination 't' where $t = 1, 2, \ldots, 8$ and

(I) = 1: (1) is designated by the number 1 and stand for lower levels of the 3 factors.

(h) = 2: denotes lower level of block and structures and upper level of housing units.

Similarly, (s) =3, (hs) =4, (b) =5, (bh) =6, (bs) =7, (bhs) =8.

Observations as recorded under each treatment combination will be denoted by the following notations:

(I) $y_{ijk(1)},\ i=1,2,...,n_{(1)},\ j=1,2,....m_{i(1)},\ k=1,2,....q_{ij(1)}$

(h) $y_{ijk(2)},\ i=1,2,...,n_{(2)},\ j=1,2,....m_{i(2)},\ k=1,2,....q_{ij(2)}$

(s) $y_{ijk(3)},\ i=1,2,...,n_{(3)},\ j=1,2,....m_{i(3)},\ k=1,2,....q_{ij(3)}$

(hs) $y_{ijk(4)},\ i=1,2,...,n_{(4)},\ j=1,2,....m_{i(4)},\ k=1,2,....q_{ij(4)}$

(b) $y_{ijk(5)},\ i=1,2,...,n_{(5)},\ j=1,2,....m_{i(5)},\ k=1,2,....q_{ij(5)}$

(hb) $y_{ijk(6)},\ i=1,2,...,n_{(6)},\ j=1,2,....m_{i(6)},\ k=1,2,....q_{ij(6)}$

(bs) $y_{ijk(7)},\ i=1,2,...,n_{(7)},\ j=1,2,....m_{i(7)},\ k=1,2,....q_{ij(7)}$

(bhs) $y_{ijk(8)},\ i=1,2,...,n_{(8)},\ j=1,2,....m_{i(8)},\ k=1,2,....q_{ij(8)}$

where a number in the parentheses stands for that particular combination. The number of observations in rows are schematically different from each other. The totals in rows are not comparable and as such averages may be computed and analysis made. However, analysis of variance can be performed as if a 2^3 factorial experiment had been designed in a completely randomized design.

The following formulae are obtained for estimating totals and their variances for each combination (scheme of 3-stage sampling):

$$y'_{(t)} = \frac{N}{n_{(t)}} \sum_{i=1}^{n} \frac{M_i}{m_{i(t)}} \sum_{j=1}^{m_{i(t)}} \frac{Q_{ij}}{q_{ij(t)}} \sum_{k=1}^{q_{is(t)}} y_{ijk(t)}, \qquad t=1,2,...8.$$

$$\text{var}\left(y'_{(t)}\right) = \frac{N\left(N-n_{(t)}\right)}{n_{(t)}} s^2_{1y(t)} + \frac{N}{n_{(t)}} \sum_{i=1}^{N} \frac{M_i\left(M_i - m_{i(t)}\right)}{m_{i(t)}} s^2_{2y(t)}$$

$$+ \frac{N}{n_{(t)}} \sum_{i=1}^{N} \frac{M_i}{m_{i(t)}} \frac{Q_{ij}\left(Q_{ij} - q_{ij(t)}\right)}{q_{ij(t)}} s^2_{2ijy(t)}, \qquad t=1,2,...8.$$

where

$$s^2_{1y(t)} = \frac{1}{N-1} \sum_{i=1}^{N} \left(Y_{i(t)} - \bar{Y}_{(t)}\right)^2$$

$$s^2_{2iy(t)} = \frac{1}{M_i - 1} \sum_{i=1}^{M_i} \left(Y_{ij(t)} - \bar{Y}_{i(t)}\right)^2, \text{ and}$$

$$s^2_{2ijy(t)} = \frac{1}{Q_{ij} - 1} \sum_{K=1}^{Q_{ij}} \left(Y_{ijk(t)} - \bar{Y}_{ij(t)}\right)^2,$$

$y'_{(t)}$ is an unbiased estimate of $Y_{(t)}$. The unbiased variance estimator of $Var\left(y'_{(t)}\right)$ is given by

$$Var\left(y'_{(t)}\right)=\frac{N\left(N-n_{(t)}\right)}{n_{(t)}}s_{1y(t)}^{2}+\frac{N}{n_{(t)}}\sum_{i=1}^{n_{(t)}}\frac{M_i\left(M_i-m_{i(t)}\right)}{m_{i(t)}}s_{2iy(t)}^{2}$$

$$+\frac{N}{n_{(t)}}\sum_{i=1}^{n_{(t)}}\frac{M_i}{m_{i(t)}}\sum_{j=1}^{m_{i(t)}}\frac{Q_{ij}\left(Q_{ij}-q_{ij(t)}\right)}{q_{ij(t)}}s_{2ijy(t)}^{2},$$

where

$$s_{1y}^{2}=\frac{1}{n_{(t)}-1}\sum_{i=1}^{n_{(t)}}\left(y_{i(t)}-\bar{y}_{(t)}\right)^{2}$$

$$s_{2iy}^{2}=\frac{1}{m_{i(t)}-1}\sum_{j=1}^{m_{i(t)}}\left(y_{ij(t)}-\bar{y}_{i(t)}\right)^{2},\ \text{and}$$

$$s_{2ijy(t)}^{2}=\frac{1}{q_{ij(t)}-1}\sum_{K=1}^{q_{ij(t)}}\left(Y_{ijk(t)}-\bar{Y}_{ij(t)}\right)^{2},$$

These formulae become simpler as all the units in a stage are equal.

If a proportion $p_{(t)}$ of solid waste is collected from each housing unit of the tth combination, the estimated totals and variances become $p_{(t)}^{-1}\left(y'_{(t)}\right)$ and $p_{(t)}^{-2}Var\left(y'_{(t)}\right)$ respectively.

A comparison of the variances may lead to the selection of a particular 3-stage sample design.

A further analysis using the analysis of variance technique is made to test if there is any difference between using different levels of the three factors. Table 1 shows that the number of observations in different combinations are schematically different. There are 8 observations in (1), 16 in (h), (s) and (b) etc. Here we shall assume that a random sample of size 8 is assigned to the combination (1), a random sample of size 16 is assigned to treatment (h) etc. The size of the random sample is not assumed to be constant for all treatments. The notations that will be used for computation of means and variances are outlined in Table 2.

Table 2
Computation Scheme for Analysis of Various Combinations

	1	2	3	4	5	6	7	8	Total
Number of Observations	$n_{(1)}$	$n_{(2)}$	$n_{(3)}$	$n_{(4)}$	$n_{(5)}$	$n_{(6)}$	$n_{(7)}$	$n_{(8)}$	$M=\sum_{t=1}^{8} n_{(t)}$
Sum of Observations	$T_{(1)}$	$T_{(2)}$	$T_{(3)}$	$T_{(4)}$	$T_{(5)}$	$T_{(6)}$	$T_{(7)}$	$T_{(8)}$	$G=\sum_{t=1}^{8} T_{(i)}$
Mean of Observations	$\bar{T}_{(1)}$	$\bar{T}_{(2)}$	$\bar{T}_{(3)}$	$\bar{T}_{(4)}$	…	…	…	$\bar{T}_{(8)}$	$\bar{G}=\frac{G}{\sum_{t=1}^{8} n_{(t)}}$
Sum of Squares of Observations	$\sum Y_{(1)}^2$	$\sum Y_{(2)}^2$	…	…	…	…	…	$\sum Y_{(8)}^2$	$\sum\ \sum Y_{(t)}^2$
$T_{(i)}^2 / n_{(i)}$	$T_{(1)}^2 / n_{(1)}$	$T_{(2)}^2 / n_{(2)}$	…	…	…	…	…	$T_{(8)}^2 / n_{(8)}$	
Within-class Variation	$SS_{(1)}^2=\sum Y_{(1)}^2-\frac{T_{(1)}^1}{8}$		…	…	…	…	…	SS_8	
Within-class Variance	$S_{(1)}^2=\frac{SS_{(1)}}{n_{(1)}-1}, S_{(2)}^2$		…	…	…	…	…	$S_{(8)}^2=\frac{SS_{(8)}}{n_{(8)}-1}$	

The estimates of variation due to error and combination effects are obtained by assuming an additive model,

$$y_{ijk(t)}=\mu+\tau_{(t)}+\epsilon_{ijk(t)}\ .$$

Least-squares estimates of the parameters μ and τ_t are obtained by minimizing $\sum_i \epsilon^2_{(t)}$ under the constraints $\sum_i n_{(t)}\hat{\tau}_{(t)}=0$. The computation of analysis of variance table is:

ANOVA TABLE

Due to	df	SS	MSE	F
Combinations	$t-1$	$\sum\frac{T_{(t)}^2}{n_{(t)}}-\frac{G^2}{M}$	$SSC/(t-1)$	$\frac{SSC/(t-1)}{SSC/(M-t)}$
Error	$n-t$	$\sum\sum Y_{(t)}^2-\sum\frac{T_{(t)}^2}{n_{(t)}}$	$SSE/(M-t)$	
Total	$M-1$	$\sum\sum Y_{(i)}^2-\frac{G^2}{M}$		

The $(t-1)$ degrees of freedom can be split up into $(t-1)$ single degrees of freedom for testing various contrasts (comparisons).

5. DATA ANALYSIS

A detailed analysis has been made by Ahmad et al. (1981).

6. RESULTS

The Table 3 shows the summary results of the experiments on blocks only.

Table 3
Means, standard deviations and estimated totals of solid waste weight, volume and density in Al-Khobar by Sample Sizes (Figure in kg.)

Sample Sizes		Means $\bar{X}$	Standard Deviations σ (of means)	Estimated Totals $\hat{Y}(10)^5$
0.5%	Weight	402.56	154.01 (97.40)	4.39
	Volume	8.24	2.46 (1.56)	0.90
	Density	400.54	36.68 (-)	----
1.0%	Weight	315.64	89.77 (47.98)	3.45
	Volume	5.32	0.98 (0.52)	0.58
	Density	126.07	65.25 (-)	----
1.5%	Weight	258.48	42.59 (17.76)	2.82
	Volume	3.96	0.66 (0.28)	0.43
	Density	122.90	57.04 (-)	
2%	Weight	224.84	60.06 (20.02)	2.45
	Volume	3.56	2.17 (0.72)	0.39
	Density	140.82	111.25 (-)	

The most efficient sample seems to have a 1.5% size which has the smallest standard deviation and standard error of means of weight and volume of solid waste. The estimated densities of solid waste are on the upper range of the values reported in the literature which is 60 -120kg/m^3. Only 0.5% sample size design renders a density of 400.54 kg/m^3. Whereas the densities for other sample sizes lie outside the established range. The estimate of density from the sample design with 2% size is well above the upper limit. In order to study this marked variation in the densities, the following table (showing the post-stratification of the areas at 2% sample size) has been constructed.

Table 4
Mean and standard deviations of densities by areas

Sample size		North	South	West	Agrabia	Tughba	Total
2%	$\bar{x}$	60.8	83.4	128.5	299.8	119.9	140.82
	σ	34.6	63.1	21.9	113.4	79.5	111.25

It is observed from the Table 4 that the density estimated at 2% in Agrabia area is 299.8 kg/m^3 with a standard deviation of 113.4. This value is quite high and when checking with the field-group notes, it was found that on the sample collection date, one block in this group contained two $5m^3$ containers full to the top with food waste (mostly cooked rice). This was an unusual event and one of the reasons for high density.

In Table 4, one also observes that as areas become more commercialized, the density decreases. An example is the comparison between North (commercial) and Agrabia (mostly residential) areas where the densities are 69.6 and 197.8 kg/m^3, respectively. The other three areas follow the same pattern of higher densities for residential areas. The solid waste generation rate based $1\frac{1}{2}\%$ on sample design and an estimated population of Al-Khobar of 1.03×10^5, is 2.489 kg/person/day. The values reported in the literature [5] are in the range of 0.91 —2.268 kg/m^3 with 1.588kg/m3 most commonly mentioned. Our estimates are at one of the extremes of this range. It should be noted that this estimate is very crude because (i) there was no daily replicate of solid waste, (ii) population estimate is very crude (and could not be checked with the figures of Central Department of Statistics, Kingdom of Saudi Arabia). However, it shows that a sample design with 1.5% sample size does provide a well-balanced sample design.

ACKNOWLEDGEMENTS

The authors are indebted to referees for helpful comments. The work is supported by the Saudi Arabian Center for Science and Technology under Project No. AR-04-049.

REFERENCES

1. Bayunus, O and Khan, M.Z. (1981). Forecasting Automobile Rate of Abandonment – A Case Study in Solid Waste Management, City of Jeddah. Presented in *10th Pakistan Statist. Conference*, Islamabad.
2. Khan, M.Z. Ali. (1979). *Community Housing and Refuse Disposal in Developing Countries*. Vol. 1 Pergamon Press, New York.
3. Khan, M.Z. and Bayunus O. (1980). *Energy Conservation by Resource Recovery and Reuse in Solid Waste Management. Energy Resources and Conservation Related to Built Environment*. Vol. I. Pergamon Press, New York.
4. Khan, M.Z. and Bayunus, O. (1981). *Factors Affecting the Solid Waste Generation and Composition in Semi Arid Coastal Zones*. Proc. Institute of Environmental Science, Energy and the Environment. Vol. II. Los Angeles, California.
5. Office of Solid Waste Management, Environmental Health Service, Department of Health Education and Welfare. (1968a). *Massive Statistics*. Public Health Service Publication No. 1867.
6. Office of Solid Waste Management, Environmental Health Service, Department of Health Education and Welfare. (1968b). *Interim Report*. National Survey of Community Solid Waste Practices.
7. Tebobanoglous, G., Theisen, H. and Elliasen, R. (1977). *Solid Wastes Engineering Principles and Management Issues*. McGraw-Hill, New York.

GENERALIZED REGRESSION-CUM-RATIO ESTIMATORS FOR TWO-PHASE SAMPLING USING MULTI-AUXILIARY VARIABLES*

Zahoor Ahmad[1], **Muhammad Hanif**[2] and **Munir Ahmad**[3]
[1] University of Gujrat, Gujrat, Pakistan. Email: zahoor_ahmed_stat@yahoo.com
[2] Lahore University of Management Sciences, Lahore, Pakistan.
Email: hanif@lums.edu.pk
[3] National College of Business Administration and Economics, Lahore, Pakistan.
Email: drmunir@brain.net.pk

ABSTRACT

In this paper we suggest three classes of regression-cum-ratio estimators for estimating population mean of variable of interest for two-phase sampling using multi-auxiliary variables for full information, partial information and no information cases. The expressions for mean square errors are derived. Theoretical comparison is given. Special cases of estimators are also identified.

KEY WORDS

Regression-cum-ratio estimator; two-phase sampling; auxiliary variable.

1. INTRODUCTION

The use of auxiliary information is a widely discussed topic in sampling theory to obtain improved designs and precise estimates of some population parameters like mean or variance. It is well known that when the auxiliary information is utilized at the estimation stage. The ratio, product and regression methods are employed in many such situations.

The estimation of the population mean is an unrelenting issue in sampling theory and several efforts have been made to improve the precision of the estimates in the presence of multi-auxiliary variables. A variety of estimators have been proposed following different ideas linking together ratio, product or regression estimators.

Olkin (1958) was the first author to deal with the problem of estimating the mean of a survey variable when auxiliary variables are made available. He suggested the use of information on more than one auxiliary variable, positively correlated with the study variable analogously to Olkin; Singh (1967a) gave a multivariate expression of Murthy's (1964) product estimator, while Raj (1965) suggested a method for using multi-auxiliary variables through a linear combination of single difference estimators. Moreover, Singh (1967b) considered the extension of the ratio-cum-product estimators to multi-auxiliary variables Shukla (1965) suggested a multiple regression estimator while Rao and Mudholkar (1967) proposed a multivariate estimator based on a weighted sum of single ratio and product estimators.

*Published in Pak. J. Statist. (2009), Vol. 25(2).

John (1969) suggested two multivariate generalizations of ratio and product estimators which actually reduce to the Olkin's (1958) and Singh's (1967a) estimators. Srivastava (1971) proposed a general ratio-type estimator which generates a large class of estimators including most of the estimators up to that time proposed.

Robinson (1994) proposed a regression estimator ignoring some of the assumptions usually adopted in the literature (see, e.g., Srivastava (1971)), Tracy et al. (1996) and Perri (2004) proposing an alternative to Singh's (1965, 1967b) ratio-cum-product estimators, when two auxiliary variables are available. Ceccon and Diana (1996) provided a multivariate extension of the Naik and Gupta (1991) univariate class of estimators. Agarwal et al. (1997), moving from Raj (1965), illustrated a new approach to form a multivariate difference estimator which does not require the knowledge of any population parameters. Abu-Dayyeh et al. (2003) introduced two estimators which are definitely members of the class proposed by Srivastava (1971), while Kadilar and Cingi (2004, 2005) analyzed combinations of regression type estimators in the case of two auxiliary variables. In the same situation, Perri (2005) proposed some new estimators obtained from Singh's (1965, 1967b) estimators. Pradhan (2005) suggested a chain regression estimator for two-phase sampling using three auxiliary variables when the population mean of one auxiliary variable is unknown and other auxiliary population means are known.

In practical surveys, the problem is to estimate population means of variables of interest. For example, in a typical socio-economic survey conducted in rural areas in Indo-Pak subcontinent, the multiple variables of interests may be size of household, monthly income and expenditure of the household, number of unemployed persons, number of illiterates, number of persons engaged in agriculture, amount of land owned, leased and leased out, number of cattle owned etc. In some situations the auxiliary information may be available through the past census data or conveniently collected. For example in a village land survey, the information on the variables such as area of the village, cultivable area, grazing grounds etc. may be easily obtained through the past census data and may be used to estimate the means of variables of interest.

If we have information on multi-auxiliary variables practically sometimes either information for all these auxiliary is available from population or available for some variables or not available for all auxiliary variables. By considering these practical situations, we suggest general classes of regression-cum-ratio estimators for estimating the population mean of study variable for two-phase sampling using multi-auxiliary variables by considering the following three cases (see Samiuddin and Hanif (2007).

1. Estimators when information on all auxiliary variables is known for population (Full Information Case).
2. Estimators when information on some auxiliary variables is known for population (Partial Information Case).
3. Estimators when information on all auxiliary variables is unknown for population (No Information Case).

Before suggesting the estimators we provide two-phase sampling scheme and some useful notations and results in the following section.

2. TWO-PHASE SAMPLING USING MULTI-AUXILIARY VARIABLES

Consider a population of N units. Let Y be the variable for which we want to estimate the population mean and $X_1, X_2, ... X_q$ are q auxiliary variables. For two-phase sampling design let n_1 and $n_2\left(n_2 < n_1\right)$ are sample sizes for first and second phase respectively. $x_{(1)i}$ and $x_{(2)i}$ denote the i^{th} auxiliary variables form first and second phase samples respectively and y_2 denote the variable of interest from second phase. $\bar{X}_i$ and C_{x_i} denote the population means and coefficient of variation of i^{th} auxiliary variables respectively and ρ_{yx_i} denotes the population correlation coefficient of Y and X_i. Further let $\theta_1 = \frac{1}{n_1} - \frac{1}{N}$, $\theta_2 = \frac{1}{n_2} - \frac{1}{N}$, $y_{(2)} = Y + e_{y_{(2)}}$, $x_{(1)i} = X_i + e_{x_{(1)i}}$ and $x_{(2)i} = X_i + e_{x_{(2)i}}$; $\left(i = 1, 2, ..., k\right)$ where $e_{y_{(2)}}$, $e_{x_{(1)i}}$ and $e_{x_{(2)i}}$ are sampling errors and are of very small quantities. We assume that $E_2\left(e_{y_{(2)}}\right) = E_1\left(e_{x_{(1)i}}\right) = E_2\left(e_{x_{(2)i}}\right) = 0$. Then for simple random sampling without replacement for both first and second phases we write by using phase wise operation of expectations as:

$$E_2\left(e_{y_2}\right)^2 = \left(1 - \frac{n_2}{N}\right) S_Y^2,\ E_2\left(\bar{e}_{y_2}\right)^2 = \left(1 - \frac{n_2}{N}\right) \frac{S_Y^2}{n_2} = \theta_2 \bar{Y}^2 C_y^2,$$

$$E_2\left(e_{y_2} e_{x_{(2)i}}\right) = \left(1 - \frac{n_2}{N}\right) S_{YX_i} = \theta_2 \overline{YX}_i C_y C_{x_i} \rho_{yx_i},$$

$$E_2\left(\bar{e}_{y_2} \bar{e}_{x_{(2)i}}\right) = \left(1 - \frac{n_2}{N}\right) \frac{S_{YX_i}}{n_2} = \theta_2 \overline{YX}_i C_y C_{x_i} \rho_{yx_i},$$

$$E_1 E_{2|1}\left[e_{y_2}\left(e_{x_{(1)i}} - e_{x_{(2)i}}\right)\right] = E_1 E_{2|1}\left(e_{y_2} e_{x_{(1)i}}\right) - E_2\left(e_{y_2} e_{x_{(2)i}}\right)$$
$$= \left(1 - \frac{n_1}{N}\right) S_{YX_i} - \left(1 - \frac{n_2}{N}\right) S_{YX_i} = \frac{1}{N}\left(n_2 - n_1\right) S_{YX_i},$$

$$E_1 E_{2|1}\left[\bar{e}_{y_2}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)\right] = \left(1 - \frac{n_1}{N}\right) \frac{S_{YX_i}}{n_1} - \left(1 - \frac{n_2}{N}\right) \frac{S_{YX_i}}{n_2} = \left(\theta_1 - \theta_2\right) \overline{YX}_i C_y C_{x_i} \rho_{yx_i}.$$

Similarly

$$E_1 E_{2|1}\left[\bar{e}_{x_{(2)i}}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)\right] = \left(\theta_1 - \theta_2\right) \sigma_{x_i}^2 = \left(\theta_1 - \theta_2\right) \bar{X}_i^2 C_{x_i}^2,$$

$$E_1 E_{2|1}\left[\bar{e}_{x_{(1)i}}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)\right] = 0,$$

$$E_1E_{2/1}\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)^2=\left(\theta_2-\theta_1\right)\sigma_{x_i}^2=\left(\theta_2-\theta_1\right)\bar{X}_i^2C_{x_i}^2,$$

$$E_1E_{2|1}\left[\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)\left(\bar{e}_{x_{(1)j}}-\bar{e}_{x_{(2)j}}\right)\right]=\left(\theta_2-\theta_1\right)\sigma_{x_ix_j}$$
$$=\left(\theta_2-\theta_1\right)\bar{X}_i\bar{X}_jC_{x_i}C_{x_j}\rho_{x_ix_j};\left(i\neq j\right),$$

and

$$E_1E_{2|1}\left[\left(\bar{e}_{x_{(2)i}}\right)\left(\bar{e}_{x_{(1)j}}-\bar{e}_{x_{(2)j}}\right)\right]=\left(\theta_1-\theta_2\right)\sigma_{x_ix_j}=\left(\theta_1-\theta_2\right)\bar{X}_i\bar{X}_jC_{x_i}C_{x_j}\rho_{x_ix_j},\left(i\neq j\right).$$

The following notations will be used in deriving the mean square errors of proposed estimators

$|\mathrm{R}|_{y\underset{\sim}{x}_q}$ Determinant of population correlation matrix of variables $y,\ x_1,\ x_2,...,x_{q-1}$ and x_q.

$\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}$ Determinant of i^{th} minor of $|\mathrm{R}|_{y\underset{\sim}{x}_q}$ corresponding to the i^{th} element of ρ_{yx_i}.

$\rho^2_{y.\underset{\sim}{x}_s}$ Denotes the multiple coefficient of determination of y on $x_1,\ x_2,...,x_{r-1}$ and x_r.

$\rho^2_{y.\underset{\sim}{x}_q}$ Denotes the multiple coefficient of determination of y on $x_1,\ x_2,...,x_{q-1}$ and x_q.

$|R|_{\underset{\sim}{x}_s}$ Determinant of population correlation matrix of variables $x_1,\ x_2,...,x_{r-1}$ and x_r.

$|R|_{\underset{\sim}{x}_q}$ Determinant of population correlation matrix of variables $x_1,\ x_2,...,x_{q-1}$ and x_q.

$|R|_{y_i\underset{\sim}{x}_s}$ Determinant of the correlation matrix of $y_i,\ x_1,\ x_2,...,x_{r-1}$ and x_r.

$|R|_{y_i\underset{\sim}{x}_q}$ Determinant of the correlation matrix of $y_i,\ x_1,\ x_2,...,x_{q-1}$ and x_q.

$|R|_{y_iy_j\underset{\sim}{x}_s}$ Determinant of the minor corresponding to $\rho_{y_iy_j}$ of the correlation matrix of $y_i,\ y_j,x_1,\ x_2,...,x_{r-1}$ and x_r, for $\left(i\neq j\right)$.

$|R|_{y_iy_j\underset{\sim}{x}_q}$ Determinant of the minor corresponding to $\rho_{y_iy_j}$ of the correlation matrix of $y_i,\ y_j,x_1,\ x_2,...,x_{q-1}$ and x_q, for $\left(i\neq j\right)$.

2.1 Result: 1

The following result will help us in deriving the mean square errors of suggested estimators

$$\frac{|\mathrm{R}|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}=\left(1-\rho^2_{y.\underset{\sim}{x}_q}\right).\qquad\text{[Arora and Lal (1989)]}$$

2.2 Regressions-Cum-Ratio Estimator (Full Information Case)

If we estimate a study variable when information on all auxiliary variables is available from population, it is utilized in the form of their means. By taking the advantage of Regression-cum-Ratio technique for two-phase sampling, a generalized estimator for estimating population mean of study variable $\bar{Y}$ with the use of multi-auxiliary variables are suggested as:

$$t_1 = \left[\bar{y}_2 + \sum_{i=1}^{r} \alpha_i \left(\bar{X}_i - \bar{x}_{(2)i}\right)\right] \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}}\right)^{\gamma_i}$$

$$= \left[\bar{Y} + \bar{e}_{y_2} - \sum_{i=1}^{r} \alpha_i \bar{e}_{x_{(2)i}}\right] \prod_{i=r+1}^{r+s=q} \left(1 + \frac{\bar{e}_{x_{(2)i}}}{\bar{X}_i}\right)^{-\gamma_i} .$$

Ignoring second and higher terms for each expansion of product and after simplification, we write $t_1 = \left[\bar{Y} + \bar{e}_{y_2} - \sum_{i=1}^{r} \alpha_i \bar{e}_{x_{(2)i}} - \sum_{i=r+1}^{r+s=q} \frac{\bar{Y}}{\bar{X}_i} \gamma_i \bar{e}_{x_{(2)i}}\right]$.

The mean square error is

$$MSE\left(t_1\right) = E_2\left[\bar{e}_{y_2} - \sum_{i=1}^{r} \alpha_i \bar{e}_{x_{(2)i}} - \sum_{i=r+1}^{r+s=q} \frac{\bar{Y}}{\bar{X}_i} \gamma_i \bar{e}_{x_{(2)i}}\right]^2 . \tag{2.2.1}$$

The optimum values of α_i and γ_i for which the mean square error of estimator t_1 is minimum to term of o(1/n) are

$$\alpha_i = \left(-1\right)^{i+1} \frac{\bar{Y}}{\bar{X}_i} \frac{C_y}{C_{x_i}} \frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}} = \left(-1\right)^{i+1} \beta_{y\underset{\sim}{x}_i \cdot \underset{\sim}{x}_q} \quad \left(i = 1, 2, \ldots. r \text{ and } r + s = q\right)$$

and

$$\gamma_i = \left(-1\right)^{i+1} \frac{C_y}{C_{x_i}} \frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}} = \left(-1\right)^{i+1} \frac{\bar{X}_i}{\bar{Y}} \beta_{y\underset{\sim}{x}_i \cdot \underset{\sim}{x}_q} \quad \left(i = r+1, r+2, \ldots. r+s \text{ and } r + s = q\right) .$$

The unknown constants are related to the partial regression coefficients of study variable and auxiliary variables. If these partial regression coefficients are not known, these will be estimated from second phase because the estimator t_1 utilizes the information on q auxiliary variables and study variable from second phase sample.

Using normal equations that are used to find the optimum values of α_i and γ_i, (2.2.1) can be written in simplified form as:

$$MSE(t_1)=E_2\left[\bar{e}_{y_2}\left(\bar{e}_{y_2}-\sum_{i=1}^{r}\alpha_i\bar{e}_{x_{(2)i}}-\sum_{i=r+1}^{r+s=q}\frac{\bar{Y}}{\bar{X}_i}\gamma_i\bar{e}_{x_{(2)i}}\right)\right]$$

$$=E_2\left(\bar{e}_{y_2}^2\right)-\sum_{i=1}^{r}\alpha_i E_2\left(\bar{e}_{y_2}\bar{e}_{x_{(2)i}}\right)-\sum_{i=r+1}^{r+s=q}\frac{\bar{Y}}{\bar{X}_i}\gamma_i E_2\left(\bar{e}_{y_2}\bar{e}_{x_{(2)i}}\right)$$

$$=\theta_2\bar{Y}^2C_y^2-\sum_{i=1}^{r}\alpha_i\theta_2\bar{Y}\bar{X}_iC_yC_{x_i}\rho_{yx_i}-\sum_{i=r+1}^{r+s=q}\frac{\bar{Y}}{\bar{X}_i}\gamma_i\theta_2\bar{Y}\bar{X}_iC_yC_{x_i}\rho_{yx_i}.$$

Using the values of α_i and γ_i and after simplification, we get:

$$MSE(t_1)=\bar{Y}^2C_y^2\left[\theta_2-\theta_2\sum_{i=1}^{r}(-1)^{i+1}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\rho_{yx_i}-\theta_2\sum_{i=r+1}^{r+s=q}(-1)^{i+1}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\rho_{yx_i}\right]$$

$$=\bar{Y}^2C_y^2\left[\theta_2+\theta_2\sum_{i=1}^{q}(-1)^{i}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\rho_{yx_i}\right]$$

$$=\frac{\theta_2\bar{Y}^2C_y^2}{|R|_{\underset{\sim}{x}_q}}\Big[|R|_{\underset{\sim}{x}_q}-\rho_{yx_1}\left|R_{yx_1}\right|_{y\underset{\sim}{x}_q}+\rho_{yx_2}\left|R_{yx_2}\right|_{y\underset{\sim}{x}_q}$$

$$-\rho_{yx_3}\left|R_{yx_3}\right|_{y\underset{\sim}{x}_q}+\ldots\ldots+(-1)^q\rho_{yx_q}\left|R_{yx_q}\right|_{y\underset{\sim}{x}_q}\Big]$$

or
$$MSE(t_1)=\theta_2\bar{Y}^2C_y^2\frac{|\mathrm{R}|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}.$$

Using Result 1, we get:

$$MSE(t_1)=\theta_2\bar{Y}^2C_y^2\left(1-\rho^2_{y.\underset{\sim}{x}_q}\right).$$

2.3 Regressions-Cum-Ratio Estimator (Partial Information Case)

In this case suppose we have no information on all q auxiliary variables but only for r auxiliary variables from population. Considering Regression-Cum-Ratio technique of estimating technique, the population mean of study variable $\bar{Y}$ can be estimated for two-phase sampling using multi-auxiliary variables as:

$$t_2=\left[\bar{y}_2+\sum_{i=1}^{r}\alpha_i''\left(\bar{x}_{(1)i}-\bar{x}_{(2)i}\right)\right]\prod_{i=r+1}^{r+s=q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\gamma_i''}\prod_{i=r+1}^{r+s=q}\left(\frac{\bar{X}_i}{\bar{x}_{(2)i}}\right)^{\delta_i''}$$

$$=\left[\bar{Y}+\bar{e}_{y_2}+\sum_{i=1}^{r}\alpha_i''\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)\right]\prod_{i=r+1}^{r+s=q}\left(1+\frac{\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}}{\bar{X}_i}\right)^{\gamma_i''}\prod_{i=r+1}^{r+s=q}\left(1+\frac{\bar{e}_{x_{(2)i}}}{\bar{X}_i}\right)^{-\delta_i''}.$$

Ignoring second and higher terms for each expansion of products and after simplification we write

$$t_2 = \left[\bar{Y} + \bar{e}_{y_2} + \sum_{i=1}^{r} \alpha_i^{''} \left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}} \right) + \sum_{i=r+1}^{r+s} \gamma_i^{''} \frac{\bar{Y}}{\bar{X}_i} \left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}} \right) - \sum_{i=r+1}^{r+s} \delta_i^{''} \frac{\bar{Y}}{\bar{X}_i} \bar{e}_{x_{(2)i}} \right].$$

The mean square error of t_2 is written as

$$MSE(t_2) = E_1 E_{2/1} \left[\bar{e}_{y_2} + \sum_{i=1}^{r} \alpha_i^{''} \left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}} \right) \right.$$

$$\left. + \sum_{i=r+1}^{r+s} \gamma_i^{''} \frac{\bar{Y}}{\bar{X}_i} \left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}} \right) - \sum_{i=r+1}^{r+s} \delta_i^{''} \frac{\bar{Y}}{\bar{X}_i} \bar{e}_{x_{(2)i}} \right]^2 \qquad (2.3.1)$$

The optimum values of $\alpha_i^{''}, \gamma_i^{''}$ and $\delta_i^{''}$ for which the mean square error of t_2 is minimum to term of o (1/n) are:

$$\alpha_i^{''} = (-1)^{i+1} \frac{\bar{Y}}{\bar{X}_i} \frac{C_y}{C_{x_i}} \frac{\left| R_{yx_i} \right|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}} = (-1)^{i+1} \beta_{y x_i . \underset{\sim}{x}_q}, \ (i = 1, 2, ...r).$$

$$\gamma_i^{''} = (-1)^{i+1} \frac{C_y}{C_{x_i}} \left[\frac{\left| R_{yx_i} \right|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}} - \frac{\left| R_{yx_i} \right|_{y\underset{\sim}{x}_s}}{|R|_{\underset{\sim}{x}_s}} \right]$$

$$= (-1)^{i+1} \frac{\bar{X}_i}{\bar{Y}} \left(\beta_{yx_i . \underset{\sim}{x}_q} - \beta_{yx_i . \underset{\sim}{x}_s} \right), \ (i = r+1, r+2, ...r+s)$$

and

$$\delta_i^{''} = (-1)^{i+1} \frac{C_y}{C_{x_i}} \frac{\left| R_{yx_i} \right|_{y\underset{\sim}{x}_s}}{|R|_{\underset{\sim}{x}_s}} = (-1)^{i+1} \beta_{y x_i . \underset{\sim}{x}_s}, \ i = r+1, r+2, ...r+s.$$

The optimum values are related to the partial regression coefficients of variable of interest and auxiliary variables. Usually these partial regression coefficients are unknown then these can be estimated from sample data. The estimator t_2 utilizes the information on *q* auxiliary variables from both first and second phase. Keenly observing the estimators t_2 the optimum values of unknown constants $\alpha_i^{''}$ and $\gamma_i^{''}$ will be estimated from first phase sample and $\delta_i^{''}$ from the second phase. Using normal equations that are used to fined the optimum values of $\alpha_i^{''}, \gamma_i^{''}$ and $\delta_i^{''}$, (2.3.1) can be written in a simplified form as:

$$MSE\left(t_2\right)=E_1E_{2/1}\left[\bar{e}_{y_2}\left(\bar{e}_{y_2}+\sum_{i=1}^{r}\alpha_i^{''}\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)\right.\right.$$

$$\left.\left.+\sum_{i=r+1}^{r+s}\gamma_i^{''}\frac{\bar{Y}}{\bar{X}_i}\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)-\sum_{i=r+1}^{r+s}\delta_i^{''}\frac{\bar{Y}}{\bar{X}_i}\bar{e}_{x_{(2)i}}\right)\right]$$

$$=E_1E_{2/1}\left(\bar{e}_{y_2}^2\right)+\sum_{i=1}^{r}\alpha_i^{''}E_1E_{2/1}\left\{\bar{e}_{y_2}\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)\right\}$$

$$+\sum_{i=r+1}^{r+s}\gamma_i^{''}\frac{\bar{Y}}{\bar{X}_i}E_1E_{2/1}\left\{\bar{e}_{y_2}\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)\right\}-\sum_{i=r+1}^{r+s}\delta_i^{''}\frac{\bar{Y}}{\bar{X}_i}E_1E_{2/1}\left(\bar{e}_{y_2}\bar{e}_{x_{(2)i}}\right)$$

$$=\theta_2\bar{Y}^2C_y^2+\left(\theta_1-\theta_2\right)\sum_{i=1}^{r}\alpha_i^{''}\overline{YX}_iC_yC_{x_i}\rho_{yx_i}$$

$$+\left(\theta_1-\theta_2\right)\sum_{i=r+1}^{r+s}\gamma_i^{''}\frac{\bar{Y}}{\bar{X}_i}\overline{YX}_iC_yC_{x_i}\rho_{yx_i}-\theta_2\sum_{i=r+1}^{r+s}\delta_i^{''}\frac{\bar{Y}}{\bar{X}_i}\overline{YX}_iC_yC_{x_i}\rho_{yx_i}.$$

Using the values of $\alpha_i^{''},\gamma_i^{''}$ and $\delta_i^{''}$ and after simplification we get:

$$MSE\left(t_2\right)=\bar{Y}^2C_y^2\left[\theta_2-\left(\theta_1-\theta_2\right)\sum_{i=1}^{r}\left(-1\right)^i\rho_{yx_i}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}}\right.$$

$$\left.-\left(\theta_1-\theta_2\right)\sum_{i=r+1}^{r+s}\left(-1\right)^i\rho_{yx_i}\left\{\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}}-\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_s}}{\left|R\right|_{\underset{\sim}{x}_s}}\right\}+\theta_2\sum_{i=r+1}^{r+s}\left(-1\right)^i\rho_{yx_i}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_s}}{\left|R\right|_{\underset{\sim}{x}_s}}\right]$$

$$=\bar{Y}^2C_y^2\left[\left(\theta_2-\theta_1\right)+\left(\theta_2-\theta_1\right)\sum_{i=1}^{r}\left(-1\right)^i\rho_{yx_i}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}}\right.$$

$$\left.+\left(\theta_2-\theta_1\right)\sum_{i=r+1}^{r+s}\left(-1\right)^i\rho_{yx_i}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}}+\theta_1+\theta_1\sum_{i=r+1}^{r+s}\left(-1\right)^i\rho_{yx_i}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_s}}{\left|R\right|_{\underset{\sim}{x}_s}}\right]$$

or

$$=\bar{Y}^2C_y^2\left[\left(\theta_2-\theta_1\right)\left(1+\sum_{i=1}^{q}\left(-1\right)^i\rho_{yx_i}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}}\right)+\theta_1\left(1+\sum_{i=r+1}^{r+s}\left(-1\right)^i\rho_{yx_i}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_s}}{\left|R\right|_{\underset{\sim}{x}_s}}\right)\right],$$

or

$$MSE\left(t_2\right)=\bar{Y}^2C_y^2\left[\theta_2\frac{\left|\mathrm{R}\right|_{y\underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}}+\theta_1\left(\frac{\left|\mathrm{R}\right|_{y\underset{\sim}{x}_s}}{\left|R\right|_{\underset{\sim}{x}_s}}-\frac{\left|\mathrm{R}\right|_{y\underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}}\right)\right].$$

Using Result 1 we get:

$$MSE\left(t_2\right)=\bar{Y}^2C_y^2\left[\theta_2\left(1-\rho^2_{y.\underset{\sim}{x}_q}\right)+\theta_1\left(\rho^2_{y.\underset{\sim}{x}_q}-\rho^2_{y.\underset{\sim}{x}_s}\right)\right].$$

2.4 Regressions-Cum-Ratio Estimator (No Information Case)

We consider the following regression-cum-ratio estimator for two-phase sampling using multi-auxiliary variables for estimating the population mean $\bar{Y}$ when information on all auxiliary variables is not available from a population as:

$$t_3=\left[\bar{y}_2+\sum_{i=1}^{r}\alpha_i'\left(\bar{x}_{(1)i}-\bar{x}_{(2)i}\right)\right]\prod_{i=r+1}^{r+s=q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\gamma_i'}$$

$$=\left[\bar{Y}+\bar{e}_{y_2}+\sum_{i=1}^{r}\alpha_i'\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)\right]\prod_{i=r+1}^{r+s=q}\left(1+\frac{\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}}{\bar{X}_i}\right)^{\gamma_i'}.$$

Ignoring second and higher terms for each expansion of product and after simplification we can write

$$t_3=\left[\bar{Y}+\bar{e}_{y_2}+\sum_{i=1}^{r}\alpha_i'\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)+\sum_{i=r+1}^{r+s=q}\frac{\bar{Y}}{\bar{X}_i}\gamma_i'\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)\right].$$

The mean square error is

$$MSE\left(t_3\right)=E_2\left[\bar{e}_{y_2}+\sum_{i=1}^{r}\alpha_i'\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)+\sum_{i=r+1}^{r+s=q}\frac{\bar{Y}}{\bar{X}_i}\gamma_i'\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)\right]^2. \qquad (2.4.1)$$

The optimum values of α_i' and γ_i' for which the mean square error of t_3 is minimum to the order o(1/n) are:

$$\alpha_i'=\left(-1\right)^{i+1}\frac{\bar{Y}}{\bar{X}_i}\frac{C_y}{C_{x_i}}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}}=\left(-1\right)^{i+1}\beta_{y\underset{\sim}{x}_i\cdot\underset{\sim}{x}_q}\ \left(i=1,2,....r\right)$$

and

$$\gamma_i'=\left(-1\right)^{i+1}\frac{C_y}{C_{x_i}}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{\left|R\right|_{\underset{\sim}{x}_q}}=\left(-1\right)^{i+1}\frac{\bar{X}_i}{\bar{Y}}\beta_{y\underset{\sim}{x}_i\cdot\underset{\sim}{x}_q}\ ,\ \left(i=r+1,r+2,....r+s\right).$$

In this case the optimum values are also related to the partial regression coefficients of study variable and auxiliary variables. Mostly these partial regression coefficients are unknown but these can be estimated from sample data. The estimator t_3 utilizes the information on *q* auxiliary variables from both first and second phase. Analyzing the estimators t_3 the optimum values of unknown constants α_i' and γ_i' will be estimated from first phase sample in the form of sample regression coefficients.

(2.4.1) is also written in a simplified form as:

$$MSE(t_3) = E_2\left(\bar{e}_{y_2}^2\right) + \sum_{i=1}^{r}\alpha_i' E_2\left(\bar{e}_{y_2}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)\right) - \sum_{i=r+1}^{r+s=q}\frac{\bar{Y}}{\bar{X}_i}\gamma_i' E_2\left(\bar{e}_{y_2}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)\right)$$

$$= \theta_2\bar{Y}^2C_y^2 + (\theta_1-\theta_2)\sum_{i=1}^{r}\alpha_i'\overline{YX}_iC_yC_{x_i}\rho_{yx_i} + (\theta_1-\theta_2)\sum_{i=r+1}^{r+s=q}\frac{\bar{Y}}{\bar{X}_i}\gamma_i'\overline{YX}_iC_yC_{x_i}\rho_{yx_i}.$$

Using the values of α_i' and γ_i' and after simplification we get:

$$MSE(t_3) = \bar{Y}^2C_y^2\left[\theta_2 + (\theta_1-\theta_2)\sum_{i=1}^{r}(-1)^{i+1}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\rho_{yx_i}\right.$$

$$\left. + (\theta_1-\theta_2)\sum_{i=r+1}^{r+s=q}(-1)^{i+1}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\rho_{yx_i}\right]$$

$$= \bar{Y}^2C_y^2\left[\theta_2 + (\theta_2-\theta_1)\sum_{i=1}^{q}(-1)^{i}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\rho_{yx_i}\right]$$

$$= \bar{Y}^2C_y^2\left[(\theta_2-\theta_1)\frac{|\mathrm{R}|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}} + \theta_1\right].$$

Using Result1, we get:

$$MSE(t_3) = \bar{Y}^2C_y^2\left[(\theta_2-\theta_1)\left(1-\rho_{y.\underset{\sim}{x}_q}^2\right)+\theta_1\right] = \bar{Y}^2C_y^2\left[\theta_2\left(1-\rho_{y.\underset{\sim}{x}_q}^2\right)+\theta_1\rho_{y.\underset{\sim}{x}_q}^2\right].$$

3. THEORETICAL COMPARISON OF NEW ESTIMATORS

The MSE criterion is most common for comparing various estimators [Lee and Peddada (1987) and Cox and Hinkley (1974)]. We suggest three estimators in this paper in which first one is for full information case, second one is for partial information case and last one is for no information case. The estimator for full information case is more efficient than the estimator for partial information case and the estimator for partial information case is more efficient than for the no information case. It can be checked by comparing their MSE's as:

$$MSE(t_1) - MSE(t_2) = -\theta_1\bar{Y}^2C_y^2\left(\rho_{y.\underset{\sim}{x}_q}^2 - \rho_{y.\underset{\sim}{x}_s}^2\right) < 0;\ \text{as q>s}$$

and

$$MSE(t_2) - MSE(t_3) = -\theta_1\bar{Y}^2C_y^2\rho_{y.\underset{\sim}{x}_s}^2 < 0.$$

4. SPECIAL CASES

We summarize the results for the three cases as follows:

Proposed Estimators	Unknown Constants			Special Cases
I. Full Information Case	α_i		γ_i	
$t_1 = \left[\bar{y}_2 + \sum_{i=1}^{r} \alpha_i \left(\bar{X}_i - \bar{x}_{(2)i}\right)\right] \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}}\right)^{\gamma_i}$	0		γ_i	A class of ratio estimators with s auxiliary variables for Full Information Case
$t_1 = \left[\bar{y}_2 + \sum_{i=1}^{r} \alpha_i \left(\bar{X}_i - \bar{x}_{(2)i}\right)\right] \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}}\right)^{\gamma_i}$	α_i		0	A class of regression estimators with r auxiliary variables for Full Information Case
II. Partial Information Case	α_i''	γ_i''	δ_i''	
$t_2 = \left[\bar{y}_2 + \sum_{i=1}^{r} \alpha_i'' \left(\bar{x}_{(1)i} - \bar{x}_{(2)i}\right)\right] \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\gamma_i''} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}}\right)^{\delta_i''}$	0	γ_i''	δ_i''	A class of ratio estimators with s auxiliary variables for Partial Information Case
$t_2 = \left[\bar{y}_2 + \sum_{i=1}^{r} \alpha_i'' \left(\bar{x}_{(1)i} - \bar{x}_{(2)i}\right)\right] \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\gamma_i''} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}}\right)^{\delta_i''}$	α_i''	0	δ_i''	A class of ratio-cum-regression estimators with q auxiliary variables for Partial Information Case
$t_2 = \left[\bar{y}_2 + \sum_{i=1}^{r} \alpha_i'' \left(\bar{x}_{(1)i} - \bar{x}_{(2)i}\right)\right] \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\gamma_i''} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}}\right)^{\delta_i''}$	α_i''	γ_i''	0	A class of ratio-cum-regression estimators with q auxiliary variables for No Information Case
$t_2 = \left[\bar{y}_2 + \sum_{i=1}^{r} \alpha_i'' \left(\bar{x}_{(1)i} - \bar{x}_{(2)i}\right)\right] \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\gamma_i''} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}}\right)^{\delta_i''}$	α_i''	0	0	A class of regression estimators with r auxiliary variables for No Information Case

Proposed Estimators	Unknown Constants			Special Cases
$t_2 = \left[\bar{y}_2 + \sum_{i=1}^{r} \alpha_i^{''}\left(\bar{x}_{(1)i} - \bar{x}_{(2)i}\right)\right] \prod_{i=r+1}^{r+s=q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\gamma_i^{''}} \prod_{i=r+1}^{r+s=q}\left(\frac{\bar{X}_i}{\bar{x}_{(2)i}}\right)^{\delta_i^{''}}$	0	$\gamma_i^{''}$	0	A class of ratio estimators with s auxiliary variables for No Information Case
$t_2 = \left[\bar{y}_2 + \sum_{i=1}^{r} \alpha_i^{''}\left(\bar{x}_{(1)i} - \bar{x}_{(2)i}\right)\right] \prod_{i=r+1}^{r+s=q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\gamma_i^{''}} \prod_{i=r+1}^{r+s=q}\left(\frac{\bar{X}_i}{\bar{x}_{(2)i}}\right)^{\delta_i^{''}}$	0	0	$\alpha_i^{''}$	A class of ratio estimators with s auxiliary variables for Partial Information Case
III. No Information Case	$\alpha_i^{'}$		$\gamma_i^{'}$	
$t_3 = \left[\bar{y}_2 + \sum_{i=1}^{r} \alpha_i\left(\bar{x}_{(1)i} - \bar{x}_{(2)i}\right)\right] \prod_{i=r+1}^{r+s=q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\gamma_i}$	0		$\gamma_i^{'}$	A class of ratio estimators with s auxiliary variables for No Information Case
$t_3 = \left[\bar{y}_2 + \sum_{i=1}^{r} \alpha_i\left(\bar{x}_{(1)i} - \bar{x}_{(2)i}\right)\right] \prod_{i=r+1}^{r+s=q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\gamma_i}$	$\alpha_i^{'}$		0	A class of regression estimators with r auxiliary variables for No Information Case

Obviously the classes of special cases are not efficient than suggested classes of estimators.

5. NUMERICAL ILLUSTRATION

Description of populations is given in Table 1 and mean square errors of suggested estimators are given in Table 2.

Table 1
Description of Population

Source	"Measurement of four characters of: Flucus Religiousament" by Pradhan (2000)						
y	Length of petiole						
x_1	Length of lamina (blade) of the leaf						
x_2	Width of the leaf at its widest paint						
x_3	Width of leaf half way along the blade						
	N	ρ_{yx_1}	ρ_{yx_2}	ρ_{yx_3}	$\rho_{x_1x_2}$	$\rho_{x_1x_3}$	$\rho_{x_2x_3}$
	160	0.5423	0.6166	0.2704	0.8568	0.7424	0.8027

Table 2:
Mean Square Errors of Estimators

Estimator		Auxiliary Variables for which Information is Known for the Population	Auxiliary Variables for which Information is unknown for the Population	Relative Efficiency when N = 160, n_1 = 50, n_2 = 20
t_3 (NIC)		-	x_1, x_2, x_3	100
t_2	t_{21}	x_2, x_3	x_1	129.34
	t_{22}	x_1, x_3	x_2	123.46
	t_{23}	x_1, x_2	x_3	156.65
PIC	t_{24}	x_1,	x_2, x_3	168.07
	t_{25}	x_2	x_1, x_3	149.43
	t_{26}	x_3	x_1, x_2	164.12
t_1(FIC)		x_1, x_2, x_3	-	135.73

In Table 2 we provide MSE's of eight estimators, first estimator is for full information case, last estimator is for no information case and other six estimators are for partial information case with all possible combinations of auxiliary variables with known and unknown information from population. In theoretical comparisons t_1 is more efficient than t_2 and t_2 is more efficient than t_3. But in empirical comparison, we see that a special case of partial information i.e. t_{24} performs better than all others. It means that population characteristics like mean, coefficient of variation, variances, sample sizes of both phases and especially correlation coefficients of study variable with auxiliary variables and correlation coefficients within auxiliary variables count a lot for suggesting an estimator for use in real life situations. This can be adequately judged by considering at least ten different types of natural populations and MSE's should be calculated for study variable in the presence of at least five auxiliary variables.

REFERENCES

1. Abu-Dayyeh, W.A., Ahmed, M.S., Ahmed, R.A. and Muttlak, H.A. (2003). Some estimators of finite population mean using auxiliary information. *App. Math. and Compu.*, 139, 287-298.
2. Agarwal, S.K., Sharma, U.K. and Kashyap, S. (1997). A new approach to use multivariate auxiliary information in sample surveys. *J. Statist. Plann. and Infer.*, 60, 261-267.
3. Arora, S. and Lal, Bansi (1989). *New Mathematical Statistics*. Satya Prakashan, New Delhi.
4. Ceccon, C. and Diana, G. (1996). Classi di stimatori di tipo rapporto in presenza di pi´u variabili ausiliarie. *Proceedings of Italian Statistical Society*, XXXVIII Scientific Meeting, Rimini, 9-13 April 1996, 119-126.
5. Cox, D.R. and Hinkley, D.V. (1974). *Theoretical Statistics*. Champman and Hall, London.

6. John, S. (1969). On multivariate ratio and product estimators. *Biometrika*, 56, 533-536.
7. Kadilar, C. and Cingi, H. (2004). Estimator of a population mean using two auxiliary variables in simple random sampling. *Int. Math. J.*, 5, 357-367.
8. Kadilar, C. and Cingi, H. (2005). A new estimator using two auxiliary variables, *App. Math. and Compu*., 162, 901-908.
9. Lee, Carl M.S. and Peddada, S.D. (1987). Comparisons of some common ratio estimators using Pitman measure of nearness. *Comm. Statist. Theo Math.*, 16(7), 2017-2027.
10. Murthy, M.N. (1964). Product method of estimation. *Sankhya*, 26, 294-307.
11. Naik, V.D. and Gupta, P.C. (1991). A general class of estimators for estimating population means using auxiliary information. *Metrika*, 38, 11-17.
12. Olkin, I. (1958). Multivariate ratio estimation for finite populations. *Biometrika*, 45, 154-165.
13. Perri, P.F. (2004). Alcune considerazioni sull'efficienza degli stimatori rapporto-cum-prodotto. *Statistica and Applicazioni*, 2(2), 59-75.
14. Perri, P.F. (2005). Combining two auxiliary variables in ratio-cum-product type estimators. *Proceedings of Italian Statistical Society, Intermediate Meeting on Statistics and Environment*, Messina, 21-23 September 2005, 193-196.
15. Pradhan, B.K. (2000). *Some problems of estimation in multi-phase sampling*. Ph.D. Dissertation (unpublished), Utkal University, Orissa, India.
16. Pradhan, B.K. (2005). A Chain Regression Estimator in Two Phase Sampling Using Multi-auxiliary Information. *Bull. Malays. Math. Sci. Soc.*, (2) 28(1), 81-86.
17. Raj, D. (1965). On a method of using multi-auxiliary information in sample surveys. *J. Amer. Statist. Assoc.*, 60, 309, 270-277.
18. Rao, P.S.R.S. and Mudholkar, G.S. (1967). Generalized multivariate estimators for the mean of a finite population. *J. Amer. Statist. Assoc.*, 62, 1009-1012.
19. Robinson, P.M. (1994). A class of estimators for the mean of a finite population using auxiliary information. *Sankhya*, B, 56, 389-399.
20. Samiuddin, M. and Hanif, M. (2007). Estimation of population mean in single and two phase sampling with or without additional information. *Pak. J. Statist.*, 23(2), 99-118.
21. Shukla, G.K. (1965). Multivariate regression estimate. *J. Indian Statist. Assoc.*, 3, 202-211.
22. Singh, M.P. (1965). On the estimation of ratio and product of population parameters. *Sankhya*, Ser. C, 27, 321-328.
23. Singh, M.P. (1967a). Multivariate product method of estimation for finite populations. *J. Ind. Soc. of Agricul. Statist*., 31, 375-378.
24. Singh, M.P. (1967b). Ratio cum product method of estimation. *Metrika*, 12, 34-43.
25. Srivastava, S.K. (1971). A generalized estimator for the mean of a finite population using multi-auxiliary information. *J. Amer. Statist. Assoc.*, 66, 404-407.
26. Tracy, D.S., Singh, H.P. and Singh, R. (1996). An alternative to the ratio-cum-product estimator in sample surveys. *J. Statist. Plann. and Infer.*, 53, 375-387.

GENERALIZED MULTIVARIATE RATIO ESTIMATORS USING MULTI-AUXILIARY VARIABLES FOR MULTI-PHASE SAMPLING*

Muhammad Hanif[1], **Zahoor Ahmed**[2] and **Munir Ahmad**[3]

[1] Lahore University of Management Sciences
Lahore, Pakistan. Email: hanif@lums.edu.pk

[2] University of Gujrat, Gujrat, Pakistan
Email: zahoor_ahmed_stat@yahoo.com

[3] National College of Business Administration and Economics
Lahore, Pakistan. Email: drmunir@brain.net.pk

ABSTRACT

In this paper we propose a number of generalized multivariate ratio estimators for two-phase and multi-phase sampling in the presence of multi-auxiliary variables for estimating population mean for a single variable and a vector of variables of interest(s). The expressions for mean square errors are also derived. The suggested estimators are theoretically compared and an empirical study has also been conducted.

KEY WORDS

Multi-Phase Sampling; Multivariate Ratio Estimator; Multi-Auxiliary Variables.

1. INTRODUCTION

The estimation of the population mean is an unrelenting issue in sampling theory and several efforts have been made to improve the precision of the estimators in the presence of multi-auxiliary variables. A variety of estimators have been proposed following different ideas of ratio, regression and product estimators.

Olkin (1958) was the first author to deal with the problem of estimating the mean of a survey variable when auxiliary variables are made available. John (1969) proposed two multivariate generalizations of ratio and product estimators which actually reduce to the Olkin's (1958) and Singh's (1967a) estimators. Srivastava (1971) proposed a general ratio-type estimator which generates a large class of estimators including most of the estimators up to that time proposed. Sen (1972) developed a multivariate ratio estimator under two-phase sampling using multi-auxiliary variables. Singh and Namjoshi (1988) discussed a class of multivariate regression estimators of population mean of study variable in two-phase sampling.

Ceccon and Diana (1996) provided a multivariate extension of the Naik and Gupta (1991) univariate class of estimators. Ahmed (2003) put forward chain based general estimators using multivariate auxiliary information under multiphase sampling. In the same situation, Perri (2005) recommended some new estimators obtained from Singh's (1965, 1967b) estimators.

In multipurpose surveys, the problem is to estimate population means of several variables simultaneously [Swain (2000)]. Tripathi and Khattree (1989) estimated means

*Published in Pak. J. Statist. (2009), Vol. 25(4).

of several variables of interest, using multi-auxiliary variables, under simple random sampling. Further Tripathi (1989) extended the results to the case of two phase sampling.

we suggest general classes of ratio estimators for estimating the population mean of study variable for two-phase and multi-phase sampling using multi-auxiliary variables when information on all multi-auxiliary variables (Full Information Case) or not on all auxiliary variables (No Information Case) is available for population (See Samiuddin and Hanif, 2007).

Before suggesting the estimators we provide Multi-phase sampling scheme and some useful notations and results in the following section.

2. MULTI-PHASE SAMPLING USING MULTI-AUXILIARY VARIABLES

Consider a population of N units. Let Y be the variable of interest and $X_1, X_2, ...X_q$ are q auxiliary variables. For multi-phase sampling design let n_h and n_k $\left(n_h < n_k\right)$ be sample sizes for h^{th} and k^{th} phase respectively. $x_{(h)i}$ and $x_{(k)i}$ denote the i^{th} auxiliary variables from h^{th} and k^{th} phase samples respectively and y_k denote the variable of interest from the k^{th} phase. Let, $\bar{X}_i$, C_{x_i} and ρ_{yx_i} denote the population mean, coefficient of variation of i^{th} auxiliary variables respectively and the population correlation coefficient of Y and X_i. Further let $\theta_h = \frac{1}{n_k} - \frac{1}{N}$, $\theta_k = \frac{1}{n_k} - \frac{1}{N}$. Also $y_{i(k)} = Y + e_{y_{i(k)}}$, $x_{(h)i} = X_i + e_{x_{(h)i}}$ and $x_{(k)i} = X_i + e_{x_{(k)i}}$; $\left(i = 1, 2, ..., k\right)$ where $e_{y_{(k)}}$, $e_{x_{(h)i}}$ and $e_{x_{(k)i}}$ are sampling errors. We assume that $E_k\left(e_{y_{(k)}}\right) = E_h\left(e_{x_{(h)i}}\right) = E_k\left(e_{x_{(k)i}}\right) = 0$ where E_h and E_k denote the expectations of errors of h^{th} and k^{th} phase sampling. Then for simple random sampling without replacement for both first and second phases, we write by using phase wise operation of expectations as:

$$E_k\left(e_{y_k}\right)^2 = \left(1 - \frac{n_k}{N}\right)\sigma_y^2,\ E_k\left(\bar{e}_{y_k}\right)^2 = \left(1 - \frac{n_k}{N}\right)\frac{\sigma_y^2}{n_k} = \theta_k \bar{Y}^2 C_y^2,$$

$$E_k\left(e_{y_k} e_{x_{(k)i}}\right) = \left(1 - \frac{n_k}{N}\right)\sigma_{yx_i} = \theta_k \overline{YX}_i C_y C_{x_i} \rho_{yx_i}$$

$$E_k\left(\bar{e}_{y_k} \bar{e}_{x_{(k)i}}\right) = \left(1 - \frac{n_k}{N}\right)\frac{\sigma_{yx_i}}{n_k} = \theta_k \overline{YX}_i C_y C_{x_i} \rho_{yx_i},$$

$$E_h E_{k|h}\left[e_{y_k}\left(e_{x_{(h)i}} - e_{x_{(k)i}}\right)\right] = E_h E_{k|h}\left(e_{y_k} e_{x_{(h)i}}\right) - E_k\left(e_{y_k} e_{x_{(k)i}}\right) = \frac{1}{N}\left(n_k - n_h\right)\sigma_{yx_i},$$

$$E_h E_{k|h}\left[\bar{e}_{y_k}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right)\right] = \left(1 - \frac{n_h}{N}\right)\frac{\sigma_{yx_i}}{n_h} - \left(1 - \frac{n_k}{N}\right)\frac{\sigma_{yx_i}}{n_k} = \left(\theta_h - \theta_k\right)\overline{YX}_i C_y C_{x_i} \rho_{yx_i}.$$

Similarly

$$E_h E_{k|h}\left[\bar{e}_{x_{(k)i}}\left(\bar{e}_{x_{(h)i}}-\bar{e}_{x_{(k)i}}\right)\right]=\left(\theta_h-\theta_k\right)\sigma_{x_i}^2=\left(\theta_h-\theta_k\right)\bar{X}_i^2 C_{x_i}^2,$$

$$E_h E_{k|h}\left[\bar{e}_{x_{(h)i}}\left(\bar{e}_{x_{(h)i}}-\bar{e}_{x_{(k)i}}\right)\right]=0,$$

$$E_h E_{k/h}\left(\bar{e}_{x_{(h)i}}-\bar{e}_{x_{(k)i}}\right)^2=\left(\theta_k-\theta_h\right)\sigma_{x_i}^2=\left(\theta_k-\theta_h\right)\bar{X}_i^2 C_{x_i}^2,$$

$$E_h E_{k|h}\left[\left(\bar{e}_{x_{(h)i}}-\bar{e}_{x_{(k)i}}\right)\left(\bar{e}_{x_{(h)j}}-\bar{e}_{x_{(k)j}}\right)\right]=\left(\theta_k-\theta_h\right)\sigma_{x_i x_j}$$

$$=\left(\theta_k-\theta_h\right)\bar{X}_i\bar{X}_j C_{x_i} C_{x_j}\rho_{x_i x_j};\left(i\neq j\right),$$

and

$$E_h E_{k|h}\left[\left(\bar{e}_{x_{(k)i}}\right)\left(\bar{e}_{x_{(h)j}}-\bar{e}_{x_{(k)j}}\right)\right]=\left(\theta_h-\theta_k\right)\sigma_{x_i x_j}=\left(\theta_h-\theta_k\right)\bar{X}_i\bar{X}_j C_{x_i} C_{x_j}\rho_{x_i x_j},\left(i\neq j\right).$$

The following notations will be used in deriving the mean square errors of proposed estimators

$\lvert R\rvert_{y\underset{\sim}{x}_q}$	Determinant of population correlation matrix of variables $y, x_1, x_2, ..., x_{q-1}$ *and* x_q.
$\lvert R_{yx_i}\rvert_{y\underset{\sim}{x}_q}$	Determinant of i^{th} minor of $\lvert R\rvert_{y\underset{\sim}{x}_q}$ corresponding to the i^{th} element of ρ_{yx_i}.
$\rho^2_{y.\underset{\sim}{x}_s}$	Denotes the multiple coefficient of determination of y on $x_1, x_2, ..., x_{r-1}$ and x_r.
$\rho^2_{y.\underset{\sim}{x}_q}$	Denotes the multiple coefficient of determination of y on $x_1, x_2, ..., x_{q-1}$ and x_q.
$\lvert R\rvert_{\underset{\sim}{x}_s}$	Determinant of population correlation matrix of variables $x_1, x_2, ..., x_{r-1}$ and x_r.
$\lvert R\rvert_{\underset{\sim}{x}_q}$	Determinant of population correlation matrix of variables $x_1, x_2, ..., x_{q-1}$ and x_q.
$\lvert R\rvert_{y_i\underset{\sim}{x}_s}$	Determinant of the correlation matrix of $y_i, x_1, x_2, ..., x_{r-1}$ and x_r.
$\lvert R\rvert_{y_i\underset{\sim}{x}_q}$	Determinant of the correlation matrix of $y_i, x_1, x_2, ..., x_{q-1}$ and x_q.
$\lvert R\rvert_{y_i y_j\underset{\sim}{x}_s}$	Determinant of the minor corresponding to $\rho_{y_i y_j}$ of the correlation matrix of $y_i, y_j, x_1, x_2, ..., x_{r-1}$ and x_r, for $(i\neq j)$.
$\lvert R\rvert_{y_i y_j\underset{\sim}{x}_q}$	Determinant of the minor corresponding to $\rho_{y_i y_j}$ of the correlation matrix of $y_i, y_j, x_1, x_2, ..., x_{q-1}$ and x_q, for $(i\neq j)$.

2.1 Result: 1

The following result will help in deriving the mean square errors of suggested estimators

$$\frac{\lvert R\rvert_{y\underset{\sim}{x}_q}}{\lvert R\rvert_{\underset{\sim}{x}_q}}=\left(1-\rho^2_{y.\underset{\sim}{x}_q}\right),$$ [Arora and Lal (1989)].

3. GENERALIZED MULTIVARIATE RATIO ESTIMATOR FOR MULTI-PHASE SAMPLING

We propose a more general multivariate ratio estimator when information on all auxiliary variables is not available for population (No Information Case) and we obtain information on variables of interests $\left(y_1, y_2, ..., y_p\right)$ and for auxiliary variables $\left(x_1, x_2, ..., x_q\right)$ form k^{th} phase and also for auxiliary variables from h^{th} phase. The proposed estimator is

$$T_{hk(1\times p)} = \left[\bar{y}_{(k)1} \prod_{i=1}^{q} \left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}} \right)^{\alpha_{i1}} \quad \bar{y}_{(k)2} \prod_{i=1}^{q} \left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}} \right)^{\alpha_{i2}} \quad \cdots \quad \bar{y}_{(k)p} \prod_{i=1}^{q} \left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}} \right)^{\alpha_{ip}} \right]. \quad (3.1)$$

$$= \Bigg[\left(\bar{Y}_1 + \bar{e}_{y_{(k)1}}\right)\left(1 + \sum_{i=1}^{q} \frac{\alpha_{i1}}{\bar{X}_i}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right)\right)$$

$$\left(\bar{Y}_2 + \bar{e}_{y_{(k)2}}\right)\left(1 + \sum_{i=1}^{q} \frac{\alpha_{i2}}{\bar{X}_i}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right)\right)$$

$$\cdots \quad \left(\bar{Y}_p + \bar{e}_{y_{(k)p}}\right)\left(1 + \sum_{i=1}^{q} \frac{\alpha_{ip}}{\bar{X}_i}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right)\right) \Bigg]$$

$$= \left[\left(\bar{Y}_1 + \bar{e}_{y_{(k)1}}\right) \quad \left(\bar{Y}_2 + \bar{e}_{y_{(k)2}}\right) \quad \cdots \quad \left(\bar{Y}_p + \bar{e}_{y_{(k)p}}\right) \right]$$

$$+ \Bigg[\sum_{i=1}^{q} \alpha_{i1} \frac{\bar{Y}_1}{\bar{X}_i}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right) \quad \sum_{i=1}^{q} \alpha_{i2} \frac{\bar{Y}_2}{\bar{X}_i}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right)$$

$$\cdots \quad \sum_{i=1}^{q} \alpha_{ip} \frac{\bar{Y}_p}{\bar{X}_i}\left(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}\right) \Bigg]$$

$$= \left[\left(\bar{Y}_1 + \bar{e}_{y_{(k)1}}\right) \quad \left(\bar{Y}_2 + \bar{e}_{y_{(k)2}}\right) \quad \cdots \quad \left(\bar{Y}_p + \bar{e}_{y_{(k)p}}\right) \right]$$

$$+ \left[\left(\bar{e}_{x_{(h)1}} - \bar{e}_{x_{(k)1}}\right) \quad \left(\bar{e}_{x_{(h)2}} - \bar{e}_{x_{(k)2}}\right) \quad \cdots \quad \left(\bar{e}_{x_{(h)q}} - \bar{e}_{x_{(k)q}}\right) \right] \left[\frac{\bar{Y}_j}{\bar{X}_i} \alpha_{ij} \right]_{(q\times p)}$$

$$\left(T_{hk(1\times p)} - \bar{Y}\right) = \left[\bar{e}_{y_{(k)1}} \quad \bar{e}_{y_{(k)2}} \quad \cdots \quad \bar{e}_{y_{(k)p}} \right]$$

$$+ \left[\left(\bar{e}_{x_{(h)1}} - \bar{e}_{x_{(k)1}}\right) \quad \left(\bar{e}_{x_{(h)2}} - \bar{e}_{x_{(k)2}}\right) \quad \cdots \quad \left(\bar{e}_{x_{(h)q}} - \bar{e}_{x_{(k)q}}\right) \right] A^*$$

or

$$T_{hk(1\times p)} - \bar{Y} = \bar{D}_y + \bar{D}_x A^*.$$

where

$$T_{hk(1\times p)} = \left[\bar{y}_{(k)1} \quad \bar{y}_{(k)2} \quad \cdots \quad \bar{y}_{(k)p} \right], \bar{Y} = \left[\bar{Y}_1 \quad \bar{Y}_2 \quad \cdots \quad \bar{Y}_p \right],$$

$$\bar{D}_y = \begin{bmatrix} \bar{e}_{y_{(k)1}} & \bar{e}_{y_{(k)2}} & \dots & \bar{e}_{y_{(k)p}} \end{bmatrix},$$

$$\bar{D}_x = \left[\left(\bar{e}_{x_{(h)1}} - \bar{e}_{x_{(k)1}}\right) \ \left(\bar{e}_{x_{(h)2}} - \bar{e}_{x_{(k)2}}\right) \ \dots \ \left(\bar{e}_{x_{(h)q}} - \bar{e}_{x_{(k)q}}\right)\right] \text{ and}$$

$$A^* = \left[\alpha^*_{ij}\right]_{(q\times p)} = \left[\frac{\bar{Y}_j}{\bar{X}_i}\alpha_{ij}\right]_{(q\times p)} ;\ \left(i = 1, 2, ..., q\ ,\ j = 1, 2, ..., p\right).$$

We use information related to auxiliary variables from h^{th} and k^{th} phase both then the variance covariance matrix of $t_{1m(1\times p)}$ is written as:

$$\Sigma_{T_{hk}(p\times p)} = E_h E_{k/h}\left(T_{hk(1\times p)} - \bar{Y}\right)'\left(T_{hk(1\times p)} - \bar{Y}\right)$$

$$= E_h E_{k/h}\left[\left(\bar{D}_y + \bar{D}_x A^*\right)'\left(\bar{D}_y + \bar{D}_x A^*\right)\right].$$

We can write

$$E_h E_{k/h}\left(\bar{D}'_y \bar{D}_y\right) = \theta_k \Sigma_y = \theta_k [\sigma_{y_i y_j}]_{(p\times p)}, \text{ for } i = j, \sigma_{y_i y_j} = \sigma^2_{y_i}$$

$$E_h E_{k/h}\left(\bar{D}'_y \bar{D}_x\right) = \left(\theta_h - \theta_k\right)\Sigma_{yx} = \left(\theta_h - \theta_k\right)[\sigma_{y_i x_j}]_{(p\times q)}$$

and

$$E_h E_{k/h}\left(\bar{D}'_x \bar{D}_x\right) = \left(\theta_k - \theta_h\right)\Sigma_x = \left(\theta_k - \theta_h\right)[\sigma_{x_i x_j}]_{(q\times q)}, \text{for } i = j, \sigma_{x_i x_j} = \sigma^2_{x_i}$$

Using above substitutions in expression of variance covariance matrix, we write:

$$\Sigma_{T_{hk}(p\times p)} = \theta_k \Sigma_{y_{(p\times p)}} + \left(\theta_h - \theta_k\right) A^{*'}_{(p\times q)} \Sigma'_{yx_{(q\times p)}}$$

$$+ \left(\theta_h - \theta_k\right)\Sigma_{yx_{(p\times q)}} A^*_{(q\times p)} + \left(\theta_k - \theta_h\right) A^{*'}_{(p\times q)} \Sigma_{x_{(q\times q)}} A^*_{(q\times p)}.$$

Given that $\Sigma^{-1}_{x(q\times q)}$ exist, the value A' of that minimizes the variance covariance matrix of $t_{(1\times p)}$ will be

$$A^*_{(q\times p)} = \Sigma^{-1}_{x_{(q\times q)}} \Sigma'_{yx_{(q\times p)}}. \tag{3.2}$$

The transpose of $A^*_{(q\times p)}$ is $\Sigma_{yx(p\times q)}\Sigma^{-1}_{x(q\times q)} = \mathrm{B}_{yx}$ it is actually the matrix of regression coefficients for a multivariate regression model in which $p\left(Y_1, Y_2, ..., Y_p\right)$ dependent variables are regressed on q $\left(X_1, X_2, ..., X_q\right)$ dependent variables. These regression coefficients are usually unknown and they are estimated from second phase sample. Then the variance covariance matrix after simplification can be written as:

$$\Sigma_{T_{hk}(p\times p)} = \theta_k \Sigma_{y(p\times p)} - \left(\theta_k - \theta_h\right)\Sigma'_{yx(p\times q)}\Sigma^{-1}_{x(q\times q)}\Sigma_{yx(q\times p)}. \tag{3.3}$$

The variance covariance matrix form of variance of y_i, covariances and correlation coefficients of x_i and y_i is written as:

$$\Sigma_{T_{hk}(p\times p)} = \left[\sigma_{y_i}\sigma_{y_j}\left(\theta_k\left(\rho_{y_iy_j} - \rho_{y_iy_j\cdot \underset{\sim}{x}_q}\right) + \theta_h\rho_{y_iy_j\cdot \underset{\sim}{x}_q}\right)\right]_{p\times p};\left(i, j = 1, 2, ..., p\right), \quad (3.4)$$

for $i = j$, $\sigma_{y_i}\sigma_{y_j}\left(\rho_{y_iy_j} - \rho_{y_iy_j\cdot \underset{\sim}{x}_q}\right) = \sigma_{y_i}^2\left(1-\rho_{y_i\cdot \underset{\sim}{x}_q}^2\right)$.

For $|R|_{\underset{\sim}{x}_q} \neq 0$, we see that

$$\left(1-\rho_{y_i\cdot \underset{\sim}{x}_q}^2\right) = \frac{|R|_{y_i\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}};\left(i = 1, 2, ..., p\right) \text{ and } \left(\rho_{y_iy_j} - \rho_{y_iy_j\cdot \underset{\sim}{x}_q}\right) = \frac{|R|_{y_iy_j\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}};\left(i, j = 1, 2, ..., p\right).$$

Now after simplification we write the variance covariance matrix as:

$$\Sigma_{T_{hk}(p\times p)} = \left[\sigma_{y_i}\sigma_{y_j}\left[\theta_k\left(\frac{|R|_{y_iy_j\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\right) + \theta_h\left(\rho_{y_iy_j} - \frac{|R|_{y_iy_j\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\right)\right]\right]_{p\times p};\left(i, j = 1, 2, ..., p\right), \quad (3.5)$$

for $i = j$, $\sigma_{y_i}\sigma_{y_j}\dfrac{|R|_{y_iy_j\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}} = \sigma_{y_i}^2\dfrac{|R|_{y_i\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}$.

Remark-1:

To develop generalized multivariate ratio estimator for two-phase sampling using multi-auxiliary variables when information on all auxiliary variables is not available for population (No Information Case), replace h by 1 and k by 2 in (3.1), we get the following estimator

$$T_{12(1\times p)} = \left[\bar{y}_{(2)1}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\alpha_{i1}} \quad \bar{y}_{(2)2}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\alpha_{i2}} \quad \cdots \quad \bar{y}_{(2)p}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\alpha_{ip}}\right] \quad (3.6)$$

The expression of unknown matrix for which the mean square error will be minimum is same as given in (3.2) and the expression for variance covariance matrix can be directly written from (3.3) just replacing h by 1 and k by 2 as:

$$\Sigma_{T_{12}(p\times p)} = \theta_2\Sigma_{y(p\times p)} - \left(\theta_2 - \theta_1\right)\Sigma'_{yx(p\times q)}\Sigma^{-1}_{x(q\times q)}\Sigma_{yx(q\times p)} \quad (3.7)$$

The variance covariance matrix in the form of variance of y_i, covariances and correlation coefficients of x_i and y_i is written as:

$$\Sigma_{T_{12}(p\times p)} = \left[\sigma_{y_i}\sigma_{y_j}\left(\theta_2\left(\rho_{y_iy_j} - \rho_{y_iy_j\cdot \underset{\sim}{x}_q}\right) + \theta_1\rho_{y_iy_j\cdot \underset{\sim}{x}_q}\right)\right]_{p\times p};\left(i, j = 1, 2, ..., p\right), \quad (3.8)$$

for $i = j$, $\sigma_{y_i}\sigma_{y_j}\left(\rho_{y_iy_j} - \rho_{y_iy_j\cdot \underset{\sim}{x}_q}\right) = \sigma_{y_i}^2\left(1-\rho_{y_i\cdot \underset{\sim}{x}_q}^2\right)$.

In the form of determinates we can write

$$\Sigma_{T_{12}(p\times p)}=\left[\sigma_{y_i}\sigma_{y_j}\left[\theta_2\left(\frac{|R|_{y_iy_j\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\right)+\theta_1\left(\rho_{y_iy_j}-\frac{|R|_{y_iy_j\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\right)\right]\right]_{p\times p}\ ;\ (i,j=1,2,...,p), \tag{3.9}$$

for $i=j$, $\sigma_{y_i}\sigma_{y_j}\dfrac{|R|_{y_iy_j\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}=\sigma_{y_i}^2\dfrac{|R|_{y_i\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}$.

Remark-2:

To develop a univariate generalized ratio estimator for multiphase sampling using multi auxiliary variable when information on all auxiliary variables is not known (No Information Case). This estimator can be made if we put $p=1$ in (3.1) as:

$$T_{hk}=\bar{y}_{(k)1}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}}\right)^{\alpha_{i1}}. \tag{3.10}$$

The expression for vector of unknown constants for which the mean square error will be minimum can be written from (3.2) as

$$A^*_{(q\times 1)}=\Sigma^{-1}_{x_{(q\times q)}}\Sigma'_{yx_{(q\times 1)}}. \tag{3.11}$$

It can be written in determinants form as:

$$\alpha_i^*=(-1)^{i+1}\frac{\bar{Y}}{\bar{X}_i}\frac{C_y}{C_{x_i}}\frac{\left|R_{yx_i}\right|_{y\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}=(-1)^{i+1}\beta_{yx_i\cdot\underset{\sim}{x}_q}\ ,\ (i=1,2,....q).$$

The expression for mean square error can be directly written from (3.3) as:

$$MSE(T_{hk})=\theta_k\sigma_y^2-(\theta_k-\theta_h)\left(\sigma_x^2\right)^{-1}\Sigma'_{yx(1\times q)}\Sigma_{yx(q\times 1)} \tag{3.12}$$

It can be written the form of multiple coefficient of determination as:

$$MSE(T_{hk})=\bar{Y}^2C_y^2\left[\theta_k\left(1-\rho^2_{y.\underset{\sim}{x}_q}\right)+\theta_h\rho^2_{y.\underset{\sim}{x}_q}\right] \tag{3.13}$$

Remark-3:

To develop a generalized univariate ratio estimator for two sampling using multi-auxiliary variables when information on all auxiliary variables is not known (No Information Case) we put $h=1$ and $k=2$ in (3.10). The required estimator becomes

$$T_{12}=\bar{y}_{(2)1}\prod_{i=1}^{q}\left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}}\right)^{\alpha_{i1}}. \tag{3.14}$$

The expression for vector of unknown constants for which the mean square error will be minimum is same as given in (3.10) and the expression for mean square error can be written from (3.12) just by replacing $h=1$ and $k=2$ as:

$$MSE\left(T_{12}\right)=\theta_2\sigma_y^2-\left(\theta_2-\theta_1\right)\left(\sigma_x^2\right)^{-1}\Sigma'_{yx(1\times q)}\Sigma_{yx(q\times 1)} \tag{3.15}$$

It can be written in the form of multiple coefficient of determination from (3.13) as:

$$MSE\left(T_{12}\right)=\bar{Y}^2C_y^2\left[\theta_2\left(1-\rho^2_{y.\underset{\sim}{x}_q}\right)+\theta_1\rho^2_{y.\underset{\sim}{x}_q}\right] \tag{3.16}$$

Remark-4:

To develop generalized multivariate ratio estimator for multi-phase sampling using multi auxiliary variables when information on all auxiliary variables is known for population (Full Information Case), replace $\bar{x}_{(h)i}$ by $\bar{X}_i$ in (3.1). For this replacement $n_h\to N$ and $\theta_h\to 0$. The estimator becomes

$$T_{k(1\times p)}=\left[\bar{y}_{(k)i}\prod_{i=1}^{q}\left(\frac{\bar{X}_i}{\bar{x}_{(k)i}}\right)^{\alpha_{i1}}\quad \bar{y}_{(k)2}\prod_{i=1}^{q}\left(\frac{\bar{X}_i}{\bar{x}_{(k)i}}\right)^{\alpha_{i2}}\quad \cdots\quad \bar{y}_{(k)p}\prod_{i=1}^{q}\left(\frac{\bar{X}_i}{\bar{x}_{(k)i}}\right)^{\alpha_{ip}}\right]. \tag{3.17}$$

The expression for the matrix of unknown constants is same as given in (3.2) and the expression for variance covariance matrix can be written from (3.3) as:

$$\Sigma_{T_k(p\times p)}=\theta_k\left(\Sigma_{y(p\times p)}-\Sigma_{yx(p\times q)}\Sigma^{-1}_{x(q\times q)}\Sigma'_{yx(q\times p)}\right) \tag{3.18}$$

The variance covariance matrix in the form of variance of y_i, covariances and correlation coefficients of x_i and y_i can be written as:

$$\Sigma_{T_k(p\times p)}=\theta_k\left[\sigma_{y_i}\sigma_{y_j}\left(\rho_{y_iy_j}-\rho_{y_iy_j.\underset{\sim}{x}_q}\right)\right]_{p\times p};\ \left(i,j=1,2,\ldots,p\right), \tag{3.19}$$

for $i=j$, $\sigma_{y_i}\sigma_{y_j}\left(\rho_{y_iy_j}-\rho_{y_iy_j.\underset{\sim}{x}_q}\right)=\sigma^2_{y_i}\left(1-\rho^2_{y_i.\underset{\sim}{x}_q}\right)$.

The variance covariance matrix in the form of determinants can be written as:

$$\Sigma_{T_k(p\times p)}=\theta_k\left[\sigma_{y_i}\sigma_{y_j}\frac{|R|_{y_iy_j\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}\right]_{p\times p},\ \left(i,j=1,2,\ldots,p\right), \tag{3.20}$$

for $i=j$, $\sigma_{y_i}\sigma_{y_j}\dfrac{|R|_{y_iy_j\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}=\sigma^2_{y_i}\dfrac{|R|_{y_i\underset{\sim}{x}_q}}{|R|_{\underset{\sim}{x}_q}}$.

Remark-5:

To develop generalized multivariate ratio estimator for two-phase sampling using multi auxiliary variables when information on all auxiliary variables is known for population (Full Information Case), replace k by 2 in (3.17). The estimator becomes

$$T_{2(1\times p)} = \left[\bar{y}_{(2)i} \prod_{i=1}^{q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}} \right)^{\alpha_{i1}} \quad \bar{y}_{(k)2} \prod_{i=1}^{q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}} \right)^{\alpha_{i2}} \quad \ldots \quad \bar{y}_{(k)p} \prod_{i=1}^{q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}} \right)^{\alpha_{ip}} \right]. \quad (3.21)$$

The expression for the matrix of unknown constants will be same as given in (3.2) and the expression for variance covariance matrix can be written from (3.18) as:

$$\Sigma_{T_2(p\times p)} = \theta_2 \left(\Sigma_{y(p\times p)} - \Sigma_{yx(p\times q)} \Sigma_{x(q\times q)}^{-1} \Sigma'_{yx(q\times p)} \right) \quad (3.22)$$

The variance covariance matrix in the form of variance of y_i, covariances and correlation coefficients of x_i and y_i and in the form of determinants can obtained from (3.19) and (3.20) respectively just by replacing k by 2.

Remark-6:

To develop a generalized univariate ratio estimator for multi-sampling using multi-auxiliary variables when information on all auxiliary variables is known for population (Full Information Case), put $p = 1$ in (3.17). The estimator becomes

$$T_k = \bar{y}_k \prod_{i=1}^{q} \left(\frac{\bar{X}_i}{\bar{x}_{(k)i}} \right)^{\alpha_i} \quad (3.23)$$

The expression for unknown constant for which the mean square error will be minimum of above estimator is same as given in (3.11). The expression of mean square error can be written from (3.18) by putting $p = 1$ as:

$$MSE(T_k) = \theta_k \left(\sigma_y^2 - \left(\sigma_x^2\right)^{-1} \Sigma'_{yx(1\times q)} \Sigma_{yx(q\times 1)} \right) \quad (3.24)$$

It can also be written as:

$$MSE(T_k) = \theta_k \bar{Y}^2 C_y^2 \left(1 - \rho_{y.\underset{\sim}{x}_q}^2 \right) \quad (3.25)$$

Remarks-7:

To develop a generalized univariate ratio estimator for two-phase sampling using multi-auxiliary variables when information on all auxiliary variables is known for population (Full Information Case), replacing k by 2 in (3.23). The estimator becomes

$$T_2 = \bar{y}_2 \prod_{i=1}^{q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}} \right)^{\alpha_i} \quad (3.26)$$

The expression for unknown constant for which the mean square error of above estimator will be minimum is same as given in (3.11). The expression of mean square error can be written from (3.24) just replacing k by 2 as:

$$MSE(T_2) = \theta_2 \left(\sigma_y^2 - \left(\sigma_x^2\right)^{-1} \Sigma'_{yx(1\times q)} \Sigma_{yx(q\times 1)} \right) \quad (3.27)$$

It can also be written as:

$$MSE\left(T_2\right)=\theta_2\bar{Y}^2C_y^2\left(1-\rho_{y.\underset{\sim}{x}_q}^2\right) \tag{3.28}$$

4. THEORETICAL COMPARISON OF NEWLY DEVELOPED ESTIMATORS

Obviously estimator for which the information on all auxiliary variables is available for population will be more efficient than that for which the information on all auxiliary variables is not available for population. It means in the case of two-phase sampling generalized ratio estimator developed for FIC T_2 is more efficient than or NIC T_{12}. It can be checked by considering the mean square errors of suggested estimators as:

$$MSE\left(T_2\right)-MSE\left(T_{12}\right)=-\theta_1\bar{Y}^2C_y^2\rho_{y.\underset{\sim}{x}_q}^2<0$$

For multiphase sampling generalized ratio estimator developed for FIC T_k is more efficient than generalized ratio estimator for NIC t_{2m}. It can be checked by considering the mean square errors of these estimators as:

$$MSE\left(T_k\right)-MSE\left(T_{hk}\right)=-\theta_h\bar{Y}^2C_y^2\rho_{y.\underset{\sim}{x}_q}^2<0$$

Also the estimators developed for multi-phase sampling will be less efficient than those which are developed for two-phase sampling because if we increase the phases the efficiency will decrease but cost will reduced. It can be checked for FIC and NIC as:

$$MSE\left(T_2\right)-MSE\left(T_k\right)=\left(\theta_2-\theta_k\right)\bar{Y}^2C_y^2\left(1-\rho_{y.\underset{\sim}{x}_q}^2\right)<0; \quad k>2,$$

and

$$MSE\left(T_{12}\right)-MSE\left(T_{hk}\right)=\bar{Y}^2C_y^2\left[\left(\theta_2-\theta_k\right)\left(1-\rho_{y.\underset{\sim}{x}_q}^2\right)+\left(\theta_1-\theta_h\right)\rho_{y.\underset{\sim}{x}_q}^2\right]<0;$$

$$k>2 \text{ and } 1<h<k$$

Theoretical comparison on the basis of generalized MSE's can be made for all multivariate estimators. These comparisons give same results as discussed above for univariate case.

5. EMPIRICAL STUDY OF NEWLY DEVELOPED ESTIMATORS

For empirical comparison of newly developed multivariate and univariate estimators using multi-auxiliary variables for no and full information cases under two and multi-phase sampling we consider five natural populations. The data is used from five districts census reports of province Punjab, Pakistan. The detail of populations and variables description is given in Table A-1.1 and Table A-1.2 respectively of Appendix A. We consider three variables of interests denoted by Y's and five auxiliary variables denoted by X's for computing the determinants of matrices of MSE's of multivariate ratio estimators and for univariate we consider Y_2 as study variable and the same five auxiliary variables as considered in multivariate case. The necessary parameters of populations for computing MSE's are given in A-1.3. We calculate pair-wise (determinant of matrices of MSE's)/MSE's for no information case and for full information case we calculate (determinant of matrices of MSE's)/MSE's for each phase for first five phases. The determinant of matrices of mean square errors of multivariate ratio estimators for multiphase sampling using pair-wise phases for no information case are given in Table A-1.4.1 and for full information case using each phase for full information case are given in A-1.4.2. The mean square errors of univariate estimators for multiphase sampling using pair-wise phases for no information case are given in Table A-1.5.1 and for full information case using each phase are given in A-1.5.2.

From Table A-1.4.1 and Table A-1.4.2, we can say that the multivariate ratio estimators for full information case are more efficient than no information case for each phase e.g. T_2 is more efficient than T_{12}, T_3 is more efficient than T_{13} & T_{23} etc. and the same is true for univariate ratio estimators (See Table A-1.5.1 and Table A-1.5.2). Furthermore we can say for no information case from Table A-1.4.1 that as we increase phase the efficiency decreases e.g. T_{12}, is more efficient than all others, T_{13} is more efficient than all others except T_{12}, T_{34} is more efficient than T_{35}, T_{45} but less efficient than all others and so on, similarly the same argument can be made for univariate case given in Table A-1.5.1. Also for full information case the estimators become less efficient as we increase phases because the sample size decreases by increasing phases, it can be seen from Table A-1.4.2 and A-1.5.2 for multivariate and univariate estimators respectively.

REFERENCES

1. Ahmed, M.S. (2003). General Chain estimators under multi phase sampling. *J. Applied Statist. Sci*., 12(4), 243-250.
2. Arora, S. and Lal. Bansi (1989). *New Mathematical Statistics*. Satya Prakashan, New Delhi.
3. Ceccon, C. and Diana, G. (1996). Classi di stimatori di tipo rapporto in presenza di pi´u variabili ausiliarie, *Proceedings of Italian Statistical Society*, XXXVIII Scientific Meeting, Rimini, 9-13 April 1996, 119-126.
4. John, S. (1969). On multivariate ratio and product estimators. *Biometrika*, 56, 533-536.
5. Naik, V.D. and Gupta, P.C. (1991). A general class of estimators for estimating population mean using auxiliary information. *Metrika*, 38, 11-17.
6. Olkin, I. (1958). Multivariate ratio estimation for finite populations. *Biometrika*, 45, 154-165.
7. Perri, P.F. (2005). Combining two auxiliary variables in ratio-cum-product type estimators. *Proceedings of Italian Statistical Society, Intermediate Meeting on Statistics and Environment*, Messina, 21-23 September 2005, 193-196.
8. Samiuddin, M. and Hanif, M. (2007). Estimation of population mean in single and two phase sampling with or without additional information. *Pak. J. Statist.*, 23(2), 99-118.
9. Sen, A.R. (1972). Successive Sampling with $p(p \geq 1)$ auxiliary variables. *Ann. Math. Statist.*, 43(6), 2031-2034.
10. Shukla, G.K. (1965). Multivariate regression estimate. *J. Indian Statist. Assoc.*, 3, 202-211.

11. Singh, M.P. (1965). On the estimation of ratio and product of population parameters. *Sankh*, Ser. C, 27, 321-328.
12. Singh, M.P. (1967a). Multivariate product method of estimation for finite populations. *J. Ind. Soc. Agri. Statist*., 31, 375-378.
13. Singh, M.P. (1967b). Ratio cum product method of estimation. *Metrika*, 12, 34-43.
14. Singh, H.P. and Namjoshi, U.D. (1988). A class of multivariate regression estimators in two-phase sampling. *Assam Statist. Rev.*, 2(2), 1-7.
15. Srivastava, S.K. (1971). A generalized estimator for the mean of a finite population using multi-auxiliary information. *J. Amer. Statist. Assoc.*, 66, 404-407.
16. Swain, A.K.P.C. (2000). *Current Developments in Survey Sampling*. Om Prakash, Bhubaneswar, India.
17. Tripathi, T.P. (1989). Optimum estimation of mean vector for dynamic population. Invited paper in the *Proceeding of International Symposium on Optimization and Statistics.*, held at AMU. Aligarh.

APPENDIX A

Table A-1.1: Detail of Populations

S#	Source of Populations
1	Population census report of Jhang district (1998), Pakistan
2	Population census report of Faisalabad district (1998), Pakistan.
3	Population census report of Gujrat district (1998), Pakistan.
4	Population census report of Kasur (1998) Pakistan
5	Population census report of Sialkot district (1998), Pakistan.

Table A-1.2: Description of variables (Each variables is taken from Rural Locality)

Description of variables			
Y_1	Literacy ratio	X_2	Population of primary but below matric
Y_2	Population of currently married	X_3	Population of matric and above
Y_3	Total household	X_4	Population of 18 years old and above
X_1	Population of both sexes	X_5	Population of women 15-49 years old

Table A-1.3: Parameters of populations for calculating the Matrices of MSE's of multivariate estimators and MSE's of univariate estimators

Districts	N	n_1	n_2	n_3	n_4	n_5	$\bar{Y}_1$	$\bar{Y}_2$	$\bar{Y}_3$	C_{y_1}	C_{y_2}	C_{y_3}
Jhang	368	184	92	46	23	12	29.705	860.11	897.71	0.270	0.595	0.512
Faisalabad	283	142	71	35	18	9	51.394	1511.260	1540.530	3.210	0.522	0.478
Gujrat	204	102	51	26	13	6	57.535	1101.280	1102.540	0.145	0.484	0.487
Kasur	181	91	45	23	11	6	31.890	1393.200	1449.020	0.747	0.551	0.530
Sailkot	269	135	67	34	17	8	52.061	1058.740	998.220	0.147	0.647	0.646

Table A-1.3 (Contd…)

Districts	σ_{y_1}	σ_{y_2}	σ_{y_3}	σ_{x_1}	σ_{x_2}	σ_{x_3}	σ_{x_4}	σ_{x_5}	$\rho_{y_1 y_2}$
Jhang	8.022	511.908	459.842	5626.450	455.060	170.670	2455.170	1064.480	.182

Faisalabad	164.950	788.380	736.395	5426.030	1677.920	525.670	6289.710	1482.170	.070
Gujrat	8.364	533.041	537.236	3507.160	940.480	381.690	8139.680	830.010	.055
Kasur	23.823	767.636	767.796	5515.420	1095.690	357.890	2719.210	1355.640	.295
Sailkot	7.641	685.019	644.886	4787.250	1172.710	603.220	2461.590	1151.320	.324

Table A-1.3 (Contd…)

Districts	$\rho_{y_1y_3}$	$\rho_{y_2y_3}$	$\rho_{y_1x_1}$	$\rho_{y_1x_2}$	$\rho_{y_1x_3}$	$\rho_{y_1x_4}$	$\rho_{y_1x_5}$	$\rho_{y_2x_1}$	$\rho_{y_2x_2}$	$\rho_{y_2x_3}$	$\rho_{y_2x_4}$	$\rho_{y_2x_5}$
Jhang	.164	.733	.131	.460	.548	.185	.129	.428	.912	.659	.484	.425
Faisalabad	.084	.943	.072	.025	.033	.039	.042	.943	.927	.599	.731	.501
Gujrat	.056	.988	.092	.334	.543	.069	.103	.995	.941	.764	.490	.996
Kasur	.301	.989	.299	.255	.352	.301	.250	.998	.758	.879	.989	.799
Sailkot	.316	.997	.323	.426	.461	.338	.313	.999	.983	.931	.996	.939

Table A-1.3 (Contd…)

District	$\rho_{y_3x_1}$	$\rho_{y_3x_2}$	$\rho_{y_3x_3}$	$\rho_{y_3x_4}$	$\rho_{y_3x_5}$	$\rho_{x_1x_2}$	$\rho_{x_1x_3}$	$\rho_{x_1x_4}$
Jhang	.474	.732	.748	.559	.489	.416	.421	.317
Faisalabad	.967	.615	.747	.520	.822	.641	.782	.513
Gujrat	.984	.933	.749	.487	.986	.954	.796	.509
Kasur	.991	.752	.878	.988	.792	.764	.889	.993
Sailkot	.996	.980	.933	.994	.938	.983	.931	.997

Table A-1.3 (Contd…)

District	$\rho_{x_1x_5}$	$\rho_{x_2x_3}$	$\rho_{x_2x_4}$	$\rho_{x_2x_5}$	$\rho_{x_3x_4}$	$\rho_{x_3x_5}$	$\rho_{x_4x_5}$	$\rho^2_{y.x_1...x_5}$
Jhang	.275	.824	.475	.432	.590	.464	.325	0.885
Faisalabad	.819	.708	.359	.559	.543	.685	.436	0.869
Gujrat	.996	.892	.500	.958	.420	.797	.505	0.996
Kasur	.802	.798	.764	.614	.896	.719	.797	0.995
Sailkot	.939	.959	.985	.928	.939	.887	.938	0.997

Table A-1.4.1 Determinants of matrices of MSE's of multivariate ratio estimators for pair-wise phases (No Information Case)

District	**T_{12} (h=1,k=2)**	**T_{13} (h=1,k=3)**	**T_{14} (h=1,k=4)**	**T_{15} (h=1,k=5)**	**T_{23} (h=2,k=3)**	**T_{24} (h=2,k=4)**
Jhang	212279.97	1223241.603	7221747.657	46128460.86	3572866.63	16011134.82
Faisalabad	212515.48	1190925.96	6725695.10	40958629.30	2777523.40	13258374.47
Gujrat	95363.18	312081.54	1102996.85	4211396.91	1123210.67	3372979.29
Kasur	203091.03	901801.71	3838230.04	16192403.14	2176260.22	8973735.68
Sialkot	9555.27	41464.31	173806.26	723929.58	126175.45	482925.70

Table A-1.4.1 (Contd…)

District	T_{25} (h=2,k=5)	T_{34} (h=3,k=4)	T_{35} (h=3,k=5)	T_{45} (h=4,k=5)
Jhang	79961159.68	38868782.02	158093587.7	2.60491E+11
Faisalabad	67439732.76	27555308.50	122925398.17	243999111.56
Gujrat	11251537.17	10702183.50	30944428.16	93069748.63
Kasur	36805031.49	19922737.86	79389873.19	170075469.89
Sialkot	1897366.87	1256890.23	4554991.24	11150157.88

Table A-1.4.2 Determinants of matrices of MSE's of multivariate ratio estimators for each phase (Full Information Case)

District	T_1 (k=1)	T_2 (k=2)	T_3 (k=3)	T_4 (k=4)	T_5 (k=5)
Jhang	1023.378901	27631.23032	351018.963	3453903.79	30487480.83
Faisalabad	1535.620269	27767.37297	311260.3873	2908814.863	25077800.89
Gujrat	27.15981853	367.7474988	3710.595857	33124.8694	279522.8025
Kasur	103.7803005	1306.215104	12804.97189	112839.0944	946342.2579
Sialkot	2.156156072	36.85754751	405.2293669	3755.836241	32256.39965

Table A-1.5.1 MSE's of univariate ratio estimators for pair-wise phases (No Information Case)

District	T_{12} (h=1,k=2)	T_{13} (h=1,k=3)	T_{14} (h=1,k=4)	T_{15} (h=1,k=5)	T_{23} (h=2,k=3)	T_{24} (h=2,k=4)
Jhang	212279.97	1223241.603	7221747.657	46128460.86	3572866.63	16011134.82
Faisalabad	212515.48	1190925.96	6725695.10	40958629.30	2777523.40	13258374.47
Gujrat	95363.18	312081.54	1102996.85	4211396.91	1123210.67	3372979.29
Kasur	203091.03	901801.71	3838230.04	16192403.14	2176260.22	8973735.68
Sialkot	9555.27	41464.31	173806.26	723929.58	126175.45	482925.70

Table A-1.5.1 (Contd…)

District	T_{25} (h=2,k=5)	T_{34} (h=3,k=4)	T_{35} (h=3,k=5)	T_{45} (h=4,k=5)
Jhang	79961159.68	38868782.02	158093587.7	2.60491E+11
Faisalabad	67439732.76	27555308.50	122925398.17	243999111.56
Gujrat	11251537.17	10702183.50	30944428.16	93069748.63
Kasur	36805031.49	19922737.86	79389873.19	170075469.89
Sialkot	1897366.87	1256890.23	4554991.24	11150157.88

Table A-1.5.2 MSE's of univariate ratio estimators for each-wise phase (Full Information Case)

District	T_1 (k=1)	T_2 (k=2)	T_3 (k=3)	T_4 (k=4)	T_5 (k=5)
Jhang	81.89064	245.6719	573.2344	1228.36	2538.61
Faisalabad	131.0829	344.0564	770.0035	1621.898	3325.686
Gujrat	213.5579	509.0065	1099.904	2281.698	4645.286
Kasur	251.1011	584.0929	1250.076	2582.043	5245.977
Sialkot	142.167	366.2247	814.34	1710.571	3503.032

A FAMILY OF ESTIMATORS FOR SINGLE AND TWO-PHASE SAMPLING USING TWO AUXILIARY ATTRIBUTES*

Inam-ul-Haq[1], **Muhammad Hanif**[2] and **Munir Ahmad**[1]

[1] National College of Business Administration & Economics, Lahore, Pakistan. Email: inam-ul-haq786@hotmail.com; drmunir@ncbae.edu.pk

[2] Lahore University of Management Sciences, Lahore, Pakistan. Email: hanif@lums.edu.pk

ABSTRACT

Jhajj et al. (2006) proposed a general family of estimators and derived a general expression for mean square error of these estimators by using single auxiliary attribute. Hanif et al. (2009) proposed a more general form of Jhajj et al. (2006) family of estimators and derived expression for mean square error of particular estimators of proposed family by using *"k"* auxiliary attributes. In this paper we suggest some new estimators. We also derive the mean square error expression of Jhajj et al. (2006) estimator for the partial information case, using two auxiliary attributes. Mathematical comparisons of these estimators are made. An empirical study has also been conducted.

KEY WORDS

Full information, partial information, no information, auxiliary attribute; efficiency, bi-serial correlation coefficient, dichotomy

1. INTRODUCTION

The use of auxiliary information in estimation process is as old as history of survey sampling. The first use of auxiliary information in survey sampling can be traced to Neyman (1938). Generally the auxiliary variables are quantitative in nature but the use of qualitative auxiliary variables has been proposed in ratio, product and regression estimators by Naik and Gupta. (1996). Jhajj et al. (2006) proposed a family of estimators using single auxiliary attribute.

In this paper we develop a set of estimators which are an improved form of Jhajj et al. (2006) as well as Shabbir and Gupta (2007) estimators. Let $\left(y_i, \tau_{i1}, \tau_{i2}\right)$ be the *i*th sample point from a population of size N, where $\tau_j \left(j=1,2\right)$ is the value of *j*th auxiliary attribute. Suppose that the complete dichotomy is recorded for each attribute, so that $\tau_{ij}=1$ if *i*th unit of population possesses *j*th attribute and 0 otherwise. Let $A_j=\sum_{i=1}^{N}\tau_{ij}$ and $a_j=\sum_{i=1}^{n}\tau_{ij}$ be the total number of units in the population and sample respectively possessing attribute τ_j. Let $P_j=N^{-1}A_j$ and $p_j=n^{-1}a_j$ be the corresponding proportions.

*Published in Pak. J. Statist. (2011), Vol. 27(1).

Using simple random sampling without replacement (SRSWOR) scheme, let us define $\bar{e}_y = \bar{y} - \bar{Y}$ and $\bar{e}_{\tau_j} = p_j - P_j$ with following properties:

$$E\left(\bar{e}_y^2\right) = \theta S_y^2,\ E\left(\bar{e}_{\tau_j}^2\right) = \theta S_{\tau_j}^2,\ E\left(\bar{e}_y \bar{e}_{\tau_j}\right) = \theta S_y S_{\tau_j} \rho_{pb_j},\ E\left(\bar{e}_{\tau_1} \bar{e}_{\tau_2}\right) = \theta S_{\tau_1} S_{\tau_2} Q_{12},$$

where $\theta = n^{-1} - N^{-1}$, $E\left(\bar{e}_y\right) = 0 = E\left(\bar{e}_{\tau_j}\right)$ and $S_{y\tau_j} = \frac{1}{N-1}\sum_{j=1}^{N}\left(y_i - \bar{Y}\right)\left(\tau_{ij} - P_j\right)$. Let $\rho_{Pbj} = S_{y\tau_j} \Big/ \left(S_y S_{\tau_j}\right)$ be the point bi-serial correlation coefficient and Q_{12} is coefficient of association. Also S_y^2 is variance of main variable and $S_{\tau_j}^2$ is variance of *j*th auxiliary attribute. Let n_1 and n_2 be the size of first-phase and second-phase sample respectively, so that $n_2 < n_1$ (n_2 is sub sample of n_1) and $p_{j(1)}$, $p_{j(2)}$ are the proportions of units possessing attribute τ_j in first-phase and second-phase sample, respectively. The mean of the main variable of interest at the second phase is denoted by $\bar{y}_2$. Also

$$\bar{e}_{y_2} = \bar{y}_2 - \bar{Y},\ \bar{e}_{\tau_j(1)} = p_{j(1)} - P_j,\ \bar{e}_{\tau_j(2)} = p_{j(2)} - P_j \quad \left(j = 1, 2\right),$$

$$E\left(\bar{e}_{y_2}\right) = \theta_2 S_y^2,\ E\left\{\bar{e}_{y_2}\left(\bar{e}_{\tau_{j(2)}} - \bar{e}_{\tau_{j(1)}}\right)\right\} = \theta_3 S_y S_{\tau_j} \rho_{pbj},\ \theta_2 = n_2^{-1} - N^{-1},$$

$$E\left(\bar{e}_{\tau_{j(1)}} - \bar{e}_{\tau_{j(2)}}\right)^2 = \theta_3 S_{\tau_j}^2,\ E\left[\left(\bar{e}_{\tau_{1(2)}} - \bar{e}_{\tau_{1(1)}}\right)\left(\bar{e}_{\tau_{2(2)}} - \bar{e}_{\tau_{2(1)}}\right)\right] = \theta_3 S_{\tau_1} S_{\tau_2} Q_{12},$$

$$\theta_3 = \theta_2 - \theta_1,\ \theta_1 = n_1^{-1} - N^{-1}\quad S_y^2 = \frac{1}{N-1}\sum_{j=1}^{N}\left(y_i - \bar{Y}\right)^2,\ S_{\tau_j}^2 = \frac{1}{N-1}\sum_{j=1}^{N}\left(\tau_{ij} - P_j\right)^2,$$

for 2×2 contingency table $Q_{12} = \frac{ad - bc}{ad + bc}$, θ_1 and θ_2 are finite population correction factors(f.p.c) for first and second-phase scheme respectively.

2. SOME PREVIOUS ESTIMATORS BASED ON AUXILIARY ATTRIBUTES

In this section we reproduce previous estimators available in literature.

2.1 Single-Phase Sampling (Full Information Case)

i) If information on a single auxiliary attribute τ_1 is known then a family of estimators suggested by Jhajj et al. (2006) is given as,

$$T_{1(1)} = g_{\omega}\left(\bar{y}, v_1\right), \tag{2.1}$$

where $v_1 = p_1 / P_1$ and $g_{\omega}\left(\bar{y}, v_1\right)$ is a parametric function of $\bar{y}$ and v_1 such that $g_{\omega}\left(\bar{Y}, 1\right) = \bar{Y}$ and satisfying following regularity conditions.

(a) What ever be sample chosen, the point $(\bar{y}, v_1)$ assumes value in a bounded closed convex subset R_2 of two-dimensional real space containing the point $(\bar{Y}, 1)$.

(b) The function $g_\omega(\bar{y}, v_1)$ is continuous and bounded in R_2.

(c) The first order and second order partial derivatives of $g_\omega(\bar{y}, v_1)$ exist and are continuous as well as bounded in R_2.

The mean square error of (2.1) up to the term of order n^{-1} is,

$$MSE\left(T_{1(1)}\right) \approx \theta\left(1-\rho_{pb1}^2\right)S_y^2 . \tag{2.2}$$

ii) An estimator suggested by Shabbir and Gupta (2007) for the full information case is,

$$t_{2(1)} = \bar{y}\left[d_1 + d_2\left(P_1 - p_1\right)\right]\frac{P_1}{p_1} . \quad \left(p_1 > 0\right) \tag{2.3}$$

The values of d_1 and d_2 that minimize $MSE\left(t_{2(1)}\right)$ are,

$$d_1 = \frac{1}{1+\theta\left(1-\rho_{Pb_1}^2\right)\bar{Y}^{-2}S_y^2} \quad \text{and} \quad d_2 = \frac{\left(\rho_{Pb_1}\bar{Y}^{-1}S_y - P_1^{-1}S_{P_1}\right)}{\left[1+\theta\left(1-\rho_{Pb_1}^2\right)\bar{Y}^{-2}S_y^2\right]S_{P_1}} .$$

The mean square error of (2.3) up to the term of order n^{-2} is,

$$MSE\left(t_{2(1)}\right) \approx \frac{\theta\left(1-\rho_{pb1}^2\right)S_y^2}{1+\theta\left(1-\rho_{pb1}^2\right)\bar{Y}^{-2}S_y^2} . \tag{2.4}$$

iii) If information on all auxiliary attributes τ_j $(j = 1, 2, ..., k)$ is known then a family of estimators suggested by Hanif et al. (2009) is given as

$$T_{3(1)} = g_\omega\left(\bar{y}, v_1, v_2, ... v_k\right) \tag{2.5}$$

where $v_j = p_j / P_j$ and $g_\omega\left(\bar{y}, v_1, v_2, ... v_k\right)$ is a parametric function of $\bar{y}$ and v_j such that $g_\omega\left(\bar{Y}, 1, 1, ...1\right) = \bar{Y}$ and satisfying following regularity conditions.

(a) Whatever be sample chosen, the point $\left(\bar{y}, v_1, v_2, ... v_k\right)$ assumes value in a bounded closed convex subset R_k of k-dimensional real space containing the point $\left(\bar{Y}, 1, 1, ...1\right)$.

(b) The function $g_\omega\left(\bar{y}, v_1, v_2, ... v_k\right)$ is continuous and bounded in R_k.

(c) All possible first order and second order partial derivatives of $g_\omega\left(\bar{y}, v_1, v_2, ... v_k\right)$ exist and are continuous as well as bounded in R_k.

The mean square error of (2.5) up to the term of order n^{-1} is:

$$MSE\left(T_{3(1)}\right) \approx \theta\left(1-\rho^2_{y.\tau_1\tau_2...\tau_k}\right)S_y^2, \tag{2.6}$$

where $\rho^2_{y.\tau_1\tau_2...\tau_k}$ is multiple bi-serial correlation coefficient.

2.2 Two-Phase Sampling (No Information Case)

i) A family of estimators for two phase sampling proposed by Jhajj et al. (2006) is,

$$T_{4(2)} = g_{\omega}\left(\bar{y}_2, v_{1d}\right), \tag{2.7}$$

where $v_{1d} = p_{1(2)}/p_{1(1)}$, such that $g_{\omega}\left(\bar{Y},1\right)=\bar{Y}$

The mean square error of (2.7) up to the term of order n^{-1} is,

$$MSE\left(T_{4(2)}\right) \approx \left(\theta_2 - \theta_3\rho^2_{pb1}\right)S_y^2. \tag{2.8}$$

ii) An estimator suggested by Shabbir and Gupta (2007) for no information is,

$$t_{5(2)} = \bar{y}_2\left[W_1 + W_2\left(p_{1(1)} - p_{1(2)}\right)\right]\frac{p_{1(1)}}{p_{1(2)}}. \quad \left(p_{1(2)} > 0\right) \tag{2.9}$$

The expressions for W_1 and W_2 that minimizes $MSE\left(t_{5(2)}\right)$ are,

$$W_1 = \frac{1}{1+\left(\theta_2 - \theta_3\rho^2_{pb1}\right)\bar{Y}^{-2}S_y^2} \text{ and } W_2 = \frac{\left(\rho_{Pb_1}\bar{Y}^{-1}S_y - P_1^{-1}S_{P_1}\right)}{\left[1+\left(\theta_2 - \theta_3\rho^2_{Pb_1}\right)\bar{Y}^{-2}S_y^2\right]S_{P_1}}.$$

Then the mean square error of (2.9) up to the term of order n^{-2} is,

$$MSE\left(t_{5(2)}\right) \approx \frac{\left(\theta_2 - \theta_3\rho^2_{pb1}\right)S_y^2}{1+\left(\theta_2 - \theta_3\rho^2_{pb1}\right)\bar{Y}^{-2}S_y^2}. \tag{2.10}$$

iii) If information on all auxiliary attributes $\tau_j\left(j=1,2,...,K\right)$ is not known then a family of estimators suggested by Hanif et al. (2009) is,

$$T_{6(2)} = g_{\omega}\left(\bar{y}_2, v_{1d}, v_{2d}, ..., v_{kd}\right), \tag{2.11}$$

where $v_{jd} = p_{j(2)}/p_{j(1)}$; $v_{jd} > 0$ and $g_{\omega}\left(\bar{y}_2, v_{1d}, v_{2d}, ..., v_{kd}\right)$ is a parametric function of $\bar{y}_2$ and v_{jd} such that $g_{\omega}\left(\bar{Y},1,1,....1\right)=\bar{Y}$ and satisfying same regularity conditions stated in (2.5). The mean square error of (2.11) up to the term of order n^{-1} is,

$$MSE\left(T_{6(2)}\right) \approx \left\{\theta_2\left(1-\rho^2_{y.\tau_1\tau_2...\tau_k}\right) + \theta_1\rho^2_{y.\tau_1\tau_2...\tau_k}\right\}S_y^2. \tag{2.12}$$

2.3 Two-Phase Sampling (Partial Information Case)

Suppose that population proportion P_j is known for $j = (1, 2 \ldots m)$ auxiliary attributes and not known for $j = (m+1,\ m+2 \ldots k)$ auxiliary attributes. Using such partial information Hanif et al. (2009) proposed following general family of estimators,

$$T_{7(2)} = g_\omega\left(\bar{y}_2, v_1, v_2, \ldots, v_m\, v_{(m+1)d}, v_{(m+2)d}, \ldots, v_{kd}\right), \tag{2.13}$$

where $v_j = p_{j(1)}/P_j;\ v_j > 0(j = 1, 2, \ldots, m)$ and $v_{jd} = p_{j(2)}/p_{j(1)}; v_{jd} > 0(j = m+1, m+2, \ldots, k)$. Also $g_\omega\left(\bar{y}_2, v_1, v_2, \ldots, v_m\, v_{(m+1)d}, v_{(m+2)d}, \ldots, v_{kd}\right)$ is a parametric function of $\bar{y}_2$, v_j and v_{jd} such that $g_\omega\left(\bar{Y}, 1, 1, \ldots. 1\right) = \bar{Y}$ and satisfying certain regularity conditions.

The mean square error of (2.13) up to the term of order n^{-1} is:

$$MSE\left(T_{7(2)}\right) \approx \left\{\theta_2\left(1 - \rho^2_{y.\tau_{m+1}\tau_{m+2}\ldots\tau_k}\right) + \theta_1\left(\rho^2_{y.\tau_{m+1}\tau_{m+2}\ldots\tau_k} - \rho^2_{y.\tau_1\tau_2\ldots\tau_m}\right)\right\} S_y^2. \tag{2.14}$$

2.4 An Estimator Proposed by Shahbaz and Hanif (2009)

Shahbaz and Hanif (2009) proposed the following general shrinkage estimator,

$$t_s = d\hat{t} = \frac{\hat{t}}{1 + T^{-2} MSE(\hat{t})}, \tag{2.15}$$

where $\hat{t}$ is any available estimator of parameter T and $d = \dfrac{1}{1 + T^{-2} MSE(\hat{t})}$ is shrinkage constant. The mean square error of t_s is

$$MSE\left(t_s\right) = \frac{MSE\left(\hat{t}\right)}{1 + T^{-2} MSE\left(\hat{t}\right)}. \tag{2.16}$$

3. A NEW ESTIMATORS FOR SINGLE AND TWO PHASES SAMPLING USING ONE ATTRIBUTE

An approximate estimator suggested by Shabbir and Gupta (2007) is not defined at $p_1 = 0$. Therefore we are proposing a new estimator, which may be considered as an alternative to that of Shabbir and Gupta (2007). This new approach has an advantage over estimator suggested by Shabbir and Gupta (2007), as it is defined for any value of sample proportion p_1 because it contained no ratio.

The estimator for full information case using a single attribute is

$$t_{8(1)} = d_0\left[\bar{y} - d_1\left(p_1 - P_1\right)\right] = d_0 t^*_{8(1)}, \tag{3.1}$$

where $d_0 = \dfrac{1}{1 + \bar{Y}^{-2} MSE\left(t^*_{8(1)}\right)}$ and $t^*_{8(1)} = \left[\bar{y} - d_1\left(p_1 - P_1\right)\right]$.

Using Shahbaz and Hanif (2009) approach given in (2.16) the mean square error of $t_{8(1)}$ is

$$MSE\left((t_{8(1)}\right) = \frac{MSE\left(t^*_{8(1)}\right)}{1+\bar{Y}^{-2}MSE\left(t^*_{8(1)}\right)}. \tag{3.2}$$

The mean square error of $t^*_{8(1)}$ is

$$MSE\left(t^*_{8(1)}\right) = \left[S_y^2 + d_1^2 S_{\tau_1}^2 - 2d_1 S_y S_{\tau_1} \rho_{Pb_1}\right]. \tag{3.3}$$

Optimum value of d_1 which minimize $MSE\left(t^*_{8(1)}\right)$ is $d_1 = \dfrac{S_y \rho_{pb_1}}{S_{\tau_1}}$.

Using the value of d_1 in (3.3), the mean square error of $t^*_{8(1)}$ is

$$MSE\left(t^*_{8(1)}\right) = \theta\left(1-\rho^2_{pb1}\right)S_y^2. \tag{3.4}$$

Using (3.4) in (3.2), the mean square error of $t_{8(1)}$ is

$$MSE(t_{8(1)}) = \frac{\theta\left(1-\rho^2_{pb1}\right)S_y^2}{1+\theta\left(1-\rho^2_{pb1}\right)\bar{Y}^{-2}S_y^2}. \tag{3.5}$$

It is not approximated like suggested that by Shabbir and Gupta (2007).

Another suggested estimator for the no information case is

$$t_{9(2)} = W_0\left[\bar{y}_2 - W_1\left(p_{1(2)} - p_{1(1)}\right)\right] = W_0 t^*_{9(2)}, \tag{3.6}$$

where $W_0 = \dfrac{1}{1+\bar{Y}^{-2}MSE\left(t^*_{9(2)}\right)}$ and $t^*_{9(2)} = \left[\bar{y}_2 - W_1\left(p_{1(2)} - p_{1(1)}\right)\right]$.

Using Shahbaz and Hanif (2009) approach given in (2.16) the mean square error of $t_{9(2)}$ is

$$MSE\left(t_{9(2)}\right) = \frac{MSE\left(t^*_{9(2)}\right)}{1+\bar{Y}^{-2}MSE\left(t^*_{9(2)}\right)}. \tag{3.7}$$

The mean square error of $t^*_{9(2)}$ is

$$MSE\left(t^*_{9(2)}\right) = \theta_2 S_y^2 + \theta_3\left(W_1^2 S_{\tau_1}^2 - 2W_1 S_y S_{\tau_1} \rho_{Pb_1}\right). \tag{3.8}$$

The optimum value of W_1, which minimize $MSE\left(t^*_{9(2)}\right)$ is $W_1 = \dfrac{S_y \rho_{pb_1}}{S_{\tau_1}}$.

Using the value of W_1 in (3.8), and upon simplification,

$$MSE\left(t_{9(2)}^{*}\right)=\left(\theta_2-\theta_3\rho_{Pb_1}^2\right)S_y^2 \,. \tag{3.9}$$

Using (3.9) in (3.7) the mean square error of $t_{9(2)}$ is

$$MSE\left(t_{9(2)}\right)=\frac{\left(\theta_2-\theta_3\rho_{Pb_1}^2\right)S_y^2}{1+\left(\theta_2-\theta_3\rho_{Pb_1}^2\right)\bar{Y}^{-2}S_y^2}\,. \tag{3.10}$$

It is not approximated like the one suggested by Shabbir and Gupta (2007) .This may be considered as an alternative to the one suggested in Shabbir and Gupta (2007).

4. SINGLE PHASE SAMPLING USING TWO AUXILIARY ATTRIBUTES (FULL INFORMATION CASE)

In this section we produce a family of estimators for full information case by putting $k = 2$ in (2.5)

$$T_{10(1)}=g_{\omega}\left(\bar{y},v_1,v_2\right), \tag{4.1}$$

Similarly putting $k = 2$ in (2.6) the mean square error of $T_{10(1)}$ up to the term of order n^{-1} is

$$MSE\left(T_{10(1)}\right)\approx\theta\left(1-\rho_{y.\tau_1\tau_2}^2\right)S_y^2\,. \tag{4.2}$$

A regression type estimator for full information case using two auxiliary attributes is

$$t_{11(1)}=\gamma_0\left[\bar{y}-\gamma_1\left(p_1-P_1\right)-\gamma_2\left(p_2-P_2\right)\right]=\gamma_0 t_{11(1)}^{*}, \tag{4.3}$$

where $\gamma_0=\dfrac{1}{1+\bar{Y}^{-2}MSE\left(t_{11(1)}^{*}\right)}$ and $t_{11(1)}^{*}=\left[\bar{y}-\gamma_1\left(p_1-P_1\right)-\gamma_2\left(p_2-P_2\right)\right]$.

Using Shahbaz and Hanif (2009) approach given in (2.16) the mean square error of $t_{11(1)}$ is

$$MSE\left(t_{11(1)}\right)=\frac{MSE\left(t_{11(1)}^{*}\right)}{1+\bar{Y}^{-2}MSE\left(t_{11(1)}^{*}\right)}\,. \tag{4.4}$$

The mean square error of $t_{11(1)}^{*}$ is

$$MSE\left(t_{11(1)}^{*}\right)=\theta\Big[S_y^2+\gamma_1^2S_{\tau_1}^2+\gamma_2^2S_{\tau_2}^2-2\gamma_1S_yS_{\tau_1}\rho_{Pb_1}$$
$$-2\gamma_2S_yS_{\tau_2}\rho_{Pb_2}+2\gamma_1\gamma_2S_{\tau_1}S_{\tau_2}Q_{12}\Big]. \tag{4.5}$$

Optimum values of γ_1 and γ_2 are,

$$\gamma_1 = \frac{P_1\left(\rho_{Pb_1} - Q_{12}\rho_{Pb_2}\right)S_y}{\left(1-Q_{12}^2\right)S_{\tau_1}} \text{ and } \gamma_2 = \frac{P_2\left(\rho_{Pb_2} - Q_{12}\rho_{Pb_1}\right)S_y}{\left(1-Q_{12}^2\right)S_{\tau_2}}.$$

Using the values of γ_1 and γ_2 in (4.5) the mean square error of $t_{11(1)}^*$ is

$$MSE\left(t_{11(1)}^*\right) = \theta\left(1-\rho_{y.\tau_1\tau_2}^2\right)S_y^2. \tag{4.6}$$

Using (4.6) in (4.4) the mean square error of $t_{11(1)}$ is

$$MSE\left(t_{11(1)}\right) = \frac{\theta\left(1-\rho_{y.\tau_1\tau_2}^2\right)S_y^2}{1+\theta\left(1-\rho_{y.\tau_1\tau_2}^2\right)\bar{Y}^{-2}S_y^2}. \tag{4.7}$$

5. TWO-PHASE SAMPLING USING TWO AUXILIARY ATTRIBUTES

In this section two cases are to be discussed one for the partial information and other for the no information.

5.1 Partial Information Case

We propose a family of estimators as

$$T_{12(2)} = g_\omega\left(\bar{y}, v_1, v_{2d}\right), \tag{5.1}$$

where $v_1 = \frac{p_{1(2)}}{P_1}$, $v_{2d} = \frac{p_{2(2)}}{p_{2(1)}}$, $v_1 > 0$, $v_{2d} > 0$, P_1 is known but P_2 is not known, $g_\omega\left(\bar{y}, v_1, v_{2d}\right)$ is parametric function such that $g_\omega\left(\bar{Y},1,1\right) = \bar{Y}$ and satisfying regularity conditions mentioned in (2.1). Consider the following estimator

$$t_{12(2)} = \bar{y}_2 + \alpha_1'\left(v_1 - 1\right) + \alpha_2'\left(v_{2d} - 1\right), \tag{5.2}$$

where α_1' and α_2' are constants to be determined. It must be noted that (5.2) is not special case of (2.13) because in (2.13) $v_j = \frac{p_{j(1)}}{P_1}$ but in (5.2) $v_j = \frac{p_{j(2)}}{P_1}$ for $j = 1$. The mean square error of $t_{12(2)}$ is

$$MSE\left(t_{12(2)}\right) = \theta_2\left[S_y^2 + \alpha_1'^2\frac{S_{\tau_1}^2}{P_1^2} + 2\alpha_1' S_y \frac{S_{\tau_1}}{P_1}\rho_{Pb_1}\right]$$
$$+\theta_3\left[\alpha_2'^2\frac{S_{\tau_2}^2}{P_2^2} + 2\alpha_2'\bar{Y}S_y\frac{S_{\tau_2}}{P_2}\rho_{Pb_2} + 2\alpha_1'\alpha_2'\frac{S_{\tau_1}}{P_1}\frac{S_{\tau_2}}{P_2}Q_{12}\right]. \tag{5.3}$$

The optimum values of α_1' and α_2' are

$$\alpha_1' = \frac{-P_1 S_y\left(\theta_2\rho_{Pb_1} - Q_{12}\theta_3\rho_{Pb_2}\right)}{S_{\tau_1}\left(\theta_2 - \theta_3 Q_{12}^2\right)} \text{ and } \alpha_2' = \frac{-\theta_2 P_2 S_y\left(\rho_{Pb_2} - Q_{12}\rho_{Pb_1}\right)}{S_{\tau_2}\left(\theta_2 - \theta_3 Q_{12}^2\right)}.$$

Using the values of α_1' and α_2' in (5.3), the mean square error of $t_{12(2)}$ is

$$MSE\left(t_{12(2)}\right) = \theta_2\left[1 - \frac{\theta_2\rho_{Pb_1}^2 + \theta_3\rho_{Pb_2}^2 - 2\theta_3 Q_{12}\rho_{Pb_1}\rho_{Pb_2}}{\left(\theta_2 - \theta_3 Q_{12}^2\right)}\right]S_y^2. \tag{5.4}$$

A regression type estimator using two auxiliary attributes has also been suggested:

$$t_{13(2)} = \delta_0\left[\bar{y}_2 - \delta_1\left(p_{1(2)} - P_1\right) - \delta_2\left(p_{2(2)} - p_{2(1)}\right)\right] = \delta_0 t_{13(2)}^*, \tag{5.5}$$

where $\delta_0 = \dfrac{1}{1+\bar{Y}^{-2}MSE(t_{13(2)}^*)}$ and $t_{13(2)}^* = \left[\bar{y}_2 - \delta_1\left(p_{1(2)} - P_1\right) - \delta_2\left(p_{2(2)} - p_{2(1)}\right)\right]$.

Using Shahbaz and Hanif (2009) approach given in (2.16) the mean square error of $t_{13(2)}$ is

$$MSE\left(t_{13(2)}\right) = \frac{MSE\left(t_{13(2)}^*\right)}{1+\bar{Y}^{-2}MSE\left(t_{13(2)}^*\right)}. \tag{5.6}$$

The mean square error of $t_{13(2)}^*$ is

$$MSE\left(t_{13(2)}^*\right) = \left\{\theta_2\left(\bar{Y}^2 S_y^2 + \delta_1^2 S_{\tau_1}^2 - 2\delta_1 S_y S_{\tau_1}\rho_{Pb_1}\right) + \theta_3\left(\delta_2^2 S_{\tau_2}^2 - 2\delta_2 S_y S_{\tau_2}\rho_{Pb_2} + 2\delta_1\delta_2 S_{\tau_1} S_{\tau_2} Q_{12}\right)\right\}. \tag{5.7}$$

The optimum value of δ_1 and δ_2 are,

$$\delta_1 = \frac{P_1 S_y\left(\theta_2\rho_{Pb_1} - Q_{12}\theta_3\rho_{Pb_2}\right)}{S_{\tau_1}\left(\theta_2 - \theta_3 Q_{12}^2\right)} \text{ and } \delta_2 = \frac{\theta_2 P_2 S_y\left(\rho_{Pb_2} - Q_{12}\rho_{Pb_1}\right)}{S_{\tau_2}\left(\theta_2 - \theta_3 Q_{12}^2\right)}.$$

Using the values of δ_1 and δ_2 in (5.7),

$$MSE\left(t_{13(2)}^*\right) = \theta_2\left[1 - \frac{\theta_2\rho_{Pb_1}^2 + \theta_3\rho_{Pb_2}^2 - 2\theta_3 Q_{12}\rho_{Pb_1}\rho_{Pb_2}}{\left(\theta_2 - \theta_3 Q_{12}^2\right)}\right]S_y^2. \tag{5.8}$$

Using (5.8) in (5.6), the mean square error of $t_{13(2)}$ is

$$MSE\left(t_{13(2)}\right)=\frac{\theta_2\left[1-\frac{\theta_2\rho^2_{Pb_1}+\theta_3\rho^2_{Pb_2}-2\theta_3Q_{12}\rho_{Pb_1}\rho_{Pb_2}}{\left(\theta_2-\theta_3Q^2_{12}\right)}\right]S^2_y}{1+\theta_2\left[1-\frac{\theta_2\rho^2_{Pb_1}+\theta_3\rho^2_{Pb_2}-2\theta_3Q_{12}\rho_{Pb_1}\rho_{Pb_2}}{\left(\theta_2-\theta_3Q^2_{12}\right)}\right]\bar{Y}^{-2}S^2_y}. \tag{5.9}$$

5.2 No Information Case

A family of estimators for the case of no information is

$$T_{14(2)}=g_{\omega}\left(\bar{y}_2,v_{1d},v_{2d}\right), \tag{5.10}$$

Putting $k=2$ in (2.12) the mean square error of $T_{14(2)}$ yields

$$MSE\left(T_{14(2)}\right)=\left\{\theta_2\left(1-\rho^2_{y.\tau_1\tau_2}\right)+\theta_1\rho^2_{y.\tau_1\tau_2}\right\}S^2_y. \tag{5.11}$$

A regression type estimator for no information case is

$$t_{15(2)}=\gamma^*_0\left[\bar{y}_2-\gamma_1\left(p_{1(2)}-p_{1(1)}\right)-\gamma_2\left(p_{2(2)}-p_{2(1)}\right)\right]=\gamma^*_0t^*_{15(2)}, \tag{5.12}$$

where $\gamma^*_0=\dfrac{1}{1+\bar{Y}^{-2}MSE\left(t^*_{15(2)}\right)}$ and $t^*_{15(2)}=\left[\bar{y}_2-\gamma_1\left(p_{1(2)}-p_{1(1)}\right)-\gamma_2\left(p_{2(2)}-p_{2(1)}\right)\right]$.

Using Shahbaz and Hanif (2009) approach given in (2.16) the mean square error of $t_{15(2)}$ is

$$MSE\left(t_{15(2)}\right)=\frac{MSE\left(t^*_{15(2)}\right)}{1+\bar{Y}^{-2}MSE\left(t^*_{15(2)}\right)}. \tag{5.13}$$

The mean square error of $t^*_{15(2)}$ is

$$MSE\left(t^*_{15(2)}\right)=\left\{\theta_2S^2_y+\theta_3\left(\gamma^2_1S^2_{\tau_1}+\gamma^2_2S^2_{\tau_2}-2\gamma_1S_yS_{\tau_1}\rho_{Pb_1}\right.\right.$$
$$\left.\left.-2\gamma_2S_yS_{\tau_2}\rho_{Pb_2}+2\gamma_1\gamma_2S_{\tau_1}S_{\tau_2}Q_{12}\right)\right\}. \tag{5.14}$$

The optimum values γ_1 and γ_2 in (5.14) is same as given in the full information case.

Using the value of γ_1 and γ_2 in (5.14), the mean square error of $t^*_{15(2)}$ is

$$MSE\left(t^*_{15(2)}\right)=\left\{\theta_2\left(1-\rho^2_{y.\tau_1\tau_2}\right)+\theta_1\rho^2_{y.\tau_1\tau_2}\right\}S^2_y. \tag{5.15}$$

Using (5.15) in (5.13) the mean square error of $t_{15(2)}$ is

$$MSE\left(t_{15(2)}\right)=\frac{\left\{\theta_2\left(1-\rho^2_{y.\tau_1\tau_2}\right)+\theta_1\rho^2_{y.\tau_1\tau_2}\right\}S_y^2}{1+\left\{\theta_2\left(1-\rho^2_{y.\tau_1\tau_2}\right)+\theta_1\rho^2_{y.\tau_1\tau_2}\right\}\bar{Y}^{-2}S_y^2}. \quad (5.16)$$

Some estimators of family proposed in (4.1) are,

i) $\bar{y}+\alpha_1\left(v_1-1\right)+\alpha_1\left(v_2-1\right)$ ii) $\bar{y}V_1^{\alpha_1}V_2^{\alpha_2}$

iii) $\bar{y}\ e^{\alpha_1(V_1-1)+\alpha_2(V_2-1)}$ iv) $\bar{y}\left(V_1e^{(V_1-1)}\right)^{\alpha_1}\left(V_2e^{(V_2-1)}\right)^{\alpha_2}$

v) $\bar{y}V_1^{\alpha_1}e^{\alpha_2(V_2-1)}$ vi) $\frac{\bar{y}}{2}\left[V_1^{\alpha_1}V_2^{\alpha_2}+e^{\alpha_1(V_1-1)+\alpha_2(V_2-1)}\right]$

vii) $\bar{y}+\alpha_1\left(V_1^{\alpha_3}-1\right)+\alpha_2\left(V_2^{\alpha_4}-1\right)$ viii) $\bar{y}+\alpha_1\left(V_1^{\alpha_3}-1\right)+\alpha_2\left(V_2-1\right)$

ix) $\frac{\bar{y}}{k_1+k_2}\left[k_1V_1^{\frac{\alpha_1}{2}}+k_2\,e^{\alpha_2(V_2-1)}\right]$ x) $\bar{y}\left[k\,e^{\alpha_1(V_1-1)}+(1-k)e^{\alpha_2(V_2-1)}\right]$

The mean square error of all these estimators has been derived.

It can be easily verified that $MSE\left(t_{11(1)}\right)\le MSE\left(t_{2(1)}\right)$. Similarly it can be shown that $MSE\left(t_{15(2)}\right)\le MSE\left(t_{5(2)}\right)$.

6. EMPIRICAL STUDY

Twelve populations are taken from the Government of Pakistan (1998). It is shown empirically in table-2 that the proposed estimator $t_{11(1)}$ outperforms other competing estimators. Also, $t_{15(2)}$ performs best in almost all the populations. We conclude that $t_{11(1)}$ and $t_{15(2)}$ are more efficient than the other estimators in single phase and two-phase sampling. It is further observed full information case is always more efficient than the no information case.

The optimum values of $\alpha_1^{/}$ and $\alpha_2^{/}$ involve population parameters, which are assumed to be known for the efficient use of proposed family $T_{12(2)}$. Usually these parameters are unknown, but they can be estimated from the sample. Following approach of Srivastava and Jhajj (1983), the estimator of proposed family, $T_{12(2)}$ will have the same minimum mean square, if we replace the unknown value of parameters involved in optimum value of $\alpha_1^{/}$ and $\alpha_2^{/}$ with their estimators. Similar is the case for other proposed estimators and families of estimators.

6.1 Conclusion

The proposed estimator $t_{11(1)}$ is recommended for estimating the population mean in the full information case as $t_{11(1)}$ outperforms all the existing estimators for the full information. Similarly $t_{15(2)}$ is recommended for estimating the population mean for the no information case as $t_{15(2)}$ outperforms all the existing estimators for the no information.

It is also recommended that full information should always be preferred if possible, otherwise estimators with partial information is the best choice, the no information case is recommended when we have no other choice.

7. ACKNOWLEDGEMENT

We are very thankful to the reviewers for valuable comments, their suggestions helped to improve the presentation of the paper.

REFERENCES

1. Hanif, M., Haq, I. and Shahbaz, M.Q. (2009). On a New Family of Estimators using Multiple Auxiliary Attributes. *World Applied Sciences Journal*, 7(11), 1419-1422.
2. Government of Pakistan (1998). *Crop Area Production by Districts (1995-96 to 1997-98)*: Ministry of Food, Agriculture and Livestock. Food, Agriculture and Livestock Division, Economic Wing, Islamabad.
3. Jhajj, H.S., Sharma, M.K. and Grover, L.K. (2006). A family of estimators of population mean using information on auxiliary attribute. *Pak. J. Statist.*, 22(1), 43-50.
4. Neyman, J. (1938). Contributions to the theory of sampling human populations. *J. Amer. Statist. Assoc.*, 33, 101-116.
5. Naik, V.D. and Gupta, P.C. (1996). A note on estimation of mean with known population of an auxiliary character. *J. Ind. Soc. Agri. Statist.*, 48(2), 151-158.
6. Shabbir, J. and Gupta, S. (2007). On estimating the finite population mean with known population proportion of an auxiliary variable. *Pak. J. Statist*., 23(1), 1-9.
7. Shahbaz, M.Q. and Hanif, M. (2009). A general shrinkage estimator in survey Sampling. *World Applied Sciences Journal*, 7(5), 593-596.
8. Srivastava, S.K. and Jhajj, H.S. (1983). A class of estimators of the population means using multi-auxiliary information. *Cal. Statist. Assoc. Bull.*, 32, 47-56.

APPENDIX
Table-1
Description of Populations and Variables

Pop #	Description	Main Variable	Attribute-I is present $\left(\tau_1 = 1\right)$ if	Attribute-II is present $\left(\tau_1 = 1\right)$ if
1	District-wise area and production of Vegetables for year 1995-96	Production of Vegetables (In tones)	Districts of N.W.F.P (including Fata Areas)	Area of Districts less than 501 hectors.
2	District-wise area and production of Vegetables for year 1996-97	Production of Vegetable (In tones)	Districts of Punjab	Area of Districts less than 401 hectors.
3	District-wise area and production of Vegetables for year 1997-98	Production of Vegetables (In tones)	Districts of Punjab	Area of Districts greater than 1000 hectors.
4	District-wise area and production of all Fruits for year 1995-96	Production of all Fruits (In tones)	Area of Districts greater than 1000 hectares	Districts of Punjab
5	District-wise area and production of all Fruits for year 1996-97	Production of all Fruits (In tones)	Districts of Sind	Area of Districts less than 1001 hectors.
6	District-wise area and production of all Fruits for year 1997-98	Production of all Fruits (In tones)	Districts of N.W.F.P (including Fata Areas)	Area of Districts less than 501 hectors.
7	District-wise area and production of Wheat for year 1995-96	Production of Wheat (In tones)	Area of Districts greater than 30 hectors.	Districts of Punjab
8	District-wise area and production of Wheat for year 1996-97	Production of Wheat (In tones)	Districts of Punjab	Area of Districts greater than 35 hectors.
9	District-wise area and production of Wheat for year 1997-98	Production of Wheat (In tones)	Districts of N.W.F.P (including Fata Areas)	Area of Districts greater than 25 hectors.
10	District-wise area and production of Onion for year 1995-96	Production of Onions (In tones)	Area of Districts greater than 40 hectors.	Districts of N.W.F.P (including Fata Areas)
11	District-wise area and production of Onion for year 1996-97	Production of Onions (In tones)	Area of Districts greater than 50 hectors.	Districts of N.W.F.P (including Fata Areas)
12	District-wise area and production of Onion for year 1997-98	Production of Onions (In tones)	Districts of Punjab	Area of Districts greater than 60 hectors.

Table-2
Relative Efficiency of Various Estimators

Pop #	Single-Phase Sampling (Full information case)					Two-Phase Sampling			
	$\bar{y}$	$T_{1(1)}$	$t_{2(1)}$	$T_{10(1)}$	$t_{11(1)}$	$T_{4(2)}$	$t_{5(2)}$	$T_{14(2)}$	$t_{15(2)}$
1	100	108.77	120.10	118.20	129.53	104.62	115.94	109.20	120.52
2	100	142.25	153.36	149.52	160.63	119.40	130.50	122.12	133.23
3	100	142.03	152.92	143.39	154.27	119.31	130.20	119.84	130.72
4	100	122.30	134.59	124.56	136.84	111.08	123.37	112.09	124.375
5	100	102.49	114.96	126.02	138.34	101.11	113.58	112.73	125.20
6	100	111.41	123.65	114.58	126.82	105.93	118.17	107.48	119.72
7	100	146.61	155.30	239.88	248.56	115.15	123.83	140.33	149.01
8	100	225.50	233.66	268.09	276.28	147.90	156.08	157.466	165.65
9	100	125.40	132.80	186.98	194.42	113.36	120.80	137.125	144.357
10	100	105.90	137.40	107.13	136.63	103.40	134.90	104.10	135.60
11	100	107.07	136.09	107.73	136.96	104.04	133.14	104.43	133.63
12	100	101.80	129.20	109.90	137.30	101.05	128.45	105.60	133.00

Table-3
Various Population Parameters

Pop #	$P_1^{-1}S_{\tau_1}$	$P_2^{-1}S_{\tau_2}$	$\bar{Y}^{-1}S_y$	ρ_{Pb_1}	ρ_{Pb_2}	Q_{12}
1	1.524	1.67	1.457	0.284	0.376	0.50
2	1.461	1.84	1.443	0.545	0.389	-0.76
3	1.4606	0.81	1.423	0.544	0.474	0.79
4	0.6759	1.46	1.518	0.427	0.35	0.59
5	2.3366	1.46	1.529	0.142	0.424	-0.05
6	1.5245	1.94	1.515	0.32	0.479	0.69
7	0.8203	1.42	1.154	0.631	0.742	0.96
8	1.4413	0.87	1.194	0.746	0.651	0.96
9	1.5045	0.745	1.139	0.45	0.645	-0.38
10	0.4994	1.55	2.071	0.236	0.225	-0.60
11	0.5243	1.53	6.298	0.257	0.249	-0.79
12	1.3988	0.551	1.930	0.133	0.249	0.13

GENERALIZED REGRESSION ESTIMATORS UNDER TWO PHASE SAMPLING FOR PARTIAL INFORMATION CASE*

Zahoor Ahmad[1], **Muhammad Hanif**[2] and **Munir Ahmad**[2]

[1] Department of Statistics, University of Gujrat, Gujrat, Pakistan. Email: zahoor.ahmed@uog.edu.pk

[2] National College of Business Administration and Economic Lahore, Pakistan.
Email: drmianhanif@gmail.com; drmunir@ncbae.edu.pk

ABSTRACT

In this paper multivariate ratio and regression estimators for estimating the population mean vector have been proposed using multi-auxiliary variables for partial information case. The expression of variance covariance matrix has been derived. Empirical study has been carried out to see the performance of proposed estimator over estimator proposed by Butt, et al. (2011).

KEY WORDS

Multivariate Ratio Estimator; Two Phase Sampling; Partial Information Case; Multiple Auxiliary Variables.

1. INTRODUCTION

The estimation of the population mean is an unrelenting issue in sampling theory and several efforts have been made to improve the precision of the estimators in the presence of multi-auxiliary variables. A variety of estimators have been proposed following different ideas of ratio, regression and product estimators.

Olkin (1958) was the first author to deal with the problem of estimating the mean of a survey variable when information on several auxiliary variables is made available. Singh and Namjoshi (1988) discussed a class of multivariate regression estimators of population mean of study variable in two-phase sampling. Robinson (1994) proposed a regression estimator ignoring some of the assumptions usually adopted in the literature (see, e.g., Srivastava (1971)). Kadilar and Cingi (2004, 2005) analyzed combinations of regression type estimators in the case of two auxiliary variables. Pradhan (2005) suggested a chain regression estimator for two-phase sampling using three auxiliary variables when the population mean of one auxiliary variable is unknown and other is known. Hidiroglou et al. (2009) and Haziza et al. (2001) have overlooked the work of Singh (2004), thus their work is based on doubtful simulation. Their estimators can never attain the minimum variance of a linear regression estimator in two-phase sampling. For detail, see Steans and Singh (2008). They have not compared their work with that of Singh (2004) estimator either through simulation or theoretically.

*Published in Pak. J. Statist. (2013), Vol. 29(1).

In multipurpose surveys, the problem is to estimate population means of several variables simultaneously (see Swain (2000)). Tripathi and Khattree (1989) estimated means of several variables of interest, using multi-auxiliary variables for simple random sampling. Further Tripathi (1989) extended the results to the case of two-phase sampling.

Following Roy (2003), Butt, et al. (2011) has proposed the following multivariate regression estimator for two-phase sampling using two-auxiliary variables X and W for Partial Information Case (see Samiuddin and Hanif (2007)) as:

$$\mathrm{t} = \bar{\mathrm{y}}_2 + \left(\bar{x}_1 - \bar{x}_2\right)\mathrm{K} + \left(W - \bar{w}_1\right)\mathrm{AK} + \left(W - \bar{w}_2\right)\mathrm{BK} \tag{1.1}$$

where K is a vector of constants and $A \& B$ are diagonal matrices with diagonal entries α_i & β_i respectively. The expression for variance covariance matrix is

$$\Sigma_{t_{(p\times p)}} = \left[S_{ij}\right]_{p\times p}, \left(i, j = 1, 2, ..., p\right); S_{ij} = S_i^2 \quad \text{for } i = j \tag{1.2}$$

where

$$S_i^2 = S_{y_i}^2\left[\theta_2\left(1-\rho_{y_i.wx}^2\right)+\theta_1\rho_{xy_i.w}^2\left(1-\rho_{wy_i}^2\right)\right] \tag{1.3}$$

and

$$S_{ij} = S_{y_i}S_{y_j}\left[\theta_2\left\{\rho_{y_iy_j} - \frac{\rho_{xy_i}\rho_{xy_j}+\rho_{wy_i}\rho_{wy_j}-\rho_{xy_i}\rho_{wy_j}\rho_{xw}-\rho_{wy_i}\rho_{xy_j}\rho_{xw}}{\left(1-\rho_{wx}^2\right)}\right\} + \rho_{xy_i.w}\rho_{xy_j.w}\sqrt{1-\rho_{wy_i}^2}\sqrt{1-\rho_{wy_j}^2}\right] \tag{1.4}$$

This paper consists of four sections along with appendix. After providing some background in section one, two-phase sampling scheme, important notations and expectations using several study and auxiliary variables are given in section two. The new estimator is defined in section three along with expression of its mean square error. The detail of empirical results are given in appendix and discussed in section 4.

2. TWO-PHASE SAMPLING USING MULTI-AUXILIARY VARIABLES

Consider a population of size N units. Let $Y_1, Y_2, ..., Y_P$ are p variables of interest and $X_1, X_2, ..., X_q$ are q auxiliary variables. For two-phase sampling design let n_1 and n_2 $\left(n_2 < n_1\right)$ be sample sizes for first and second phase respectively, $x_{(1)i}$ and $x_{(2)i}$ denote the i^{th} auxiliary variables from first and second phase samples respectively and $y_{(2)i}$ denote i^{th} study variable from second phase sample. Let $\bar{X}_i$, C_{x_i}, C_{y_i}, $\rho_{y_ix_i}$, $\rho_{y_iy_j}$ and $\rho_{x_ix_j}$ denote the population mean, coefficient of variation of i^{th} auxiliary variable, coefficient of variation of i^{th} variable of interest, correlation coefficient of i^{th} variable of interest and i^{th} auxiliary variable, correlation coefficient of i^{th} and j^{th} variable of interest and correlation coefficient of i^{th} and j^{th} auxiliary variables respectively. Further let $\theta_1 = \frac{1}{n_1} - \frac{1}{N}$ and $\theta_2 = \frac{1}{n_2} - \frac{1}{N}$ are sampling fractions for first and second phase respectively. Also assume $y_{(2)i} = Y + e_{y_{(2)i}}$, $x_{(1)i} = X_i + e_{x_{(1)i}}$ and $x_{(2)i} = X_i + e_{x_{(2)i}}$ $\left(i = 1, 2, ..., k\right)$, where $e_{y_{(2)}}$, $e_{x_{(1)i}}$ and $e_{x_{(k)i}}$ are sampling errors. Further it is assumed that $E_2\left(e_{y_{(2)i}}\right) = E_1\left(e_{x_{(1)i}}\right) = E_2\left(e_{x_{(2)i}}\right) = 0$ where E_1 and E_2 denote the expectations over first and second phase respectively. Then for simple random sampling without replacement for both first and second phases, we write by using phase wise operation of expectations as:

$$E_2\left(e_{y_{(2)i}}\right)^2 = \left(1-\frac{n_2}{N}\right)\sigma^2_{y_i},\ E_2\left(\bar{e}_{y_{(2)i}}\right)^2 = \left(1-\frac{n_2}{N}\right)\frac{\sigma^2_{y_i}}{n_2} = \theta_2 \bar{Y}_i^2 C^2_{y_i},$$

$$E_2\left(e_{y_{(2)i}} e_{x_{(2)i}}\right) = \left(1-\frac{n_2}{N}\right)\sigma_{y_i x_i} = \theta_2 \bar{Y}_i \bar{X}_i C_{y_i} C_{x_i} \rho_{y_i x_i}$$

$$E_2\left(\bar{e}_{y_{(2)i}} \bar{e}_{x_{(2)i}}\right) = \left(1-\frac{n_2}{N}\right)\frac{\sigma_{y_i x_i}}{n_2} = \theta_2 \bar{Y}_i \bar{X}_i C_{y_i} C_{x_i} \rho_{y_i x_i},$$

$$E_1 E_{2|1}\left[e_{y_{(2)i}}\left(e_{x_{(1)i}} - e_{x_{(2)i}}\right)\right] = E_1 E_{2|1}\left(e_{y_{(2)i}} e_{x_{(1)i}}\right) - E_2\left(e_{y_{(2)i}} e_{x_{(2)i}}\right) = \frac{1}{N}\left(n_2 - n_1\right)\sigma_{y_i x_i}$$

and

$$E_1 E_{2|1}\left[\bar{e}_{y_{(2)i}}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)\right] = \left(1-\frac{n_1}{N}\right)\frac{\sigma_{y_i x_i}}{n_1} - \left(1-\frac{n_2}{N}\right)\frac{\sigma_{y_i x_i}}{n_2}$$

$$= \left(\theta_1 - \theta_2\right)\overline{YX}_i C_{y_i} C_{x_i} \rho_{y_i x_i}.$$

Similarly

$$E_1 E_{2|1}\left[\bar{e}_{x_{(2)i}}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)\right] = \left(\theta_1 - \theta_2\right)\sigma^2_{x_i} = \left(\theta_1 - \theta_2\right)\bar{X}_i^2 C^2_{x_i},$$

$$E_1 E_{2|1}\left[\bar{e}_{x_{(1)i}}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)\right] = 0,\ E_1 E_{2/1}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)^2 = \left(\theta_2 - \theta_1\right)\sigma^2_{x_i} = \left(\theta_2 - \theta_1\right)\bar{X}_i^2 C^2_{x_i},$$

$$E_1 E_{2|1}\left[\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)\left(\bar{e}_{x_{(1)j}} - \bar{e}_{x_{(2)j}}\right)\right] = \left(\theta_2 - \theta_1\right)\sigma_{x_i x_j} = \left(\theta_2 - \theta_1\right)\bar{X}_i \bar{X}_j C_{x_i} C_{x_j} \rho_{x_i x_j};\left(i \neq j\right),$$

and

$$E_1 E_{2|1}\left[\left(\bar{e}_{x_{(2)i}}\right)\left(\bar{e}_{x_{(1)j}} - \bar{e}_{x_{(2)j}}\right)\right] = \left(\theta_1 - \theta_2\right)\sigma_{x_i x_j} = \left(\theta_1 - \theta_2\right)\bar{X}_i \bar{X}_j C_{x_i} C_{x_j} \rho_{x_i x_j},\left(i \neq j\right).$$

3. GENERALIZED MULTIVARIATE REGRESSION ESTIMATOR FOR TWO-PHASE SAMPLING (PARTIAL INFORMATION CASE)

Let $\bar{X}_i$ denote the known mean of i^{th} auxiliary variable, $x_{(1)i}$ and $x_{(2)i}$ denote the i^{th} auxiliary variables form first and second phase samples respectively and $y_{(2)i}$ denote the i^{th} variable of interest for second phase sample.

We are now suggesting a generalized multivariate regression estimator for two-phase sampling in which q auxiliary variables are used. The information on first r variables is known and is not known for the remaining $s = q - r$ variables. The proposed estimator is

$$t_{m(1\times p)} = \left[\bar{y}_{(2)1} \quad \bar{y}_{(2)2} \quad \dots \quad \bar{y}_{(2)p}\right]$$

$$+\left[\sum_{i=1}^{r}\alpha_{i1}\left(\bar{x}_{(1)i}-\bar{x}_{(2)i}\right)+\sum_{i=1}^{r}\beta_{i1}\left(\bar{X}_i-\bar{x}_{(2)i}\right)+\sum_{i=r+1}^{r+s=q}\alpha_{i1}\left(\bar{x}_{(1)i}-\bar{x}_{(2)i}\right)\right.$$

$$\sum_{i=1}^{r}\alpha_{i2}\left(\bar{x}_{(1)i}-\bar{x}_{(2)i}\right)+\sum_{i=1}^{r}\beta_{i2}\left(\bar{X}_i-\bar{x}_{(2)i}\right)+\sum_{i=r+1}^{r+s=q}\alpha_{i2}\left(\bar{x}_{(1)i}-\bar{x}_{(2)i}\right)$$

$$\left.\dots \quad \sum_{i=1}^{r}\alpha_{ip}\left(\bar{x}_{(1)i}-\bar{x}_{(2)i}\right)+\sum_{i=1}^{r}\beta_{ip}\left(\bar{X}_i-\bar{x}_{(2)i}\right)+\sum_{i=r+1}^{r+s=q}\alpha_{ip}\left(\bar{x}_{(1)i}-\bar{x}_{(2)i}\right)\right]. \qquad (3.1)$$

In matrix notation we can write

$$t_{m(1\times p)} = \bar{y}_{(1\times p)} + \bar{D}_{x_1'(1\times r)}A_{1(r\times p)} - \bar{D}_{x_1''(1\times r)}B_{(r\times p)} + \bar{D}_{x_2(1\times s)}A_{2(s\times p)}. \qquad (3.2)$$

where

$$\bar{D}_{x_1'(1\times r)} = \left[\left(\bar{x}_{(1)1}-\bar{x}_{(1)2}\right) \quad \left(\bar{x}_{(2)1}-\bar{x}_{(2)2}\right) \quad \dots \quad \left(\bar{x}_{(r)1}-\bar{x}_{(r)2}\right)\right]$$

$$= \left[\left(\bar{e}_{x_{(1)1}}-\bar{e}_{x_{(1)2}}\right) \quad \left(\bar{e}_{x_{(2)1}}-\bar{e}_{x_{(2)2}}\right) \quad \dots \quad \left(\bar{e}_{x_{(r)1}}-\bar{e}_{x_{(r)2}}\right)\right]_{1\times r},$$

$$\bar{D}_{x_1''(1\times r)} = \left[\left(\bar{x}_{(1)2}-\bar{X}_{(1)}\right) \quad \left(\bar{x}_{(2)2}-\bar{X}_{(2)}\right) \quad \cdots \quad \left(\bar{x}_{(r)2}-\bar{X}_{(r)}\right)\right]$$

$$= \left[\bar{e}_{x_{(1)2}} \quad \bar{e}_{x_{(2)2}} \quad \dots \quad \bar{e}_{x_{(r)2}}\right]_{1\times r},$$

$$\bar{D}_{x_2(1\times s)} = \left[\left(\bar{x}_{(r+1)1}-\bar{x}_{(r+1)2}\right) \quad \left(\bar{x}_{(r+2)1}-\bar{x}_{(r+2)2}\right) \quad \dots \quad \left(\bar{x}_{(r+s)1}-\bar{x}_{(r+s)2}\right)\right]$$

$$= \left[\left(\bar{e}_{x_{(r+1)1}}-\bar{e}_{x_{(r+1)2}}\right) \quad \left(\bar{e}_{x_{(r+2)1}}-\bar{e}_{x_{(r+2)2}}\right) \quad \dots \quad \left(\bar{e}_{x_{(r+s)1}}-\bar{e}_{x_{(r+s)2}}\right)\right]_{1\times s},$$

$$A_{1(r\times p)} = \left[\alpha_{ij}\right]_{r\times p}, \left(i=1,2,\dots,r\ ;\ j=1,2,\dots,p\right), B_{(r\times p)} = \left[\beta_{ij}\right]_{r\times p}, \left(i=1,2,\dots,r\ ;\ j=1,2,\dots,p\right),$$

and

$$A_{2(s\times p)} = \left[\alpha_{ij}\right]_{s\times p}, \left(i=r+1, r+2, \dots, r+s\ ;\ j=1,2,\dots,p\right).$$

Let, $\bar{y}_{(1\times p)} = \bar{Y}_{(1\times p)} + \bar{D}_{y(1\times p)}$, where $\bar{Y}_{(1\times p)} = \begin{bmatrix} \bar{Y}_1 & \bar{Y}_2 & \dots & \bar{Y}_p \end{bmatrix}$ and $\bar{D}_{y(1\times p)} = \begin{bmatrix} \bar{e}_{y_{(1)2}} & \bar{e}_{y_{(2)2}} & \dots & \bar{e}_{y_{(p)2}} \end{bmatrix}$.

Then (3.2) can be written as:

$$t_{m(1\times p)} = \bar{Y} + \bar{D}_y + \bar{D}_{x_1'} A_1 - \bar{D}_{x_1''} B + \bar{D}_{x_2} A_2 \text{ or } t_{(1\times p)} - \bar{Y} = \bar{D}_y + \bar{D}_{x_1'} A_1 - \bar{D}_{x_1''} B + \bar{D}_{x_2} A_2 .$$

As we use information related to auxiliary variables from first and second phases both then the mean square error of $t_{54(1\times p)}$ can be written as:

$$\Sigma_{t_m(p\times p)} = E_1 E_{2/1}\left(t_{54} - \bar{Y}\right)'\left(t_{54} - \bar{Y}\right)$$

$$= E_1 E_{2/1}\left[\left(\bar{D}_y + \bar{D}_{x_1'} A_1 - \bar{D}_{x_1''} B + \bar{D}_{x_2} A_2\right)'\left(\bar{D}_y + \bar{D}_{x_1'} A_1 - \bar{D}_{x_1''} B + \bar{D}_{x_2} A_2\right)\right]$$

or

$$\begin{aligned} \Sigma_{t_m(p\times p)} = {} & E_1 E_{2/1}\left[\bar{D}_y'\left(\bar{D}_y + \bar{D}_{x_1'} A_1 - \bar{D}_{x_1''} B + \bar{D}_{x_2} A_2\right)\right] \\ & + E_1 E_{2/1}\left[A_1' \bar{D}_{x_1'}'\left(\bar{D}_y + \bar{D}_{x_1'} A_1 - \bar{D}_{x_1''} B + \bar{D}_{x_2} A_2\right)\right] \\ & - E_1 E_{2/1}\left[B' \bar{D}_{x_1''}'\left(\bar{D}_y + \bar{D}_{x_1'} A_1 - \bar{D}_{x_1''} B + \bar{D}_{x_2} A_2\right)\right] \\ & + E_1 E_{2/1}\left[A_2' \bar{D}_{x_2}'\left(\bar{D}_y + \bar{D}_{x_1'} A_1 - \bar{D}_{x_1''} B + \bar{D}_{x_2} A_2\right)\right]. \end{aligned} \tag{3.3}$$

We can write

$$E_1 E_{2/1}\left(\bar{D}_y' \bar{D}_{x_1''}\right) = \theta_2 \Sigma_{yx(p\times r)} = \theta_2 [\sigma_{y_i x_j}]_{(p\times r)},$$

$$E_1 E_{2/1}\left(\bar{D}_y' \bar{D}_{x_2}\right) = \left(\theta_2 - \theta_1\right) \Sigma_{yx(p\times s)} = [\sigma_{y_i x_j}]_{(p\times s)},$$

$$E_1 E_{2/1}\left(\bar{D}_{x_1'}' \bar{D}_{x_1'}\right) = \left(\theta_2 - \theta_1\right) \Sigma_{x(r\times r)} = \left(\theta_2 - \theta_1\right)[\sigma_{x_i x_j}]_{(r\times r)}, \text{ for } i = j,\ \sigma_{x_i x_j} = \sigma_{x_i}^2,$$

$$E_1 E_{2/1}\left(\bar{D}_{x_1''}' \bar{D}_{x_1''}\right) = \theta_2 \Sigma_{x(r\times r)} = \theta_2 [\sigma_{x_i x_j}]_{(r\times r)}, \text{ for } i = j, \sigma_{x_i x_j} = \sigma_{x_i}^2 ,$$

$$E_1 E_{2/1}\left(\bar{D}_{x_2}' \bar{D}_{x_2}\right) = \left(\theta_2 - \theta_1\right) \Sigma_{x(s\times s)} = \left(\theta_2 - \theta_1\right)[\sigma_{x_i x_j}]_{(s\times s)}, \text{ for } i = j, \sigma_{x_i x_j} = \sigma_{x_i}^2 ,$$

$$E\left(\bar{D}_y' \bar{D}_y\right) = \theta_2 \Sigma_{y(p\times p)} = \theta_2 [\sigma_{y_i y_j}]_{(p\times p)}, \text{ for } i = j, \sigma_{y_i y_j} = \sigma_{y_i}^2 ,$$

$$E\left(\bar{D}_y' \bar{D}_{x_1'}\right) = \left(\theta_2 - \theta_1\right) \Sigma_{yx(p\times r)} = [\sigma_{y_i x_j}]_{(p\times r)},$$

$$E_1 E_{2/1}\left(\bar{D}_{x_1'}' \bar{D}_{x_1''}\right) = \left(\theta_1 - \theta_2\right) \Sigma_{x(r\times r)} = \left(\theta_1 - \theta_2\right)[\sigma_{x_i x_j}]_{(r\times r)}, \text{ for } i = j, \sigma_{x_i x_j} = \sigma_{x_i}^2 ,$$

$$E_1 E_{2/1}\left(\bar{D}_{x_1'}' \bar{D}_{x_2}\right) = \left(\theta_2 - \theta_1\right) \Sigma_{x(r\times s)} = \left(\theta_2 - \theta_1\right)[\sigma_{x_i x_j}]_{(r\times s)}, \text{ for } i = j, \sigma_{x_i x_j} = \sigma_{x_i}^2 ,$$

and

$$E_1 E_{2/1}\left(\bar{D}_{x_1''}' \bar{D}_{x_2}\right) = \left(\theta_1 - \theta_2\right) \Sigma_{x(r\times s)} = \left(\theta_1 - \theta_2\right)\left[\sigma_{x_i x_j}\right]_{(r\times s)}, \text{ for } i = j, \sigma_{x_i x_j} = \sigma_{x_i}^2 . \tag{3.4}$$

Using (3.4) in (3.3), we get:

$$\Sigma_{t_m(p\times p)} = \theta_2\Sigma_{y_{(p\times p)}} + (\theta_1-\theta_2)\Sigma_{yx_{(p\times r)}} A_{1_{(r\times p)}} - \theta_2\Sigma_{yx_{(p\times r)}} B_{(r\times p)}$$
$$+(\theta_1-\theta_2)\Sigma_{yx_{(p\times s)}} A_{2_{(s\times p)}} + (\theta_1-\theta_2) A'_{1_{(p\times r)}} \Sigma'_{yx_{(r\times p)}}$$
$$+(\theta_2-\theta_1) A'_{1_{(p\times r)}} \Sigma_{x_{(r\times r)}} A_{1_{(r\times p)}} - (\theta_1-\theta_2) A'_{1_{(p\times r)}} \Sigma_{x_{(r\times r)}} B_{(r\times p)}$$
$$+(\theta_2-\theta_1) A'_{1_{(p\times r)}} \Sigma_{x_{(r\times s)}} A_{2_{(s\times p)}} - \theta_2 B'_{(p\times r)} \Sigma'_{yx_{(r\times p)}}$$
$$-(\theta_1-\theta_2) B'_{(p\times r)} \Sigma_{x_{(r\times r)}} A_{1_{(r\times p)}} + \theta_2 B'_{(p\times r)} \Sigma_{x_{(r\times r)}} B_{(r\times p)}$$
$$-(\theta_1-\theta_2) B'_{(p\times r)} \Sigma_{x_{(r\times s)}} A_{2_{(s\times p)}} + (\theta_1-\theta_2) A'_{2_{(p\times s)}} \Sigma'_{yx_{(s\times p)}}$$
$$+(\theta_2-\theta_1) A'_{2_{(p\times s)}} \Sigma'_{x_{(s\times r)}} A_{1_{(r\times p)}} - (\theta_1-\theta_2) A'_{2_{(p\times s)}} \Sigma'_{x_{(s\times r)}} B_{(r\times p)}$$
$$+(\theta_2-\theta_1) A'_{2_{(p\times s)}} \Sigma_{x_{(s\times s)}} A_{2_{(s\times p)}} . \quad (3.5)$$

For optimum values of unknown matrices, differentiating (3.5) w.r.t. $A_{1_{(r\times p)}}$, $B_{(r\times p)}$ and $A_{2_{(s\times p)}}$, equating all equations to zero, we get:

$$B_{(r\times p)} = \Sigma^{-1}_{x_{(r\times r)}} \Sigma'_{yx_{(r\times p)}} , \quad (3.6)$$

$$A_{2_{(s\times p)}} = W^{-1}_{x_{(s\times s)}} \left(\Sigma'_{yx_{(s\times p)}} - \Sigma_{x_{(s\times r)}} \Sigma^{-1}_{x_{(r\times r)}} \Sigma'_{yx_{(r\times p)}} \right) \quad (3.7)$$

and

$$A_{1_{(r\times p)}} = -\Sigma^{-1}_{x_{(r\times r)}} \Sigma_{x_{(r\times s)}} W^{-1}_{x_{(s\times s)}} \left(\Sigma'_{yx_{(s\times p)}} - \Sigma_{x_{(s\times r)}} \Sigma^{-1}_{x_{(r\times r)}} \Sigma'_{yx_{(r\times p)}} \right). \quad (3.8)$$

After simplification of (3.3), we get

$$\Sigma_{t_m(p\times p)} = E_1 E_{2/1} \left[\bar{D}'_y \left(\bar{D}_y + \bar{D}_{x'_1} A_1 - \bar{D}_{x''_1} B + \bar{D}_{x_2} A_2 \right) \right]$$

or

$$\Sigma_{t_m(p\times p)} = E_1 E_{2/1} \left(\bar{D}'_y \bar{D}_y \right) + E_1 E_{2/1} \left(\bar{D}'_y \bar{D}_{x'_1} \right) A_1$$
$$- E_1 E_{2/1} \left(\bar{D}'_y \bar{D}_{x''_1} \right) B + E_1 E_{2/1} \left(\bar{D}'_y \bar{D}_{x_2} \right) A_2 . \quad (3.9)$$

Using results written after eq. (3.3) in (3.9), we get:

$$\Sigma_{t_m(p\times p)} = \theta_2\Sigma_{y_{(p\times p)}} + (\theta_1-\theta_2)\Sigma_{yx_{(p\times r)}} A_{1_{(r\times p)}}$$
$$-\theta_2\Sigma_{yx_{(p\times r)}} B_{(r\times p)} + (\theta_1-\theta_2)\Sigma_{yx_{(p\times s)}} A_{2_{(s\times p)}} \quad (3.10)$$

Now by putting the values of $B_{(r\times p)}$, $A_{2_{(s\times p)}}$ and $A_{1_{(r\times p)}}$ from (3.6), (3.7) and (3.8) respectively in (3.10), we get:

$$\Sigma_{t_m(p\times p)} = \theta_2\left(\Sigma_{y_{(p\times p)}} - \Sigma_{yx_{(p\times r)}}\Sigma^{-1}_{x_{(r\times r)}}\Sigma'_{yx_{(r\times p)}}\right) - \left(\theta_2 - \theta_1\right)\left\{\left(\Sigma_{yx_{(p\times s)}} - \Sigma_{yx_{(p\times r)}}\Sigma^{-1}_{x_{(r\times r)}}\Sigma_{x_{(r\times s)}}\right)\right.$$

$$\left. W^{-1}_{x_{(s\times s)}}\left(\Sigma'_{yx_{(s\times p)}} - \Sigma_{x_{(s\times r)}}\Sigma^{-1}_{x_{(r\times r)}}\Sigma'_{yx_{(r\times p)}}\right)\right\}, \text{ where } W^{-1}_{x_{(s\times s)}} = \left(\Sigma_{x_{(s\times s)}} - \Sigma_{x_{(s\times r)}}\Sigma^{-1}_{x_{(r\times r)}}\Sigma_{x_{(r\times s)}}\right)^{-1}. \quad (3.11)$$

4. EMPIRICAL STUDY

We have used the data from Population Census Report of Sialkot District (1998), Pakistan for empirical study from which three study variables denoted by Y's and two auxiliary variables denoted by X's have been considered for computing the Eigen values of variance covariance matrices of suggested estimator and Butt, et al. (2011). The variables description is given in Table A-1 of Appendix A. We have used two auxiliary variables in empirical study because Butt, et al. (2011) constructer the estimators for two-auxiliary variables. The necessary parameters of populations for computing the Eigen values for variance covariance matrices are given in Table A-2. Table A-3 and A-4 contains the Eigen values of variances covariance matrices for different values of θ_1 and θ_2 of proposed and Butt, et al. (2011) estimators respectively. The ratios of sum of Eigen values of proposed estimator to Butt, et al. (2011) estimator are given in Table A-5. This table clearly shows that our proposed estimator for partial information case is efficient for all considered combinations of θ_1 and θ_2 then Butt, et al. (2011) and it has an additional advantage of making use of multi-auxiliary information.

REFERENCES

1. Butt, N.S., Kamal, S. and Shahbaz, M.Q. (2011). Multivariate estimators for two-phase sampling. *World Applied Science Journal*, 13(10), 2216-2223.
2. Haziza, D., Hidiroglou, M.A. and Rao, J.N.K. (2011). Comparisons of variance estimators in two-phase sampling: an empirical investigation. *Pak. J. Statist.,* 27, 477-492.
3. Hidiroglou, M.A., Rao, J.N.K. and Haziza, D. (2009). Variance estimation in two-phase sampling. *Aust. N.Z.J. Statist.*, 51(2), 127-141.
4. Kadilar, C. and Cingi, H. (2004). Estimator of a population mean using two auxiliary variables in simple random sampling. *International Mathematical Journal*, 5, 357-367.
5. Kadilar, C. and Cingi, H. (2005). A new estimator using two auxiliary variables. *Applied Mathematics and Computation*, 162, 901-908.
6. Olkin, I. (1958). Multivariate ratio estimation for finite populations. *Biometrika*, 45, 154-165.
7. Pradhan, B.K. (2005). A Chain Regression Estimator in Two Phase Sampling using Multi-auxiliary Information. *Bull. Malays. Math. Sci. Soc.*, (2) 28(1), 81-86.
8. Robinson, P.M. (1994). A class of estimators for the mean of a finite population using auxiliary information. *Sankhya*, B, 56, 389-399.
9. Roy, D.C. (2003). A regression type estimator in two-phase sampling using two auxiliary variables. *Pak. J. Statist.*, 19(3), 281-290.

10. Samiuddin, M. and Hanif, M. (2007). Estimation of population mean in single and two phase sampling with or without additional information. *Pak. J. Statist.*, 23(2), 99-118.
11. Singh, S. (2004). Repair of two-phase calibration methodology in survey sampling. SSC Annual Meeting, May 2004, *Proceedings of the Survey Methods Section*, pp. 115-121.
12. Singh, H.P. and Namjoshi, U.D. (1988). A class of multivariate regression estimators in two-phase sampling. *Assam Statist. Rev.*, 2(2), 1-7.
13. Srivastava, S.K. (1971). A generalized estimator for the mean of a finite population using multi-auxiliary information. *J. Amer. Statist. Assoc.*, 66, 404-407.
14. Stearns, M., and Singh, S. (2008). On the estimation of the general parameter. *Journal of Computational Statistics and Simulation*, 52(9), 4253-4271.
15. Swain, A.K.P.C. (2000). *Current Developments in Survey Sampling*. Om Prakash, Bhubaneswar, India.
16. Tripathi, T.P. (1989). Optimum estimation of mean vector for dynamic population. Invited paper in the *Proceeding of International Symposium on Optimization and Statistics*., held at AMU. Aligarh.
17. Tripathi, T.P. and Khattree, R. (1989). Simultaneous estimation of several means using multivariate auxiliary information, *Tech. Report* No. 13/89, Stat/Math, ISI, Calcutta.

APPENDIX A

Table A-1: Description of variables
(Each variable is taken from Rural Locality)

Description of variables			
Y_1	Literacy ratio	X_1	Population of both sexes
Y_2	Population of currently married	X_2	Population of primary but below metric
Y_3	Total Households		

Table A-2: Covariance and Correlation Matrices

Covariance Matrix					
	Y1	**Y2**	**Y3**	**X1**	**X2**
Y1	58.379	1694.029	1555.41	11831.968	3818.858
Y2	1694.029	469251.1	440381	3274827.083	789524.592
Y3	1555.411	440380.8	415878	3076130.189	741371.717
X1	11831.96	3274827	3076130	22917781.56	5519188.648
X2	3818.858	789524.6	741372	5519188.648	1375257.472
Correlation Matrix					
Y1	1	0.324	0.316	0.323	0.426
Y2	0.324	1	0.997	0.999	0.983
Y3	0.316	0.997	1	0.996	0.98
X1	0.323	0.999	0.996	1	0.983
X2	0.426	0.983	0.98	0.983	1

Table A-3: Eigen values of variance covariance matrices of proposed estimator

θ_2	θ_1							
	0.1	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.2	8.3091	10.351	12.365	14.352	16.312	18.245	20.15	22.028
	661.89	663.4	193.77	194.58	195.41	196.25	197.1	197.96
	192.18	192.97	664.92	666.45	668	669.56	671.14	672.73
Total	**862.379**	**866.721**	**871.055**	**875.382**	**879.722**	**884.055**	**888.39**	**892.718**
0.3	11.434	13.489	15.526	17.545	19.546	21.529	23.493	25.439
	992.09	993.59	995.1	996.62	291.06	291.88	292.7	293.53
	287.88	288.67	289.46	290.25	998.14	999.68	1001.2	1002.8
Total	**1291.4**	**1295.75**	**1300.09**	**1304.42**	**1308.75**	**1313.09**	**1317.39**	**1321.77**
0.4	14.557	16.618	18.666	20.701	22.722	24.73	26.724	28.705
	1322.3	1323.8	1325.3	1326.8	1328.3	387.54	388.35	389.17
	383.59	384.36	385.15	385.94	386.74	1329.8	1331.4	1332.9
Total	**1720.45**	**1724.78**	**1729.12**	**1733.44**	**1737.76**	**1742.07**	**1746.47**	**1750.78**
0.5	17.678	19.744	21.799	23.843	25.877	27.899	29.911	31.912
	1652.5	1654	1655.5	1657	1658.5	1660	1661.5	484.83
	479.29	480.06	480.85	481.63	482.43	483.22	484.02	1663.1
Total	**2149.47**	**2153.8**	**2158.15**	**2162.47**	**2166.81**	**2171.12**	**2175.43**	**2179.84**
0.6	20.799	22.868	24.927	26.978	29.019	31.052	33.075	35.09
	1982.7	1984.2	1985.7	1987.2	1988.7	1990.2	1991.7	1993.2
	574.99	575.77	576.55	577.33	578.12	578.91	579.71	580.51
Total	**2578.49**	**2582.84**	**2587.18**	**2591.51**	**2595.84**	**2600.16**	**2604.49**	**2608.8**
0.7	23.92	25.991	28.053	30.108	32.156	34.195	36.227	38.251
	2312.9	2314.4	2315.9	2317.4	2318.9	2320.4	2321.9	2323.4
	670.69	671.47	672.25	673.03	673.81	674.6	675.4	676.19
Total	**3007.51**	**3011.86**	**3016.2**	**3020.54**	**3024.87**	**3029.2**	**3033.53**	**3037.84**
0.8	27.041	29.113	31.178	33.236	35.288	37.333	39.371	41.402
	2643.1	2644.6	2646.1	2647.6	2649.1	2650.6	2652.1	2653.6
	766.4	767.17	767.95	768.73	769.51	770.3	771.09	771.88
Total	**3436.54**	**3440.88**	**3445.23**	**3449.57**	**3453.9**	**3458.23**	**3462.56**	**3466.88**
0.9	30.162	32.235	34.302	36.363	38.418	40.467	42.51	44.547
	2973.3	2974.8	2976.3	2977.8	2979.3	2980.8	2982.3	2983.8
	862.1	862.87	863.65	864.43	865.21	866	866.78	867.57
Total	**3865.562**	**3869.905**	**3874.252**	**3878.593**	**3882.928**	**3887.267**	**3891.59**	**3895.917**

Table A-4: Eigen values of variance covariance matrices of Butt, et al. (2011)

θ_2	θ_1							
	0.1	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
	-4102300	-8049600	-12002000	-15958000	-19920000	-23886000	-27857000	-31833000
0.2	6.3849	6.3888	6.3927	6.3965	6.4004	6.4043	6.4082	6.4121
	4105900	8056000	12011000	15971000	19935000	23904000	27878000	31857000
Total	**3606.3849**	**6406.3888**	**9006.3927**	**13006.3965**	**15006.4004**	**18006.4043**	**21006.4082**	**24006.4121**
	-9198200	-18045000	-26899000	-35760000	-44628000	-53504000	-62386000	-71276000
0.3	9.5721	9.576	9.5799	9.5838	9.5877	9.5916	9.5954	9.5993
	9202300	18052000	26909000	35773000	44644000	53522000	62407000	71299000
Total	**4109.5721**	**7009.576**	**10009.5799**	**13009.5838**	**16009.5877**	**18009.5916**	**21009.5954**	**23009.5993**
	-16323000	-32020000	-47726000	-63441000	-79166000	-94901000	-110640000	-126400000
0.4	12.759	12.763	12.767	12.771	12.775	12.779	12.783	12.787
	16328000	32027000	47736000	63454000	79182000	94919000	110670000	126420000
Total	**5012.759**	**7012.763**	**10012.767**	**13012.771**	**16012.775**	**18012.779**	**30012.783**	**20012.787**
	-25478000	-49974000	-74482000	-99002000	-123530000	-148080000	-172630000	-197200000
0.5	15.947	15.95	15.954	15.958	15.962	15.966	15.97	15.974
	25483000	49982000	74493000	99016000	123550000	148100000	172660000	197230000
Total	**5015.947**	**8015.95**	**11015.954**	**14015.958**	**20015.962**	**20015.966**	**30015.97**	**30015.974**
	-36662000	-71908000	-107170000	-142440000	-177730000	-213040000	-248350000	-283680000
0.6	19.134	19.138	19.142	19.145	19.149	19.153	19.157	19.161
	36667000	71916000	107180000	142460000	177750000	213050000	248370000	283710000
Total	**5019.134**	**8019.138**	**10019.142**	**20019.145**	**20019.149**	**10019.153**	**20019.157**	**30019.161**
	-49875000	-97821000	-145780000	-193760000	-241760000	-289770000	-337800000	-385850000
0.7	22.321	22.325	22.329	22.333	22.337	22.34	22.344	22.348
	49881000	97830000	145800000	193780000	241780000	289790000	337820000	385870000
Total	**6022.321**	**9022.325**	**20022.329**	**20022.333**	**20022.337**	**20022.34**	**20022.344**	**20022.348**
	-65118000	-127710000	-190330000	-252960000	-315620000	-378290000	-440980000	-503690000
0.8	25.508	25.512	25.516	25.52	25.524	25.528	25.532	25.535
	65124000	127720000	190340000	252980000	315630000	378310000	441000000	503720000
Total	**6025.508**	**10025.512**	**10025.516**	**20025.52**	**10025.524**	**20025.528**	**20025.532**	**30025.535**
	-82390000	-161590000	-240800000	-320040000	-399300000	-478590000	-557890000	-637210000
0.9	28.696	28.699	28.703	28.707	28.711	28.715	28.719	28.723
	82397000	161600000	240820000	320060000	399320000	478610000	557910000	637240000
Total	**6692.5**	**9492.16**	**12291.8**	**14991.44**	**17891.08**	**20690.73**	**23490.38**	**26290.05**

Table-A5: Relative Efficiency of proposed estimator over estimator proposed by Butt, et al. (2011)

θ_2	θ_1							
	0.1	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
0.2	0.23913	0.13529	0.09672	0.06730	0.05862	0.04910	0.04229	0.03719
0.3	0.31424	0.18485	0.12988	0.10027	0.08175	0.07291	0.06270	0.05744
0.4	0.34321	0.24595	0.17269	0.13321	0.10852	0.09671	0.05819	0.08748
0.5	0.42853	0.26869	0.19591	0.15429	0.10825	0.10847	0.07248	0.07262
0.6	0.51373	0.32208	0.25822	0.12945	0.12967	0.25952	0.13010	0.08690
0.7	0.49939	0.33382	0.15064	0.15086	0.15107	0.15129	0.15151	0.15172
0.8	0.57033	0.34321	0.34365	0.17226	0.34451	0.17269	0.17291	0.11546
0.9	0.54997	0.38588	0.19343	0.19365	0.19387	0.19408	0.19430	0.12974

A NEW QUALITY MEASURE FOR PRODUCT DESIGN*

A.K. Chaudhuri[1], **Muhammad Hanif**[1] and **Munir Ahmad**[2]
[1] El-Fateh University, Tripoli, Libya
[2] King Fahd University of Petroleum and Minerals
Dhahran, Saudi Arabia.

ABSTRACT

The object of the manufacturers is to produce products which arc robust with respect to likely variations. Thus, at the design stage, quality index becomes necessary which measures the degree of robustness of the product design. This paper outlines the concept of Signal to Noise ratio which is effectively used to evaluate a product design for its non-dynamic characteristics.

KEY WORDS

Noise Factors, Quality Characteristics, Signal-Noise ratio (SN Ratio).

1. INTRODUCTION

Efforts have been devoted to customer orientation of the product and as consequence the emphasis has been most extensive on the parameter "Quality of product design" Chaudhuri and Chaudhuri (1981), Chaudhuri and Rao (1980). Chaudhuri and Hanif (1989). Recently applications have been given to various areas of engineering sciences (Ahmad et al. 1989).

One of the objectives of the manufacturers is to produce products which are robust, with respect all noise factors. A product designer can take care of quality product by determining the optimum level of individual design variants. The nominal values of the design variants are chosen in such a way that the effect of noises on the performance characteristics is kept minimum. Thus, at the design stage, a quality index becomes necessary which measures the degree of robustness of the product.

2. SIGNAL TO NOISE. RATIO AS A QUALITY MEASURE

A product design will be called robust, if its performance characteristic is close to the desired mean value and is least sensistive to variations in noise factors. This phenomenon can be explored mathematically in the following manner.

Let X be a random variable, representing a particular performance characteristic of a product and X has a finite mean m and variance σ^2 then the condition that the observed value of X is with high probability, within a pre-assigned deviation from its desired mean value may be obtained from Chebyshev's inequality for any $\varepsilon > 0$.

$$P\left[\left|X-m\right| \leq \in\right] \geq 1-\sigma^2 / \in^2 . \qquad (2.1)$$

*Published in Pak. J. Statist. (1990 A), Vol. 6(1).

Chebyshev's inequality accordingly may be reformulated in terms of the relative deviation for any $\delta > 0$.

$$P\left[\left|\frac{X-m}{m}\right| \le \delta\right] \ge 1 - \frac{1}{\delta^2}\frac{\sigma^2}{m^2} \tag{2.2}$$

If X is normally distributed one can obtain the following conclusions

$$P\left[\left|\frac{X-m}{m}\right| \le \delta\right] \ge 95\% \; If \; \delta \ge 1.96\, m/\sigma \tag{2.3}$$

and

$$P\left[\left|\frac{X-m}{m}\right| \le \delta\right] \ge 99\% \; If \; \delta \ge 2.58\, m/\sigma \tag{2.4}$$

For non-zero mean (m) and finite variance σ^2 the percentage deviation of X from its desired mean `*m*' may, with high probability, be small, it is sufficient that the ratio m/σ or m^2/σ^2 be large. The quantity m^2/σ^2 is called the signal to Noise ratio (S/N ratio). The terminology S/N ratio originated in communication theory. The mean '*m*' of a random variable X is regarded as a signal. The difference between the desired mean `*m*' and the received value $X$ is called Noise. [Freeman (1958)]. As a measure of signal strength to noise strength one takes the S/N ratio. Higher the S/N ratio, closer the observed value X will be to the desired mean value `*m*'. The S/N ratio, denoted by η is expressed here in decibels, by adopting the transformation,

$$\eta = 10\log\left[m^2/\sigma^2\right] \tag{2.5}$$

This transformation makes η meanable for various statistical treatment like analysis of variance and other parametric tests (see 3,7).

3. MEASURES FOR DIFFERENT NON-DYNAMIC CHARACTERISTICS

The following classification had been made of the non-dynamic quality characteristics, depending on the nature and requirement of the end use. [Parsad (1983) and Kackar (1985)].

N-Type: Nominal (or the **target)** the better e.g. shut-off head of a pump, output voltage of a circuit, count of yarn etc.

S-Type: Smaller the better e.g. positive suction head required for a pump to pump the fluid; tangent of loss angle for insulation system in H.V. coil, etc.

L-Type: Larger the better, e.g. break down voltage of an electrical system; breaking strength of concrete blocks, etc.

3.1 N-Type Characteristics

In this case, the objective is to make the characteristic value as close as possible to the desired mean value '*m*'. Let $X_1, ..., X_n$ be the actual observations on the characteristic under consideration. The Quality measure η

$$\eta = 10\log\left[\frac{S_m - V_0}{nV_0}\right] \tag{3.1}$$

is a good measure of quality product as

$$E\left[\frac{S_m - V_0}{nV_0}\right] = \frac{m^2}{\sigma^2} = S/N\ Ratio \tag{3.2}$$

where

$$S_m = \frac{\left(\sum_{i=1}^{m} x_i\right)^2}{n} .and\, V_0 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2 .$$

3.2 S-Type Characteristics

For smaller, the better type of characteristics, the ideal target is zero. The closer X to zero, (from the right side) better is the achievement of the object. The S/N ratio cannot be used here, as $n = 0$. A different measure is used instead of m^2/σ^2 for comparing different design prepositions. and the proportion with smallest of this measure can be selected as the optimum. Thus if m^2/σ^2 is minimised, both variation and the absolute value get minimised concurrently. Thus, this measure is also termed as concurrent measure and is expressed in decibels, by logarithmic transformation, and for maintaining the same objective as of S/N ratio, The quality measure, η_s

$$\eta_s = -10\log\left(\frac{1}{n}\sum_{i=0}^{n} X_i^2\right)$$

is a good measure of a quality product as

$$E\left[\sum^{n} X_i^2 / n\right] = \frac{m^2}{\sigma^2}$$

3.3 L-Type Characteristics

In this case, the objective is to have the characteristics values as large as possible. Thus, if X represents the performance characteristic, X should be as large as possible or in other words $1/X$ should be as small as possible.

The concurrent measure is defined as

$$\eta_L = 10\log_{10}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_1^2}\right] \tag{3.3}$$

Let $f(x) = 1/X^2$ and expanding it in the neighborhood of 'm' in Taylor's series expansion, one gets,

$$f(X) = f(m) + \left[\frac{f'(X)}{1!}\right]_{X-m} (X-m) + \left[\frac{f''(X)}{2!}\right]_{X-m} (X-m)^2 + ... \quad (3.4)$$

$$E\left[f(X)\right] \approx \frac{1}{m^2} + 0 + \frac{3\sigma^2}{m^4} \quad (3.5)$$

Neglecting the higher order terms,

Thus

$$E\left[\frac{1}{n}\sum^{n}\frac{1}{X^2}\right] = \frac{1}{m^2} + 0 + \frac{3\sigma^2}{m^4} = \frac{1}{m^2}\left[1+\frac{3\sigma^2}{m^2}\right] \quad (3.6)$$

Thus, as η_L is maximized, variance gets minimized and concurrently the absolute value gets maximized.

ACKNOWLEDGEMENT

We are thankful to the referees for useful comments.

REFERENCES

1. Ahmad, M., Zubair, S.M., Sheikh, A.K., Raouf, A. and Abbasi, A.G. (1989). Statistical experimental design for product improvement: An application of signal to noise ratio to a concrete mix problem. *Proc. First. Islamic Countries Conference on Statistical Sciences*, Lahore, Pakistan. (Ed. Munir Ahmad.) Vol. III, 1976-1960.
2. Chaudhuri, A.K. and Hanif, M. (1989). Detecting and diagnosing latent quality losses. *Proc. First. Islamic Countries Conference on Statistical Sciences*. Lahore, Pakistan (Ed, Munir Ahmad) Vol.II, 841-849.
3. Chaudhuri, A.K. and Chaudhuri, K.K. (1981). Latent Quality Losses, *Q.R. Journal* Vol. 8(12).
4. Chaudhuri, A.K. and Rao, A.R. (1980). Improving quality/yield through process specification developments. *Proc. 2nd Inter. Conference of I.A.Q.C.*, San Fransisco, USA.
5. Freeman, J.J. (1958). *Principle of Noise*. John Wiley and Sons. U.S.A.
6. Parsad, C.R. (1983). An introduction to off-line quality control methods for improvements to product and process design. SQC and OR Division, Indian Statistical Institute, India.
7. Kackar, Raghu, N. (1985). Off-line Quality Control, parameter design and the Taguchi methods. In *Quality Control, Robust Design, and the Taguchi Method* (pp. 51-76). Springer US.

PARAMETER DESIGN: A METHOD FOR IMPROVING QUALITY*

Muhammad Hanif[1]**, A.K. Chaudhri**[2] and **Munir Ahmad**[3]

[1] Department of Mathematics and Statistics
Quaid-i-Azam University Islamabad, Pakistan

[2] El-Fatch University Tripoli, Libya.

[3] King Fahd University of Petroleum and Minerals
Dhahran, Saudi Arabia.

ABSTRACT

In this paper, Taguchi parameter design methodology is illustrated with an example of induction motor design.

KEY WORDS

Taguchi method, orthogonal array, parameter design, design variants, signal and noise ratio.

1. INTRODUCTION

The quality of a product is quantified in terms of total loss of a product from the time the product is manufactured to the time it is shipped to the customer. The loss may be due to undesirable side effects or deviation of the functional quality from the targeted performance. The variation in the product parameters also varies from unit to unit which are inevitable in a manufacturing process. The objective of a design engineer is to maximise the performance of a product and reduce the variability in performance from product to product. A product design will be called *robust* if its performance characteristics are close to the intended value and are least sensitive to noise factors. {See Chaudhuri *et al.* (1990), Kacker (1985), Taguchi (1981), Taguchi *et al.* (1989), Burgam (1985) and Leon *et al.* (1987)}.

In this paper, thirteen factors each at three levels for SPDP Induction Motor and four noise factors also at three levels are investigated on the basis of 36 orthogonal performance treatment combinations at each of 18 orthogonal noise treatment conditions. The optimal design is arrived at using Taguchi approach.

2. THE PARAMETER DESIGN

For better competition, improvement in performance of motor is necessary with no extra cost. A 30 kw, 4 pole SPDP motor is selected for application of this method.

Based on technological considerations 13 design factors each at 3 levels that influence motor performance are selected as *design variants.* Table 1 gives the list of design variants taken up for the study. In order to consider the effect of manufacturing and material variations on motor performance, four factors each at three levels are selected as noise factors. Table 2 gives the list of noise factors and their levels.

---

*Published in Pak. J. Statist. (1990 A), Vo1. 6(2).

A design variants listed in Table 1 arc allocated in orthogonal Array (O.A.) Table $L_{36},\left(3^{13}\right)$ called inner O.A. (Appendix-1). Thus there are 36 combinations of design variants.

Table 1
Design variants (All at 3 levels)

Factor Code	Description
A	Start or stamping - outside diameter
B	Start or stamping - inside dia/outside dia
C	Start or stamping- tooth width
D	Start or stamping - yoke height
E	Rotor stamping - tooth width
F	Rotor slot area/start or slot area
G	Top cage area/bottwn cage area
H	Middle cage width
I	Middle cage height
J	S.C. Ring area
K	Conductors per slot
L	Core length
M	Air gap

Table 2
Noise Factor

Factor Code	Description	Level		
		1	2	3
L	Core length	+ 3%	Nominal value	- 3%
M	Air gap	+ 10%	-do-	- 10%
O	Stamping grade (Watts/kg)	+ 20%	-do-	- 20%
P	Conductivity of Al	+ 10%	-do-	- 10%

Noise factors were allocated in O.A. Table $L_{18},\left(3^{4}\right)$ called outer OA. (Appendix-II). For each combination of inner O.A. there are 18 combinations of auter O.A. The numbers, 1, 2, 3 in the tables of Appendices 1 and II indicate levels of respective factors.

Responses selected are efficiency, power factor, slip starting Torque, starting current and cost of stampings.

3. COMPUTATION OF SIGNAL-NOISE RATIO

A well debugged computer programme for *indication motor design* is used to obtain responses. Using the simulated software model, for each of 36 combinations of the design variants the above responses are calculated with noise factor level combinations indicated in L_{18} O.A. The Table 3 gives a typical response table for experiment No. 15 of inner O.A. given in Appendix-I.

As our aim is to optimise various performance characteristics and simultaneously minimise their variation, we have to account for the variation in the performance characteristics for each combination of design variants of inner O.A. $\left(L_{36}\right)$ over the range of noise factors considered in outer O.A. $\left(L_{18}\right)$. This is achieved by expressing these varying data in a single measure, S/N Ratio. For efficiency S/N ratio is given by (see Taguchi *et al.* 1989)

$$\eta = -10 Log_{10}\left\{\sum_{i=1}^{18} X_1^2 / 18\right\},$$

where $X_1 = Loss = 1$ efficiency for the ith combination of any outer O.A. $\left(L_{18}\right)$. While minimising X_i, we maximise efficiency. Maximising *n* will maximise efficiency as well as minimise its variation in noise factor space. Thus we have second set of derived responses viz. S/N Ratios. These measures remained same for Power factor (1-power factor), slip, starting Tarque, as well as starting current as everywhere the intended target value is as low as possible. To calculate S/N ratio of any characteristic, we have 18 data from outer O.A., for each combination of inner O.A. (L_{36}). The S/N ratios are worked out for efficiency, power factor, slip, starting Torque, and starting current at Table 3. The Table 4 shows S/N ratios for each of 36 combinations of the inner O.A. A typical calculation of S/N ratio for efficiency of experimental combination No. 15 of inner O.A. is shown in Appendix-3. This amounts to 21.359 dB. This value cart also be seen from Table 4.

4. ANALYSIS OF VARIANCE

ANOVA is performed on the primary responses (for cost data) and on S/N ratio to obtain the contribution of each design variant. The total response for S/N ratio of the efficiency is shown in Table 5 and the ANOVA for the same is shown in Table 6.

Table 3
Typical Response Table

Expt. No.	Efficiency	Power factor	Slip	Starting torque	Starting current	Cost index
1.	0.91187	0.88233	0.01971	1.65006	4.98885	4.4567
2.	0.91389	0.88717	0.02174	1.47844	4.71127	4.4596
3.	0.91637	0.89678	0.02398	1.29799	4.39461	4.4625
4,	0.9(1965	0.87348	0.02128	1.67813	4.97783	4.3518
5.	0.91313	0.89027	0.02320	1.49056	4.73854	4.3545
6.	0.92127	0.90006	0.01926	1.33569	4.67053	4.3573
7.	0.91659	0.87486	0.01870	1.72530	5,23323	4.2468
8.	0.91932	0.88441	0.02071)	1.54264	4.93714	4.2495
9.	0.90913	0.89430	0.02269	1.36317	4.59622	4.2522
10.	0.91618	0.88059	0.02379	1.61603	4.83715	4.4567
11.	0.91161	0.88818	0.01977	1.49187	4.77221	4.4596
12.	0.91417	0.89765	0.02128	1.30647	4.4625	4.4625
13.	0.91195	0.87246	0.02323	1.66167	4.91591	4.3517
14.	0.92088	0.88909	0.01922	1.51629	4.92507	4.3545
15.	0.91116	0.90195	0.02129	1.32659	4.55214	4.3573
16.	0.91895	0.87411	0.02068	1.70743	5.13778	4.2468
17.	0.90926	0.88647	0.02260	1.52993	4.83583	4.2495
18.	0.91687	0.89317	0.01879	1.38094	4.76882	4.2522

Table 4
SN Ratios

Expt. No.	SN (Eff.)	SN (Pf.)	SN (Slip)	SN (Tst.	SN (Ist.)
1.	16.588	19.068	24.442	-0.346	-9.043
2.	18.953	19.107	28.585	2.827	-12.597
3.	17.400	8.023	30.884	6.623	-13.192
4.	19.008	11.463	31.551	6.560	-15.137
5.	15.644	16.988	22.254	-0.262	-7.479
6.	19.375	19.625	29.573	2.967	-13.423
7.	20.391	12.458	33.103	6.328	-13.321
8.	18.264	20.264	26.396	4.930	-13.977
9.	14.932	21.057	23.167	0.060	-9.983
10.	20.869	15.001	32.711	6.495	-14.790
11.	18.505	17.543	29.486	0.996	-12.300
12.	12.120	20.974	18.433	1,584	-8.533
13.	16.510	20.859	24.260	0.068	-9.951
14.	17.242	19.223	26.178	3.238	-12.923
15.	21.359	18.916	33.436	3.438	-13.640
16.	19.410	21.289	28.368	-0.106	-11.1%
17.	19.977	17.494	31.194	5.272	-14.119
18.	17.095	19.587	25.265	1.509	-11.694
19.	19.825	14.469	31.172	3.606	-13.547
20.	16.153	22.706	24.592	2.415	-11.959
21.	18.421	19,762	26.722	0.057	-10.242
22.	21.031	15.472	33.538	4.999	-14.831
23.	13.825	20.906	21.943	0.090	-9.576
24.	16 .207	18.912	22.872	-1.170	-8.298
25.	15.109	22.835	22.369	-0.403	-10.022
26.	19.299	8.792	34.638	6.012	-13.742
27.	16.239	16.640	23.252	-2.637	-6.683
28.	17.448	20.757	26.277	0.123	-10 147
29.	18.481	17.764	26.388	-5.225	-7.214
30.	17.565	21.981	26.041	2.229	-12.419
31.	16.575	17.117	23.316	-3.425	-6.743
32.	18.474	20.776	27.470	0.575	-10.946
33.	19.550	13.814	32.210	3.968	-15.125
34.	18.824	23.170	27.762	-0.127	-12.123
35.	19.140	20.718	27.317	-1.855	-9.815
36.	18.030	19.117	28.255	2.573	-12.235

Table 5
Total Response Table for S/N Ratio of Efficiency

Factor	Levels		
	1	2	3
A	221.59	213.96	208.29
B	225.33	219.53	198.98
C	213.39	213.99	216.46
D	198.74	216.80	228.30
E	216.35	217.86	209.30
F	220.60	212.30	210.94
G	214.30	214.61	214.93
H	218.49	212.01	213.34
I	212.88	220.55	210.41
J	214.19	215.45	214.20
K	200.92	213.62	229.30
L	205.86	220.36	217.62
M	212.05	217.06	214.73

Table 6
ANOVA for S/N ratio of Efficiency

Sources of variation	D.P.	Sum of squares	Mean sum of squares	% contribution
A	2	7.42	3.17	3.94
B	2	31.95	15.98	20.97.
C	(2)	(0.44)	(0.22)	(Pooled)
D	2	37.00	18.50	23.99
E	2	3.19	1.60	1.08
F	2	4.56	2.28	2.01
G	(2)	(0.01)	(0.00)	(Pooled)
H	(2)	(1.95)	(0.97)	(Pooled)
1	2	4.66	2.33	2.07
J	(2)	(0.09)	(.04)	(Pooled)
K	2	33.67	18.84	21.74
L	2	9.88	4.94	5.61
M	(2)	(1.04)	(0.52) '	(Pooled)
Error	9	11.69	1.30	
Pooled error	(19)	(15.22)	(0.80)	(19.00)
Total	35	147.55		

Table 7 is summary of percentage contribution of design variants for different responses. The level indicated in the Table for each factor is the best level. Whenever, the contribution is less than 1% the same in omitted from the table.

Table 7
Percentage contribution of design parameters for different responses (only the best levels are indicated)

Responses	SN (Eff)	SN (Pf)	SN (Slip)	SN (T.st)	SN (C.st)	SN (Cost)
Factor						
A	Al (3.94)	-	Al (2.38)	-	-	A3 -
B	B1 (20.57)	B3 (5.95)	B1 (12.47)	-	B3 (1.14)	B3 (7.48)
C	-	-	-	-	-	DI (3.85)
D	D3 (23.99)	D1 (14.70)	D3 (19.96)	-	-	D1 (3.85)
E	E2 (1.08)	-	-	-	-	
F	F2 (2.01)	-	Fl	-	F2	-
G	-	-	-	-	-	-
H	-	-	-	-	H3 (1.25)	-
I	12 (2.07)	12 (1.61)	-	I 1 (3.78)	13 1.31)	
J	-	-	-	-	-	-
K	K3 (21.74)	K1 (9.61)	K3 (34.40)	K3 (46.11)	K1 (51.39)	-
L	L2 (5.61)	L2 (24.72)	L3 (18.52)	L3 (23.96)	Ll (26.84)	L3 (68.5)
M	-	M2 (5.30)	-	M1 (18.02	M3 (5.71)	-

5. CONCLUSION

Based on the contribution percent of the design variant, relative importance of the performance characteristics, manufacturing convenience and so on the trade off was done. The optimum design was arrived at as A1, B2, C2, D3, D2, F1, C1, H1, I2, J2, K2, L3, M2.

With the above factor level combinations, analysis programme was run and performance data were obtained. Table 8 shows the performance of the existing design and the optimised design.

Table 8
Performance Comparison

Characteristics	Existing	Optimized Design
Efficiency	0.8181	0.9116
Power Factor	0.9180	0.8670
Slip (%)	2.7900	1.9600
Starting Torque	1.5900	1.6500
Starting Current	5.1300	5.5800
Cost index	3.5620	3.5550

Analysis programme was run for the optimized design as per the noise factors allocated in outer O.A. The variability observed in the performance characteristics were well within allowable limits of national and international standards, which was not the case with the existing design.

As can be seen from Table 8, parameter design has helped in improving the performance (efficiency, slip, and starting Torque) as well as quality (with reduced variation across all the performance characteristics) with marginal reduction in cost. No doubt, there is reduction in Power factor but increase in efficiency is preferred to some sacrifice in Power factor: Moreover, as the care length in optimised design is less, output can be increased within the same frame size. Further as the losses are lower, the temperature rise will be lower.

6. ACKNOWLEDGEMENT

Authors are indebted to El-Fateh University, Tripoli, Libya and King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia for excellent facilities provided for this research.

7. REFERENCES

1. Ahmad, M., Zubair, S.M., Sheikh, A.K., Raouf A and Abbasi A.G. (1988). Statistical experimental design for product improvement: An application of Signal-Noise ratio to a concrete mix problem. *Proc. First Islamic Countries Conference on Statistical Sciences*, III 1976-1690.
2. Bargain, P.M., (1985). Design of Experiment - The Taguchi Way. *Manufacturing Engineering*, 94, 44-47.
3. Chaudhri, A.K., Hanif, M. and Ahmad, M. (1990). A new quality measure of product design. *Pak. J. Statist.*, 6(1) 55-60.
4. Kackar, Raghu N. (1985). Off-line Quality Control, Parameter Design and the Taguchi Method. *Journal of Quality Technology*.
5. Leon, R.V., Shoemaker, A.C. and Kacker. R.N. (1987). Performance Measures Independent of adjustment: An explanation and extension of Taguchi, Signal to Noise Ratio. *Technometrics*, 29(3), 253-265.
6. Taguchi, G. (1981). *On-Line Quality Control during Production*. Japanese Standards Association, Tokyo. Japan.
7. Taguchi, G., El-Sayed, A. and Hsiang, T. (1989). Quality Engineering in Production Systems. Mc Graw Hill Bonk Company, New York, NY, USA.

Appendix-1
Inner O.A. Table L_{36}
(Allocation of Design Factors)

Factor Expt. No.	A	B	C	D	E	F	G	H	I	J	K	L	M
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	1
3	3	3	3	3	3	3	3	3	3	3	3	3	1
4	1	1	1	1	2	2	2	2	3	3	3	3	1
5	2	2	2	2	2	3	3	3	1	1	1	1	1
6	3	3	3	3	1	1	1	1	2	2	2	2	1
7	1	1	2	3	1	2	3	3	1	2	2	3	1
8	2	2	3	1	2	3	1	1	2	3	3	1	1
9	3	3	1	2	3	1	2	2	3	1	1	2	1
10	1	1	3	2	1	3	2	3	2	1	3	2	1
11	2	2	1	3	2	1	3	1	3	2	1	3	1
12	3	3	2	1	3	2	1	2	1	3	2	1	1
13	1	2	3	1	3	2	1	3	3	2	1	2	2
14	2	3	1	2	1	3	2	1	1	3	2	3	2
15	3	1	2	3	2	1	3	2	2	1	3	1	2
16	1	2	3	2	1	1	3	2	3	3	2	1	1
17	2	3	1	3	2	2	1	3	1	1	3	2	2
18	3	1	2	1	3	3	2	1	2	2	1	3	2
19	1	2	1	3	3	3	1	2	2	1	2	3	2
20	2	3	2	1	1	1	2	3	3	2	3	1	2
21	3	1	3	2	2	2	3	1	1	3	1	2	2
22	1	2	2	3	3	1	2	1	1	3	3	2	2
23	2	3	3	1	1	2	3	2	2	1	1	3	2
24	3	1	1	2	2	3	1	3	3	2	2	1	2
25	1	3	2	1	2	3	3	1	3	1	2	2	3
26	2	1	3	2	3	1	1	2	1	2	3	3	3
27	3	2	1	3	1	2	2	3	2	3	1	1	3
28	1	3	2	2	2	1	1	3	2	3	1	3	3
29	2	1	3	3	3	2	2	1	3	1	2	1	3
30	3	2	1	1	1	3	3	2	1	2	3	2	3
31	1	3	3	3	2	3	2	2	1	2	1	1	3
32	2	1	1	1	3	1	3	3	2	3	2	2	3
33	3	2	2	2	1	2	1	1	3	1	3	3	3
34	1	3	1	2	3	2	3	1	2	2	3	1	3

Appendix-2
Outer O.A. Table L18
(Allocation of Error Factor)

Factor Expt. No.	L	M	N	O
1	1	1	1	1
2	1	2	2	2
3	1	3	3	3
4	2	1	1	2
5	2	2	2	3
6	2	3	3	1
7	3	1	2	1
8	3	2	3	2
9	3	3	1	3
10	1	1	3	3
11	1	2	1	1
12	1	3	2	2
13	2.	1	2	3
14	2	2	3	1
15	2	3	1	2
16	3	1	3	2
17	3	2	1	3
18	3	3	2	1

Appendix-3
Typical Calculation of
SN Ratio of Efficiency

Expt. No.	Efficiency	X =1 -- Efficiency
1.	0.91187	0.08813
2.	0.91389	0.08611
3.	0.91637	0.08363
4.	0.90969	0.09031
5.	0.91313	0.08687
6.	0.92127	0.07873
7.	0.91659	0.08341
8.	0.91932	0.08068
9.	0.90913	0.09087
10.	0.91618	0.08382
11.	0.91161	0.08839
12.	0.91417	0.08583
13.	0.91195	0.08805
14.	0.92088	0.07912
15.	0.91116	0.08884
16.	0.91895	0.08105
17.	0.90926	0.09074
18.	0.91687	0.08313

$$\text{SN Ratio} = -10\log_{10}\left\{\frac{1}{18}\left(\sum_{i=1}^{18} X_1^2\right)\right\}$$

$$= -10\log_{10}\left\{\frac{1}{18}\left(.09913^2 + .08611^2 +, +.08313^2\right)\right\}$$

$$= -10\log_{10}\left(7.3125\times 10^{-3}\right)$$

= 21.359(dB)

A NOTE ON MALCOLM BALDRIGE NATIONAL QUALITY AWARD*

Munir Ahmad
Institute of Quality Systems & Standards, Lahore, Pakistan

1. INTRODUCTION

The U.S. government instituted the Malcolm Baldrige national quality award that may cause American companies to improve quality and productivity. The award was created by Public Law 100-107 on August 20, 1987. In this paper, an overview of the award is given along with the name of companies which received the awards since 1988 and the key quality words they used.

2. AWARD ELIGIBILITY

The National Institute of Standards and Technology (NIST), US Department of Commerce was made responsible to arrange the award in association with American Society for Quality Control. Awards are to be offered annually to U.S. companies that excel in quality management and quality achievement. The first award was given in 1988.

Manufacturing, service, and small business are the three eligible categories for award. Up to two awards may be given in each of the three categories. The business that is eligible for the award is to be located in the United States or its territories. The eligibility criteria for the award are:

1. A company or its subsidiary with more than 50% employees in US.
2. At least 50 percent of a subsidiary's customer base is free of direct financial and line organization control by the parent company.
3. Non-chain organizations.
4. Business without support functions of the company.

Parent company and its subsidiary may not apply for the award in the same year. If a company receives an award, the company and all its subsidiaries are ineligible to apply for another award for a period of five years.

3. AWARD CRITERIA

The award criteria depend on the core values:

i) Customer-oriented quality.
ii) Leadership of senior management.
iii) Continuous improvement.
iv) Full participation of employees.
v) Fast response through reduction of cycle time.
vi) Emphasis on corrective and preventive actions.
vii) Preparation of plans.
viii) Data based decisions.
ix) Inter action between employees and management.

*Published in Pak. J. Statist. (1997), Vol. 13(2).

There are seven essential elements to be documented for the core values.

1. Leadership.
2. Information and analysis.
3. Quality planning.
4. Human resource development.
5. Control of process quality.
6. Inspection and testing procedures.
7. Customer's satisfaction.

4. PROCESS CAPABILITY INDEX

Process Capability Index (PCI) is criterion of selection of sub-contractors. A company would like to enhance its subcontractor's potential for improved quality. Each subcontractor should indicate its PCI C_{pk} performance within acceptable value of PCI to be larger than one but should have programme to achieve a PCI of 2. The sub-contractor is evaluated on the basis of the quality of product delivered and the timeliness of those deliveries. Sub-contractors with higher ratings earn more business; company may continue to compute C_{pk} on monthly or quarterly basis and communicate the value of index to the sub-contractors. (See Kane 1986).

Awarding institutions compute C_{pk} and offer an award on six sigma base. The six sigma is one of the objective criteria for the award.

When first Malcolm Baldrige award was offered to Motorola in 1988, the summary contained the following six sigma description:

"To accomplish its quality and total customer satisfaction goals, Motorola concentrates on several key operational initiatives. At the top of the list is Six Sigma Quality, a statistical measure of variation from a desired result. In concrete terms Six Sigma translates into a target of no more than 3.4 defects per million products, customer services included. At the manufacturing end, this required designs that accommodate reasonable variation in component parts but production process that yield consistently uniform final products. Motorola employees record the defects found in every function of the business, and statistical technologies are increasingly made part of each and every employee's job." (See Mitra, 1993 and Pena, 1990)

For 1992 Malcolm Baidrige award, the following examination categories were considered:

Table 1
Examination categories for the 1992 Malcolm Baldrige Award

Examination Category/Items			
1.0	**Leadership**		**90**
	1.1 Senior executive leadership	45	
	1.2 Management for quality	25	
	1.3 Public responsibility	20	
2.0	**Information and analysis**		**80**
	2.1 Scope and management of quality and performance data and information.	15	
	2.2 Competitive comparisons and benchmarks	25	
	2.3 Analysis and uses of company-level data	40	
3.0	**Strategic quality planning**		**60**
	3.1 Strategic quality and company performance planning process	35	
	3.2 Quality and performance plans	25	
4.0	**Human resource development and management**		**150**
	4.1 Human resource management	20	
	4.2 Employee involvement	40	
	4.3 Employee education and training	40	
	4.4 Employee performance and recognition	25	
	4.5 Employee well-being and morale	25	
5.0	**Management of process quality**		**140**
	5.1 Design and introduction of quality products and services	40	
	5.2 Process management-product and service production and delivery processes	35	
	5.3 Process management business processes and support services	30	
	5.4 Supplier quality	20	
	5.5 Quality assessment	15	
6.0	**Quality and operational results**		**180**
	6.1 Product and service quality results	75	
	6.2 Company operational results	45	
	6.3 Business process and support service results	25	
	6.4 Supplier quality results	35	
7.0	**Customer focus and satisfaction**		**300**
	7.1 Customer relationship management	65	
	7.2 Commitment to customers	15	
	7.3 Customer satisfaction determination	35	
	7.4 Customer satisfaction results	75	
	7.5 Customer satisfaction comparison	75	
	7.6 Future requirements and expectations of customers	35	
	Total Points		**1000**

5. KEY QUALITY WORDS

Companies that received the Malcolm Baldrige Award had used some key quality statements. A list of companies in chronical order with key quality words is given below:

Table 2
Malcolm Baldrige Awarded: Companies with Key Quality Words

Year	Name of the Company	Key Quality Word
1988	Globe Metallurgical Inc.	Elements of Quality Efficiency and Cost
1988	Motoral Inc.	Key Quality Initiates
1988	Westing House Electric Corporation Commercial Nuclear Fuel Division.	Total Quality
1988	Xerox Corporation Business Products & Systems	Leadership Through Quality Bench-Marking System
1989	Milliken & Company	Pursuit of Excellencecustomer responsiveness
1990	Cadillac Motor Car Company	Customer Satisfaction as the Master Plan
1990	Federal Express Corporation	People-Service-Profit
1990	IBM, Rochester	Rochester ExcellenceCustomer Satisfaction
1990	Wallace Co. Inc.	Continuous Quality Improvement
1991	Marlow Industries	Total Systems
1991	Solectron Corporation	Customer Needs Drive Results
1991	Zytec Corporation	Total Quality Commitment.
1992	AT & T Network Systems Groups	Quality Approach
1992	AT & T Universal Card Services	Delight Customer and Employees are Key
1992	Granite Rock Company	Total Quality
1992	The Ritz-Carlton Hotel Company	Gold Standards-Detailed Planning-Quality Data
1992	We are the Best...and Getting Better Texas Instruments Defense Systems & Electronic Group	Total Strategy
1993	Ames Rubber Corporation	Excellence Through Total Quality
1993	Eastman Chemical Company	Exceeding Customer's Expectations.

REFERENCES

1. Kane, V.E. (1986). Process Capability Indices. *J. Quality Technology*, 18(1), 41-52.
2. Mitra, Amitava (1953). *Fundamentals of Quality Control and Improvement. Macmillan* Publishing Co. New York, NY, USA.
3. Pena, E. (1990). Motorola's Secrete to Total Quality Control. *Quality Progress*, 23(10), 43-45.
4. US Department of Commerce (1994). Malcolm Baldrige Award: Profiles of Winners: 1988-93 NIST, *U.S. Commerce,* U.S.A. Washington, DC.
5. US Department of Commerce (1992). *1992 Award Criteria* Malcolm Baldrige National Quality Award, NIST, Gaithersbcrg, MD. USA.

ON INSTALLING BS-5750/ISO-9000 AT INSTITUTE OF EDUCATION AND RESEARCH, UNIVERSITY OF THE PUNJAB, LAHORE*

Munir Ahmad[1] and **Muhammad Abiodullah**[2]
[1] Islamic Society of Statistical Sciences (ISOSS)
Institute of Quality Systems & Standards (IQSAS)
Lahore, Pakistan
[2] Institute of Education & Research,
University of the Punjab, Lahore, Pakistan

1. INTRODUCTION

The Institute of Education and Research (IER) is located at Quaid-e-Azam Campus, University of the Punjab, Lahore, Pakistan. The institute was set up in September 1960 in cooperation with the School of Education, Indiana University, USA. It is the pioneer Institute in Pakistan for advanced studies in the field of Education. It enjoys the status of the premier and prestigious Institute of Education in the country. The major aims and objectives of the Institute include; (a) to provide and promote facilities for advanced study and research in education. (b) to provide teaching, training and guidance in order to prepare candidates for the Master's M.Phil and Ph.D. in Education degree of the University, and such other diplomas in Education, (c) to provide opportunities for professional educators to improve their knowledge and ability through summer and evening classes, short courses, seminars and other means. (d) to conduct research in several branches of education, publish the results of such research, and act as an educational informational dissemination center and (e) to render other services to educational institutions at all levels as and when necessary *(1ER 1996).*

2. BS-5750/ISO-9000 AND PRODUCT QUALITY

The 'product' of the IER, University of Punjab, Lahore is defined as student learning experiences, or as some form of added value to the student. It is likely that the same process and procedure will be included in the quality system and that the purpose of the quality system will be to improve the institution's effectiveness and consistency in delivering its products. Hingley and Sallis (1991) state that BS-5750 sets the standard for the system, not the standard which an institution should be achieving. The institution customer (parents or Government) is the judge of standards and that customers and potential customers will demand high standards.

There is no doubt that the process of installing BS-5750/1S0-9000 at the Institute of Education and Research would result in considerably improved internal services, but it, as yet, unproved that these administrative improvement have materially affected the quality of the student learning experience. (Storey, 1994). This would be because documented procedure is not yet adopted as required under ISO-9000. It involves financial resources and time at the institute. It will put extra work load on the academic staff.

ISO-9000 installs a quality system and monitors it through inspection. The process would result in an improved and cost - effective product.

---

*Published in Pak. J. Statist. (1997), Vol. 13(2).

3. BS-57S0/1S0-91100 IN EDUCATION

The Institute of Education and Research, Punjab University Lahore claims to produce competent teachers who are back bone of schools. On installing BS-5750, it can achieve quality culture, technique improvements, facts based decision-making and a real staff commitment to parents satisfaction and belief that the Institute is parent-and research-orientated.

To achieve the objectives through the currently used tool of TQM and 1S0-9000, it is necessary to understand some of its components/clauses.

3.1 Responsibility and Authority (Clause 4.1)

The responsibility, authority and the interrelation of all personnel who manage, perform and verify work affecting quality arc to be defined. This is a key requirement for the Institute. It defines the responsibility and authority of staff. The most difficult is the area of 'interrelation'. The Institute has the following traditional, hierarchical management structure.

3.1.1 Job Titles and Job Descriptions

Staff generally has a rough idea of what their job titles are but because, in many cases, their jobs had changed since their original appointment they had not usually had an updated job description. Accurate and precise job titles illustrate job description but in 1ER, a lecturer has to perform three fold duties of administration, teaching and research.

3.2 Management Review (Clause 4.1.3)

The Management Review Team (MRT) which may consist of the Dean, the Director of the Institute, Head of Departments, the finance and personnel Services Managers and the Head of the Quality Assurance Unit, is formed to meet the requirement of clause 4.1.3 of the Standard. The MRT has two routine functions: it receives and considers the reports of quality system, normally reflected in the Quality Manual, a new generic procedures. The whole point of management review is that the individual or group has the power to make necessary changes to the quality system.

3.3 Quality System (Clause 4.2)

The Institute with all its traditional and structural growth has not established a documented quality system. It is essential to set up a quality manual which provides directional intent of its work-force; prepare quality assurance procedural details; and describe job description for search of excellence in teaching and learning.

3.4 Contract Review (Clause 4.3)

The Institute of Education and Research (IER) provides services to general schools and their students. It also provides services to Government of the Punjab for:

i) teaching special courses to their nominees;
ii) conducting research on behalf of Governments.

Contracts arc made between IER and Government Departments on individual projects. The contracts arc documented and maintained by IER Director. Contracts arc reviewed by Director and his concerned faculty who are involved in the specific project. Contracts can be changed if both IER and the Government agree to the changes.

3.5 Document and Data Control (Clause 4.5)

IER shall establish procedures for documents and data control.

3.6 Identification and Traceability of Student Learning Experiences (Clause 4.8)

This is the most difficult clause of the system. First, it is necessary to identify student's level of learning at a particular point of time during his studies and second to trace subject matter of the course material. Suppose a teacher is covering course material which is to be completed at the end of 4 months period. It is not difficult to find the proportion of course material completed at the end of second month but it is difficult to assess how much students have grasped the material, though it is roughly assessed through mid-term examination. Some procedures need to be developed for identification and traceability of the teaching and research activities at a particular point of time during an academic year.

3.7 Process Control (Clause 4.9)

In academic activities, control of teaching, learning and research process is not easy. An institution has to develop:

a) Documented procedure defining the manner of teaching, learning and research, where absence of such procedures would adversely affect its quality.
b) Aids, equipments and working environment.
c) Methods of checking or measuring the compliance of documented procedures
d) Criteria for holding positions in the three categories of teaching, learning and research.
e) Methods of monitoring and control of teaching, learning and research parameters.

3.8 Inspection and Testing (Clause 4.10)

The institute shall establish and maintain documented procedures for examinations, assignments, quizzes, projects etc. in order to verify that teachers and students have achieved the specified level.

Tests and interview can be held at the time of appointment of teachers and admission of students to specify a particular level at the time of entrance to IER.

Similarly, test, assignments, quizzes etc. can be administered during the academic year to judge the teacher's and student's achievements.

At the end of an academic session, an examination can be administered whether students have achieved a specified level. Students can be graded, either passed, passed with concession or failed. Some may be reprocessed or may go through a part of the process to achieve a specified level.

3.9 Control of Examination, Test and Results Instruments (Clause 4.11)

Proper control and security of examination and test papers, result records have to be maintained.

3.10 Corrective and Preventive Actions (Clause 4.14)

The Institute has to define parameters of a non-conformity and develop and maintain corrective and preventive measures. Details of the nonconformity parameters have to be worked out and for each parameter, corrective and/or preventive action defined.

3.11 Maintenance of Quality Records (Clause 4.16)

Records of all activities of students and teachers have to be maintained to verify its conformance to required level of excellence.

3.12 Internal Quality Audit (Clause 4.17)

This audit is also not very easy in the sense that teachers are allergic to their checking by their colleagues. However, internal audit of students performance can be carried out.

The institute has to establish and maintain documented procedures for auditing. It demonstrates the effectiveness of the system that is being installed in the institution.

3.13 Training (Clause 4.18)

From time to time, teachers are to attend refresher courses and participate in national and international seminars and conferences as a part of their training.

Institute shall have to keep the training records.

4. FUTURE DIRECTION

A comprehensive study is essential to document the IER activities and follow the procedures adopted by its faculty and staff. Quality Manual and procedures may be written documenting ways and means of implementing and monitoring proposed quality system.

REFERENCES

1. BSI (1987). *BS-5750: Part*, BSI Standards, BSI, London, UK.
2. BSI (1992). *Application of ISO 9000/EN 29000/BS-5750, BSI Quality Assurance*. BSI, London, U.K.
3. BSI (1994). *Quality Systems-Model for quality assurance in production, installation and servicing*, BSI, London, UK.
4. BSI (1983). *Quality Assurance, BSI Handbook*, 22, BSI, London. UK.
5. Change, R.Y. (1993). *When TQM goes nowhere*, Training & development.
6. Deming, W.E. (1964). *Statistical Adjustment of data*, London: Constable.
7. Doherty, G.D. (1993). *Towards Total Quality Management in higher education: A case study of the University of Wolverhampton*, Higher Education 25.
8. I.E.R. (1996). *Prospectus: Institute of Education & Research*, Punjab University. IER. P.U., Lahore Pakistan pp 1.
9. Ishikawa, K. (1985). *What is Total Quality Control? The Japanese Way*, Englewood. NJ: Prentice-Hall.
10. Juran, J.M. (1988). *Juran on planning for Quality New York*: The Free Press.
11. Storey, Susan (1994). *Doing Total Quality Management the hard way*: Installing BS-5750/1S0-9001 at the University of Wolverhampton in 'Developing Quality System in Education' Edited by Geoffrey D. Doherty. Routledge, 11-New Fetter Lane. London EC4P 4EE, U.K.

A GROUP SAMPLING PLAN BASED ON TRUNCATED LIFE TEST FOR GAMMA DISTRIBUTED ITEMS*

Muhammad Aslam[1], **Chi-Hyuck Jun**[2] and **Munir Ahmad**[1]

[1] National College of Business Administration & Economics
Lahore, Pakistan.
Email: aslam_ravian@hotmail.com
drmunir@brain.net.pk

[2] Department of Industrial and Management Engineering, POSTECH
Pohang 790-784, Republic of Korea. Email: chjun@postech.ac.kr

ABSTRACT

In practice, a multiple number of items as a group can be tested simultaneously in a tester. This study deals with a group acceptance sampling plan under the truncated life test assuming that the lifetime of an item is distributed as Gamma with known shape parameter. The plan parameters such as the number of groups and the acceptance number will be determined by satisfying the consumer's and producer's risks at the specified ratio of true average life to the specified life, termination time and the number of testers. The tables are constructed and results are explained with examples.

KEY WORDS

Group acceptance sampling; Consumer's risk; Operating characteristics; Producer's risk; Truncated life test

1. INTRODUCTION

In practice, testers accommodating multiple items are available, where more than one item can be tested simultaneously. Items in a tester can be regarded as a group and the number of items in a group is called as the group size. The acceptance sampling plan based on these groups of items will be called a group acceptance sampling plan (GASP). The GASP is used to test the items in a group simultaneously and therefore it can be used to save the time of experiment and cost as compared with the test in which a single item is put on test in a tester. If the GASP implemented on the truncated life test we may call it a GASP based on truncated life test when a lifetime of a product is assumed to follow a certain statistical distribution. In this type of tests, determining the sample size is equivalent to determining the number of groups. This type of testers is frequently used in sudden death testing. The sudden death tests are discussed by Pascual and Meeker (1998) and Vlcek et al. (2003). Recently, Jun et al. (2006) proposed the sudden death test under the assumption that the lifetime of items follows the Weibull distribution with known shape parameter. They developed the single and double group acceptance sampling plans in sudden death testing.

The acceptance sampling plans based on truncated life test having single-item group using the different statistical distributions have been developed by many authors. For

*Published in Pak. J. Statist. (2009). Vol. 25(3).

example, one may refer to Epstein (1954), Goode and Kao (1961), Kantam and Rosaiah (1998), Kantam et al. (2001), Baklizi (2003), Rosaiah and Kantam (2005), Rosaiah et al. (2006), Tsai and Wu (2006), Rosaiah et al. (2007), Aslam (2007) and Balakrishnan et al. (2007). In these plans, the sample size is usually determined which satisfies the consumer's risk only and used the single point on the OC curve, so it may not always satisfy the producer's risk. More recently, Aslam and Jun (2009) proposed the group acceptance sampling plan based on truncated life test when the lifetime of an item follows the inverse Rayleigh or log-logistics distribution. Again in their paper, they used the single point approach on operating characteristics OC curve to find the design parameter (number of groups) which satisfies only the consumer's risk.

Two risks will be attached with the current GASP. If a good lot is rejected on the basis of the information from the sample, it will be a wrong decision and the probability of committing this wrong decision is called the producer's risk. If the bad lot is accepted, then this probability is termed as consumer's risk. The purpose of a well design acceptance sampling plan is to minimize the both risks, which is called the two point approach. The two point approach on the OC curve for designing the variable acceptance sampling plans has been adopted by Fertig and Mann (1980), Balasooriya et al. (2000), and Balamurali and Jun (2006).

The purpose of this paper is to propose a GASP based on truncated life tests when the lifetime of a product follows the gamma distribution with known shape parameter. Further, we obtain the number of groups and the acceptance number simultaneously for given values of both risks using the two point approach. The rest of the paper is organized as follows: The proposed GASP along with the operating characteristics is described in Section 2. The results and conclusion are given in Section 3.

2. THE GROUP ACCEPTANCE SAMPLING PLAN (GASP)

Let μ represent the true average life of a product and μ_0 denote the specified life. A product is considered as good and accepted for consumer's use if the sample information supports the hypothesis $H_0 : \mu \geq \mu_0$; otherwise, the lot of the products is rejected. In acceptance sampling schemes, this hypothesis is tested based on the number of failures from a sample in a pre-fixed time. If the number of failures exceeds the action limit c we reject the lot. We will accept the lot if there is enough evidence that $\mu \geq \mu_0$ at certain levels of both risks. Let us propose the following GASP based on the truncated life test:

1) Select the number of groups g and allocate predefined r items (r will be called as group size) to each group so that the sample size for a lot will be $n = gr$.
2) Select the acceptance number (or action limit) c for a group and the experiment time t_0.
3) Perform the experiment for the g groups simultaneously and record the number of failures for each group.
4) Accept the lot if at most c failures occur in each of all groups. Truncate the experiment if more than c failures occur in any group and reject the lot.

It is important to note that the single sampling plan based on truncated life tests is a special case of our proposed GASP. The GASP reduces to the single sampling plan if r =1, when $n = g$. The purpose is to find the actions limit c and the number of groups g which satisfy both the consumer's and producer's risks at the same time, whereas the group size r and the termination time t_0 are pre-assigned.

Suppose that the lifetime of an item or a product follows a gamma distribution with known shape parameter. The cumulative distribution function (cdf) of gamma distribution for integer value of γ (shape parameter) and scale parameter θ is given by

$$F_T(t,\theta) = 1 - \sum_{j=0}^{\gamma-1} e^{-\frac{t}{\theta}} \left(t/\theta\right)^j \Big/ j!,\ t \geq 0\,. \tag{2.1}$$

If the shape parameter is not known, an estimated value from failure data can be used. In practice, the shape parameter for a particular type of items is usually known from the past engineering knowledge. Note that $\gamma = 1$ corresponds to an exponential distribution. Note also that the cdf depends on the scale parameter θ only through t/θ. The mean life of gamma distributed products is given by

$$\mu = \gamma\theta\,. \tag{2.2}$$

If the lot size is large enough and decision about the lot lies in two categories (accept or reject), we can use the binomial distribution to develop GASP. For more justification one may refer to Stephens (2001). According to GASP the lot of products is accepted only if the number of failures are less than or equal to c in each of g groups. So, the lot acceptance probability will be

$$L(p) = \left[\sum_{i=0}^{c} \binom{r}{i} p^i (1-p)^{r-i}\right]^g, \tag{2.3}$$

where p is the probability that an item in any group fails before the termination time t_0. It would be convenient to determine the termination time t_0 as a multiple of the specified life μ_0. That is, we will consider $t_0 = a\mu_0$ for a constant a and p is given by

$$p = F_T(t_0) = 1 - \sum_{j=0}^{\gamma-1} e^{-\frac{a\gamma}{\mu/\mu_0}} \left(\frac{a\gamma}{\mu/\mu_0}\right)^j / j! \tag{2.4}$$

The ratio $\left(\mu/\mu_0\right)$ of the true average life μ to the specified life μ_0 is called the quality level of a product. As mentioned earlier, two risks are associated with GASP. If a product is good and is accepted on the basis of sample information, it is a right decision to accept the product. If the information obtained from the sample supports the hypothesis $H_1 : \mu < \mu_0$ (that is, the product is bad), we will reject the lot, which is also a right decision. If a good lot is rejected, this will be a loss to a producer, while on the other hand if bad lot/product is accepted it will be a loss to the consumers. The probability of rejecting a good lot is called the producer's risk α and the probability of accepting a bad lot is termed as the consumer's risk β . The consumer demands that the lot acceptance probability should be smaller than the given value of β at a lower quality level especially at the level when the true average life μ is equal to the specified life μ_0 . On the other hand, a producer desires that the lot rejection probability should be smaller than the specified α at a higher quality level (the ratio $\mu/\mu_0 > 1$). As in GASP, the sample taken from the lot is distributed into different groups therefore it provides the tight inspection of the items taken from the lot than the ordinary acceptance sampling plan approach. The GASP is helpful to reach a good decision about the product (to minimize the both risks). Therefore, the purpose of this study is to develop the GASP based on truncated life test which can be used to control the producer's and consumer's risks simultaneously. The proposed two-point approach is to determine the number of groups and the acceptance number that satisfy the following two inequalities simultaneously.

$$L(p \mid \mu/\mu_0 = r_1) \le \beta \tag{2.5}$$

$$L(p \mid \mu/\mu_0 = r_2) \ge 1-\alpha , \tag{2.6}$$

where r_1 and r_2 are the mean ratios that will be specified at the consumer's and producer's risks, respectively. Larger mean ratio represents a higher quality requirement. Usually, r_1 =1 is adopted. Let p_1 and p_2 be the failure probabilities corresponding to consumer's and producer's risks, respectively. Then, the design parameters can be determined through the following inequalities.

$$L(p_1) = \left[\sum_{i=0}^{c} \binom{r}{i} p_1^i (1-p_1)^{r-i} \right]^g \le \beta \tag{2.7}$$

$$L(p_2) = \left[\sum_{i=0}^{c} \binom{r}{i} p_2^i (1-p_2)^{r-i} \right]^g \ge 1-\alpha , \tag{2.8}$$

The design parameters in terms of integers can be found by using a search, which can be implemented easily on an Excel sheet.

3. DESCRIPTION OF TABLES AND EXAMPLES

The number of groups and the acceptance number are found using (2.7) and (2.8) and presented in Tables 1-2. The design parameters of GASP are found at the various values of the consumer's risk (β=0.25, 0.10, 0.05, 0.01) when $r_1 = 1$ and 5% of the producer's risk when the true mean is r_2 (=2, 4, 6, 8, 10) times μ_0 . Two levels of group size (r=5, 10) and two levels of the test termination time multiplier (a=0.5, 1.0) are considered. We consider two values of the shape parameter of the gamma distribution: $\gamma = 2$ in Table 1 and $\gamma = 3$ in Table 2. Other choices can be easily employed in a similar fashion. It should be noted that if one needs the sample size, it can be obtained by $n = r \times g$.

Table 1

Minimum number of groups and acceptance number for GASP when $\gamma = 2$

β	$\mu/\mu_0 = r_2$	r=5						r=10					
		a=0.5			a=1.0			a=0.5			a=1.0		
		g	c	$L(p_2)$	g	c	$L(p_2)$	g	c	$L(p_2)$	g	c	$L(p_2)$
0.25	2	72	3	0.9781	19	4	0.9758	5	3	0.9563	4	6	0.9804
	4	3	1	0.9802	2	2	0.9873	1	1	0.9726	1	3	0.9911
	6	3	1	0.9955	1	1	0.9818	1	1	0.9935	1	2	0.9916
	8	1	0	0.9646	1	1	0.9933	1	1	0.9978	1	1	0.9726
	10	1	0	0.9768	1	1	0.9970	1	0	0.9542	1	1	0.9874
0.10	2	119	3	0.9641	30	4	0.9621	23	4	0.9768	6	6	0.9708
	4	5	1	0.9672	3	2	0.9810	4	2	0.9923	1	3	0.9911
	6	5	1	0.9925	1	1	0.9818	2	1	0.9870	1	2	0.9916
	8	5	1	0.9975	1	1	0.9933	2	1	0.9955	1	1	0.9726
	10	2	0	0.9542	1	1	0.9970	1	0	0.9542	1	1	0.9874
0.05	2	155	3	0.9535	*	*	*	30	4	0.9698	7	6	0.9660
	4	6	1	0.9607	3	2	0.9810	5	2	0.9903	2	3	0.9823
	6	6	1	0.9910	2	1	0.9639	2	1	0.9870	1	2	0.9916
	8	6	1	0.9969	2	1	0.9867	2	1	0.9955	1	1	0.9726
	10	2	0	0.9542	2	1	0.9970	1	0	0.9542	1	1	0.9874
0.01	2	-	-	-	*	*	*	46	4	0.9541	27	7	0.9832
	4	37	2	0.9934	5	2	0.9685	7	2	0.9865	2	3	0.9823
	6	10	1	0.9850	2	1	0.9639	3	1	0.9806	2	2	0.9832
	8	10	1	0.9949	2	1	0.9867	3	1	0.9933	1	1	0.9726
	10	10	1	0.9978	2	1	0.9941	3	1	0.9971	1	1	0.9874

Note: The cells with hyphens (-) indicate that g and c are found to be large.
The cells with hyphens (*) indicates that g and c can not satisfy the conditions.

Table 2
Minimum number of groups and acceptance number for GASP when $\gamma = 3$

β	$\mu/\mu_0 = r_2$	r=5						r=10					
		a=0.5			a=1.0			a=0.5			a=1.0		
		g	c	$L(p_2)$	g	c	$L(p_2)$	g	c	$L(p_2)$	g	c	$L(p_2)$
0.25	2	27	2	0.9833	4	3	0.9776	4	2	0.9745	1	4	0.9727
	4	5	1	0.9978	1	1	0.9849	2	1	0.9962	1	2	0.9936
	6	2	0	0.9786	1	1	0.9980	1	0	0.9786	1	1	0.9914
	8	2	0	0.9905	1	0	0.9672	1	0	0.9905	1	1	0.9981
	10	2	0	0.9950	1	0	0.9821	1	0	0.9950	1	0	0.9646
0.10	2	44	2	0.9729	7	3	0.9611	7	2	0.9558	3	5	0.9850
	4	9	1	0.9961	2	1	0.9700	3	1	0.9942	1	2	0.9936
	6	3	0	0.9681	2	1	0.9960	2	0	0.9576	1	1	0.9914
	8	3	0	0.9858	1	0	0.9672	2	0	0.9811	1	1	0.9981
	10	3	0	0.9925	1	0	0.9821	2	0	0.9900	1	0	0.9646
0.05	2	57	2	0.9650	9	3	0.9502	27	3	0.9875	4	5	0.9800
	4	11	1	0.9952	2	1	0.9700	4	1	0.9923	1	2	0.9936
	6	3	0	0.9681	2	1	0.9960	2	0	0.9576	1	1	0.9914
	8	3	0	0.9858	1	0	0.9672	2	0	0.9811	1	1	0.9981
	10	3	0	0.9925	1	0	0.9821	2	0	0.9900	1	0	0.9646
0.01	2	-	-	-	70	4	0.9823	42	3	0.9807	6	5	0.9702
	4	17	1	0.9926	3	1	0.9553	6	1	0.9885	2	2	0.9872
	6	17	1	0.9992	3	1	0.9940	6	1	0.9988	1	1	0.9914
	8	5	0	0.9764	3	1	0.9987	3	0	0.9717	1	1	0.9981
	10	5	0	0.9975	2	0	0.9646	3	0	0.9850	1	0	0.9646

Note: The cells with hyphens (-) indicate that g and c are found to be large.

In these tables, note that, as the ratio r_2 increases, the number of groups and the acceptance numbers decrease at the same time. We need a smaller number of groups if the termination ratio increases at a fixed group size. For an example, from Table 1, if a changes from 0.5 to 1.0 for r=5, the number of groups has been changed from g =72 to g =19 when r_2 =2. It is also noted that when r_2 =2 we found high values of g and c at some conditions, and we cannot find them to satisfy the conditions given in equations (2.7) and (2.8) in some cases. It is observed that the number of groups tends to decrease as γ increases, r increases or a increases. However, the trend is not monotonic since it depends on the acceptance number as well. The probability of acceptance for the lot at the mean ratio corresponding to the producer's risk is also given in Table 1 and Table 2.

Suppose that the lifetime of a product follows the gamma distribution with shape parameter of 2. It is desired to design a GASP to test that the mean life is greater than 1,000 hours and manufacturer wants to run an experiment for 500 hours using testers equipped with 5 products each. Let us assume that the producer's risk is 5% when the true mean is 4,000 hours and the consumer's risk is 25% when the true mean is 1,000 hours. Since γ =2, β =0.25, r=5, a=0.5 and $r_2 = 4$ for this example, the minimum number of groups and acceptance number can be found as g =3 and c =1 from Table 1. This indicates that a total of 15 products are needed and that 5 products are allocated to each of 3 testers. We will accept the lot if no more than 1 failure occurs before 500 hours in each of 3 groups. For this proposed sampling plan the probability of acceptance is 0.9802 when the true mean is 4,000 hours.

4. CONCLUSION

We proposed a group acceptance sampling plan based on a truncated life test under the assumption that the lifetime of a product follows the gamma distribution with known shape parameter. The two-point approach was adopted for determining the design parameters such as the number of groups and the acceptance number. This GASP can be utilized when a multi-item is adopted for a life test and it would be beneficial in terms of test time and cost because a group of items will be tested simultaneously.

Our proposed approach can be easily applied to some other underlying lifetime distributions and may be extended to develop a related sampling plan.

ACKNOWLEDGEMENTS

We thank the editor and the referees for their useful comments.

REFERENCES

1. Aslam, M. (2007). Double acceptance sampling based on truncated life tests in Rayleigh distribution. *European J. Scien. Res.*, 17(4), 605-611.
2. Aslam, M., and Jun, C.-H. (2009). A group acceptance sampling plans for truncated life tests based on the inverse Rayleigh and log-logistic distributions. *Pak. J. Statist.*, 25(2), 107-119.

3. Baklizi, A. (2003). Acceptance sampling based on truncated life tests in the Pareto distribution of the second kind. *Advances and Appl. Statist.*, 3(1), 33-48.
4. Balakrishnan, N., Leiva, V. and Lopez, J. (2007). Acceptance sampling plans from truncated life tests based on the generalized Birnbaum-Saunders distribution. *Comm. Statist.-Simu. and Compu.*, 36, 643-656.
5. Balamurali, S. and Jun, C.-H. (2006). Repetitive group sampling procedure for variables inspection. *J. App. Statist.*, 33(3), 327-338.
6. Balasooriya, U., Saw, L. and Gadag, V. (2000). Progressively censored reliability sampling plans for the Weibull distribution. *Technometrics*, 42, 160-167.
7. Epstein, B. (1954). Truncated life tests in the exponential case. *Ann. Math. Statist.*, 25, 555-564.
8. Fertig, F.W. and Mann, N.R. (1980). Life-test sampling plans for two-parameter Weibull populations. *Technometrics*, 22(2), 165-177.
9. Goode, H.P. and Kao, J.H.K. (1961). Sampling plans based on the Weibull distribution. *Proc. Seventh National Symposium on Reliability and Quality Control*, 24-40, Philadelphia.
10. Jun, C.-H., Balamurali, S. and Lee, S.-H. (2006). Variables sampling plans for Weibull distributed lifetimes under sudden death testing. *IEEE Transactions on Reliability*, 55(1), 53-58.
11. Kantam, R.R.L., Rosaiah, K. (1998). Half logistic distribution in acceptance sampling based on life tests. *IAPQR Transactions*, 23(2), 117-125.
12. Kantam, R.R.L., Rosaiah, K. and Rao, G.S. (2001). Acceptance sampling based on life tests: Log-logistic models. *J. App. Statist.*, 28(1), 121-128.
13. Pascual, F.G. and Meeker, W.Q. (1998). The modified sudden death test: planning life tests with a limited number of test positions. *J. Testing and Evaluation*, 26(5), 434-443.
14. Rosaiah, K. and Kantam, R.R.L. (2005). Acceptance sampling based on the inverse Rayleigh distribution. *Economic Quality Control*, 20(2), 277-286.
15. Rosaiah, K., Kantam, R.R.L. and Santosh Kumar, Ch. (2006). Reliability of test plans for exponentiated log-logistic distribution. *Econ. Quality Control*, 21(2), 165-175.
16. Rosaiah, K., Kantam, R.R.L. and Santosh Kumar, Ch. (2007). Exponentiated log-logistic distribution-An economic reliability test plan. *Pak. J. Statist.*, 23(2), 147-146.
17. Stephens, K.S. (2001). *The Handbook of Applied Acceptance Sampling: Plans, Procedures and Principles*. Milwaukee, WI: ASQ Quality Press.
18. Tsai, T.-R. and Wu, S.-J. (2006). Acceptance sampling based on truncated life tests for generalized Rayleigh distribution. *J. App. Statist.*, 33(6), 595-600.
19. Vlcek, B.L., Hendricks, R.C. and Zaretsky, E.V. (2003). *Monte Carlo Simulation of Sudden Death Bearing Testing*. NASA, Hanover, MD, USA.

A NEW VARIABLES SAMPLING PLAN FOR LIFE TESTING IN A CONTINUOUS PROCESS UNDER WEIBULL DISTRIBUTION*

Muhammad Aslam[1], **Saminathan Balamurali**[2], **Chi-Hyuck Jun**[3] and **Munir Ahmad**[4]

[1] Department of Statistics, Forman Christian College University, Lahore, Pakistan. Email: aslam_ravian@hotmail.com

[2] Department of Mathematics, Kalasalingam University, Krishnankoil 626190, TN, India. Email: sbmurali@rediffmail.com

[3] Department of Industrial and Management Engineering, POSTECH, Pohang 790-784, Republic of Korea. Email: chjun@postech.ac.kr

[4] National College of Business Administration and Economics, Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

A new conditional variables sampling plan called multiple dependent state (or deferred state) sampling plan, is proposed for a failure-censored life testing when the lifetime follows a Weibull distribution with known shape parameter. In the proposed sampling plan, acceptance or rejection of a lot is based not only on the sample from that lot, but also on sample results from past lots or from future lots. The design parameters of the proposed sampling plan are determined by the two-point approach considering the consumer's and the producer's risks at the specified acceptable reliability level and the lot tolerance reliability level, respectively. It was found that the proposed plan reduces the sample size required when compared with a variables single sampling plan.

INDEXED TERMS

Acceptable reliability level; Conditional sampling plan; Consumer's risk; Lot tolerance reliability level; OC curve; Producer's risk; Sampling by variables

1. INTRODUCTION

A manufacturer of products performs a life testing whether the quality level of their products meets the customer's requirements such as the minimum lifetime or reliability. In most life testing a common restriction is the duration of the total time spent on testing. In order to reduce the test time of the experiment, many types of censoring schemes such as type-I (or time-censored), type-II (or failure-censored), mixed of type-I and type-II, and progressive censoring are usually adopted.

Although various sampling plans including single, double and sequential plans are available for normally distributed quality characteristics (see Schilling, 1982), most of plans for a life testing are based on the single sampling plan. A single sampling plan based on a failure censored (type-II) or a time censored (type-I) scheme has been usually adopted for a life testing. Fertig and Mann (1980) and Schneider (1989) developed failure-censored sampling plans under a Weibull distribution having unknown shape

*Published in Pak. J. Statist. (2010), Vol. 26(3).

parameter. Their approach is to utilize the extreme value distribution and the maximum likelihood estimation. As a result, designing a sampling plan is quite complicated for being used in practice. Jun et al. (2006) proposed the single and double sampling plans for a Weibull distribution with known shape parameter under a sudden death testing.

In this study, we propose a new conditional variables sampling plan for a failure censoring scheme under a Weibull distribution having known shape parameter. Weibull distributions are popularly adopted as a life distribution since the real failure data are known to fit these distributions quite well. The assumption of known shape parameter in a Weibull distribution sometimes makes theoretical statisticians uncomfortable but it enables us to design various more efficient sampling plans. This assumption is not unrealistic because an estimate of the shape parameter can be readily available in practice from the past failure data and engineering experience.

The concept of multiple dependent (or deferred) state (MDS) sampling was introduced by Wortham and Baker (1976). The MDS plan is applicable to a continuous process where lots are submitted for inspection serially in the order of production. In this procedure, acceptance or rejection of a lot is based not only on the sample from that lot, but also on sample results from past lots (in the case of dependent state sampling) or from future lots (in the case of deferred state sampling). So, the MDS plan has an advantage over the ordinary sampling plan in terms of the minimum sample size. The operating procedure and characteristics of the attributes MDS sampling plan can be found in Wortham and Baker (1976), Vaerst (1982), Soundararajan and Vijayaraghavan (1990), and Balamurali and Kalyanasundaram (1999). More recently, Balamurali and Jun (2007) developed a variables MDS plan for normal distributions with known or unknown standard deviation. The basic advantage of a variables sampling plan is that the same operating characteristic curve can be obtained with a smaller sample size as compared to an attributes sampling plan. Therefore variables sampling plans are more economical than attributes sampling plans in terms of the cost and time. However, there are no studies on variables MDS plans for a reliability testing.

In this paper, a variables MDS sampling plan will be proposed for a failure-censored life testing under a Weibull distribution. The sampling plan will be described in Section 2 and the method of determining the design parameters will be explained in Section 3. Some comparisons are made in Section 4 and Section 5 concludes with remarks.

2. VARIABLES MDS SAMPLING PLAN

Suppose that the quality characteristic of interest is the time to a failure and that it follows a Weibull distribution with known shape parameter m and scale parameter λ such that the cumulative distribution function is given by

$$F(x) = 1 - \exp\left(-(\lambda x)^m\right), \quad x \geq 0 \tag{2.1}$$

Also, it is assumed that there is a lower specification L so that an item having the life smaller than L is regarded as non-conforming. The fraction non-conforming (or unreliability at time L) is obtained by

$$p = 1 - \exp\left(-\left(\lambda L\right)^{m}\right) \tag{2.2}$$

As mentioned in Balamurali and Jun (2007), the following assumptions are valid for the proposed MDS plan:

i) Lots are submitted for inspection serially in the order of production from a process having a constant proportion non-conforming.
ii) The consumer has confidence in the supplier and there should be no reason to believe that a particular lot is poorer than the preceding lots.

Let us consider the following procedure for multiple dependent state (MDS) sampling plan:

1) Draw a random sample of size *n* from a lot.
2) Put *n* items on test and perform testing until r $(\leq n)$ failures are observed by recording $X_{(i)}$, the *i*-th failure time (*i*=1,…,*r*).
3) Calculate the quantity

$$v = \sum_{i=1}^{r} X_{(i)}^{m} + (n-r) X_{(r)}^{m}. \tag{2.3}$$

4) Accept the lot if $v \geq k_a L^m$ and reject the lot if $v < k_r L^m$. If $k_r L^m \leq v < k_a L^m$, then accept the current lot provided that the preceding g lots were accepted on the condition that $v \geq k_a L^m$ but reject the lot, otherwise.

In the above plan, k_a and k_r are parameters related to the lot acceptance and rejection, respectively, whereas L^m is constant. We use the acceptance constant in the form of $k_a L^m$ for the mathematical convenience when deriving the lot acceptance probability. Thus, the proposed MDS plan is characterized by three parameters, namely r, k_a and k_r when *g* is specified. The value of *g* can be chosen by considering the data availability. If $k_a = k_r$, then the proposed plan is reduced to an ordinary variables single sampling plan. The sample size n can be chosen from the parameter r by considering the degree of censoring (Schneider, 1989) and the test time. In general, the test time decreases as the sample size increases. In a *multiple deferred state sampling*, the forthcoming *g* lots will be considered for acceptance of the current lot, so accept or reject decision is effectively postponed. But the mechanism is same as the proposed MDS sampling plan.

3. DETERMINATION OF DESIGN PARAMETERS

The design parameters of the proposed sampling plan can be determined by the two-point approach as in Fertig and Mann (1980). In this approach, its operating characteristics (OC) curve passes through two points $\left(p_1, 1-\alpha\right)$ and $\left(p_2, \beta\right)$, where p_1 is the acceptable reliability level (ARL), p_2 is the lot tolerance reliability level (LTRL),

α is the producer's risk and β is the consumer's risk. A producer wants that the probability of acceptance of a lot should be at least $(1-\alpha)$ when the fraction non-conforming is at ARL and a consumer demands that this probability should not be greater than β when the fraction non-conforming is at LTRL.

Now, the OC function of the variables MDS sampling plan for a given lot quality p is obtained by

$$L(p) = P\{v \geq k_a L^m\} + P\{k_r L^m \leq v < k_a L^m\}\left[P\{v \geq k_a L^m\}\right]^g \tag{3.1}$$

Note that v follows a Gamma distribution with parameters (r, λ^m) and that the quantity $2\lambda^m v$ follows a chi-square distribution with degree of freedom $2r$ (Jun et al. 2006). So, (3.1) can be rewritten by

$$L(p) = \{1 - G_{2r}(2k_a w)\}\left[1 + \{G_{2r}(2k_a w) - G_{2r}(2k_r w)\}\{1 - G_{2r}(2k_a w)\}^{g-1}\right] \tag{3.2}$$

where G_φ is the distribution function of a Chi-square random variable with degree of freedom ϕ and w is given by

$$w = (\lambda L)^m = -\ln(1-p) \tag{3.3}$$

We will determine the design parameters (r, k_a, k_r) by the two-point approach described earlier when g is specified. Several values of g were considered when constructing the tables. So, the following two inequalities should be considered:

$$L(p_1) \geq 1 - \alpha \tag{3.4}$$

$$L(p_2) \leq \beta \tag{3.5}$$

or

$$\{1 - G_{2r}(2k_a w_1)\}\left[1 + \{G_{2r}(2k_a w_1) - G_{2r}(2k_r w_1)\}\{1 - G_{2r}(2k_a w_1)\}^{g-1}\right] \geq 1 - \alpha \tag{3.6}$$

$$\{1 - G_{2r}(2k_a w_2)\}\left[1 + \{G_{2r}(2k_a w_2) - G_{2r}(2k_r w_2)\}\{1 - G_{2r}(2k_a w_2)\}^{g-1}\right] \leq \beta \tag{3.7}$$

Values of parameters (r, k_a, k_r) can be determined by a simple search in an Excel sheet for given values of (g, p_1, p_2). These design parameters for variables MDS plans are presented in Table 1 for g=1, 2, 3 according to different combinations of ARL (p_1) and LTRL (p_2) when α=0.05 and β=0.10. The values of p_2 were chosen as 2 p_1, 5 p_1, 10 p_1, 15 p_1, 20 p_1 and 30 p_1 until reaching 0.5 at maximum.

Table 1
Variables MDS sampling plans indexed by ARL and LTRL

p_1 (ARL)	p_2 (LTRL)	g=1			g=2			g=3		
		r	k_a	k_r	r	k_a	k_r	r	k_a	k_r
0.001	0.002	11	8249	4570	11	7778	3699	12	8301	4700
	0.005	3	1107	777	3	1067	750	3	1063	712
	0.010	2	392	352	2	388	351	2	388	348
	0.015	1	184	26	1	157	14	2	258	257
	0.020	1	130	40	1	116	35	1	115	26
	0.030	1	81	48	1	77	47	1	76	45
0.005	0.010	11	1635	959	11	1549	800	12	1653	988
	0.025	3	219	155	3	212	150	3	211	143
	0.050	2	77	70	2	76	70	2	76	69
	0.100	1	25	8	1	23	7	1	22	5
	0.150	1	15	9	1	15	9	1	15	9
0.010	0.020	11	830	360	11	770	428	12	823	505
	0.050	3	108	77	3	105	75	3	104	72
	0.100	2	38	35	2	37	35	2	37	35
	0.200	1	12	4	1	11	3	1	11	1
	0.300	1	7	4	1	7	4	1	7	4
0.050	0.100	11	153	107	11	149	45	12	158	114
	0.250	2	17	1	2	14	1	3	19	15
	0.500	2	7	6	2	6	5	2	6	5
0.100	0.200	10	68	43	10	65	38	11	70	45
	0.500	2	7	1	2	6	2	2	6	1

It is seen that for a fixed value of ARL, the values of r and k_a decrease as LTRL increases, but there is no specific trends in values of k_r. It is also observed that design parameters remain similar according to a different value of g when LTRL>>ARL. It should be noted that the above table can be utilized independently of the shape parameter of a Weibull distribution.

Example 1:

Suppose that a certain type of bearing is regarded as conforming if its lifetime is greater than 10 (thousand cycles). For the decision of a lot acceptance the manufacturer wants to use the MDS sampling plan with g =1. The lifetime of a bearing is known to follow a Weibull distribution with m=2. The ARL is selected as 0.01 at which the producer's risk is 5 percent and the LTRL is selected as 0.05 at which the consumer's risk is 10 percent. The design parameters for this example are obtained from Table 1 as

$r = 3$, k_a =108 and k_r =77. The MDS sampling plan operates as follows. When the degree of censoring is chosen as 0.5, six bearings out of a lot will be put on test initially and three failures will be observed. Suppose now that three failures were observed at 12.7, 19.5, and 25.2. The test is terminated at the last failure. The quantity v in (2.3) is calculated by

$$v = (12.7)^2 + (19.5)^2 + (25.2)^2 + 3(25.2)^2 = 3081.7$$

which is smaller than $k_r L^m = 77(10)^2 = 7700$. So, the lot will be rejected.

Example 2:

Consider Example 1 again. Suppose now that the same three failure times were observed if 15 bearings had been put on test (In reality, the failure times should be different according to the sample size initially put on test). Then, the quantity v in (2.3) is calculated by

$$v = (12.7)^2 + (19.5)^2 + (25.2)^2 + 12(25.2)^2 = 8797.06$$

which is larger than $k_r L^m = 77(10)^2 = 7700$ but smaller than $k_a L^m = 108(10)^2 = 10800$. So, the current lot will be accepted if the preceding one lot has been accepted but rejected otherwise.

4. COMPARISON OF PLANS

Under a failure censoring scheme, a plan requiring a smaller number of failures to be observed would be better if other conditions remain the same. So, it may be meaningful to compare with plans in terms of the number of failures to be observed. Table 2 summarizes the number of failures to be observed under the MDS plan with different g and under the variables single sampling plan.

Table 2
Comparison of number of failures in each plan

p_1	p_2	*MDS with g=1*	*MDS with g=2*	*MDS with g=3*	**Variables single sampling plan**
0.001	0.002	11	11	12	19 (12437)
	0.005	3	3	3	4 (1333)
	0.010	2	2	2	3 (530)
	0.015	1	1	2	2 (258)
	0.020	1	1	1	2 (193)
	0.030	1	1	1	2 (128)
0.005	0.010	11	11	12	19 (2464)
	0.025	3	3	3	4 (264)
	0.050	2	2	2	3 (104)
	0.100	1	1	1	2 (37)
	0.150	1	1	1	2 (24)
0.010	0.020	11	11	12	19 (1226)
	0.050	3	3	3	4 (131)
	0.100	2	2	2	2 (51)
	0.200	1	1	1	2 (18)
	0.300	1	1	1	2 (11)
0.050	0.100	11	11	12	18 (225)
	0.250	2	2	3	4 (24)
	0.500	2	2	2	2 (6)
0.100	0.200	10	10	11	17 (101)
	0.500	2	2	2	4 (10)

(note) The number in parenthesis is the acceptance parameter of the variables single sampling plan.

It is seen that the number of failures for the MDS plan does not much vary according to the value of g and that it is smaller than that for the variables single sampling plan. Obviously, the number of failures decreases as the value of p_2 increases. However, it seems that r does not depend on the value of p_1 as long as the ratio of p_2 / p_1 remains the same. It is observed that r =1 when the ratio $p_2 / p_1 \geq 20$. It is interesting to see that r for the MDS plan with g =1 is same for that with g =2 and that k_a for the MDS plan with g =1 is smaller than k_a for the MDS plan with g =2. This may be interpreted as follows: the MDS plan with g =2 requires shorter failure times than the MDS plan with g =1 because the former considers one more preceding lot than the latter.

It was found that the operating characteristics of the MDS plans with three values of g considered here are quite similar to each other although the results were not included here. In this regard, the MDS plan with $g=1$ can be recommended for use in practice. It was also found that variables single sampling plans having the same r do not meet the producer's risk at the ARL or consumer's risk at the LTRL.

5. CONCLUDING REMARKS

A MDS sampling plan by variables was proposed for a failure-censored life testing when the lifetime of an item follows a Weibull distribution with known shape parameter. The design parameters were determined by the two-point approach considering the consumer's and producer's risk simultaneously, which are not dependent on the shape parameter as long as the ARL and the LTRL are specified. It was found that the MDS plans do not much differ by the number of preceding lots in terms of the number of failures to be observed and the operating characteristics. When compared with variables single sampling plans, the proposed MDS plan requires smaller number of failures to be observed.

ACKNOWLEDGMENT

The authors would like to thank the reviewers and the Editor for their valuable comments which led to improve the manuscript.

REFERENCES

1. Balamurali, S. and Jun, C.-H. (2007). Multiple dependent state sampling plans for lot acceptance based on measurement data. *Euro. J. Oper. Res.*, 180, 1221-1230.
2. Balamurali, S. and Kalyanasundaram, M. (1999). Determination of conditional double sampling scheme. *J. App. Statist.*, 26(8), 893-902.
3. Fertig, K.W. and Mann, N.R. (1980). Life-test sampling plans for two-parameter Weibull populations. *Technometrics*, 22, 165-177.
4. Jun, C.-H., Balamurali, S. and Lee, S.-H. (2006). Variables sampling plans for Weibull distributed lifetimes under sudden death testing. *IEEE Transactions on Reliability*, 55(1), 53-58.
5. Schilling, E.G. (1982). *Acceptance Sampling in Quality Control.* Marcel Dekker, New York.
6. Schneider, H. (1989) Failure-censored variables-sampling plans for lognormal and Weibull distributions. *Technometrics*, 31, 199-206.
7. Soundararajan, V. and Vijayaraghavan, R. (1990). Construction and selection of multiple dependent (deferred) state sampling plan. *J. App. Statist.*, 17, 397-409.
8. Vaerst, R. (1982). A procedure to construct multiple deferred state sampling plan. *Meth. of Oper. Res.*, 37, 477-485.
9. Wortham, A.W. and Baker, R.C. (1976). Multiple deferred state sampling inspection. *Int. J. Prod. Res.*, 14, 719-731.

OPTIMAL DESIGNING OF A SKIP LOT SAMPLING PLAN BY TWO POINT METHOD*

Muhammad Aslam[1], **Saminathan Balamurali**[2]
Chi-Hyuck Jun[3] and **Munir Ahmad**[4]

[1] Department of Statistics, Forman Christian College University
Lahore, Pakistan. Email: aslam_ravian@hotmail.com

[2] Department of Mathematics, Kalasalingam University
Krishnankoil 626190, TN, India. Email: sbmurali@rediffmail.com

[3] Department of Industrial and Management Engineering, POSTECH
Pohang 790-784, Republic of Korea. Email: chjun@postech.ac.kr

[4] National College of Business Administration and Economics
Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

In this paper, we propose a designing methodology to determine the parameters of a skip-lot sampling plan using two points on the operating characteristic curve. The plan parameters are determined so as to minimize the average sample number subject to satisfying simultaneously the producer's and consumer's risks at the acceptable and limiting quality levels respectively. Tables are constructed and the results are compared with the single sampling plans.

KEY WORDS

Binomial sampling; Consumer's risk; Producer's risk; Single sampling; Skip-lot sampling.

1. INTRODUCTION

Dodge (1955) introduced the concept of skip-lot sampling by applying the principles of continuous sampling plan CSP-1 (Dodge, 1943) to a series of lots or batches of material. This plan is designated as SkSP-1 plan and is specifically applicable for bulk materials or products produced in successive lots. Skip-lot sampling means that a fraction of the submitted lots is inspected for acceptance or rejection of the lot. The skip-lot sampling is very effective in reducing the cost and time of the inspection on products that has an excellent quality history. For more details about the skip-lot sampling one can refer Schilling (1982), ISO 2859-3 (2005), Balamurali et al. (2008).

There are situations in practice that each lot to be inspected is sampled according to a lot inspection plan which is called the reference plan. Based on this, Perry (1973) formalized the application of skip-lot sampling to the particular kind of situation. This plan is designated as SkSP-2 plan. But it is to be pointed out that in SkSP-1 plan, no reference plan concept is used. Perry (1973) has studied the properties of SkSP-2 plan with single sampling plan as the reference plan. He has also provided operating characteristics of SkSP-2 plan with some selected parameters.

*Published in Pak. J. Statist. (2010), Vol. 26(4).

Skip-lot sampling is used for sampling chemical and physical processes in order to bring about substantial savings on inspection of products, which normally conform to specification. This particular sampling plan is useful when the lots are small or where inspection is slow and costly. SkSP-2 plan is considered as more reliable than the single sampling plan in that the required sample size to be inspected can be reduced. Another advantage of this plan is that we can obtain higher probability of acceptance at good quality levels than the single sampling plans. An SkSP-2 plan is operated as follows.

Step 1: Start with normal inspection (inspecting every lot), using the reference plan.

Step 2: When i consecutive lots are accepted on normal inspection, switch to skipping inspection. During the skipping inspection, only a fraction f of the lots are inspected.

Step 3: When a lot is rejected on skipping inspection, immediately revert to the normal inspection.

As the fate of a lot under SkSP-2 plan depends on the basis of few items taken from the lot, there are two risks associated with it. If a good lot is rejected, then it is called the producer's risk. On the other hand, the probability of accepting a bad lot is called the consumer's risk. An SkSP-2 plan is considered to be a good plan if it minimizes the average sample number (ASN) at the same time minimizing or at least maintaining the risks at the corresponding quality levels.

In the literature, there have been no attempts to determining the design parameters of a skip-lot sampling plan including the reference plan. Thus the purpose of this paper is to find the parameters of an SkSP-2 plan using two-point approach by minimizing ASN and satisfying the producer's and consumer's risks. A simulation experiment is performed to find the plan parameters such that both the risks are satisfied. The design parameters of the single sampling plan are also determined by minimizing the ASN subject to satisfying producer's and consumer's risks. Tables are also constructed for the specified quality levels and results are compared with the single sampling plans in terms of ASN. The rest of the paper is organized as: the design methodology to determine the plan is given in Section 2. Some examples are given in Section 3. Concluding remarks are given in Section 4.

2. DESIGNING METHODOLOGY

An SkSP-2 plan with the single sampling plan as the reference plan is characterized by four parameters namely i, f, n and c. This plan as the reference plan has the following parameters:

n – the sample size in a single sampling plan.
c – acceptance number in a single sampling plan.
i – clearance number.
f – fraction of lots inspected in the skipping inspection mode and in general $0 < f < 1$.

It is to be noted that i must be an integer and for the practical use of this plan, it would be better to have the value of i between 1 and 10. We have to decide the acceptance or rejection of the lot on the basis of results of inspection of a random sample of size n drawn from the infinite lot. These two assumptions support the application of the binomial distribution to the operating characteristic values for the SkSP-2 plan. The lot acceptance probability from a single sampling plan or the reference plan under binomial model is given by

$$P = \sum_{j=0}^{c} \binom{n}{j} p^j (1-p)^{n-j} \qquad (2.1)$$

where p is the true quality level.

According to Perry (1973), the operating characteristics (OC) function of SkSP-2 plan is given by,

$$P_a(p) = \frac{fP + (1-f)P^i}{f + (1-f)P^i} \qquad (2.2)$$

where P is the probability of acceptance under normal inspection. It is to be noted that when $f = 1$, the above OC function of the SkSP-2 plan reduced to the OC function of single sampling plan as given in (2.1). As we have stated earlier, there are two risks always associated with a sampling plan. The producer wants that the lot acceptance probability should be at least $(1-\alpha)$ for the producer's risk α if the process fraction nonconforming is at the acceptable quality level (AQL) and the consumer wants the probability of acceptance less than the consumer's risk of β if the process fraction nonconforming is at the limiting quality level (LQL).

The two-point approach that uses two points on the OC curve is frequently adopted for designing a sampling plan to maintain both the producer's and the consumer's risks (See for example Balamurali et al., 2005). We have to determine the parameters of the SkSP-2 plan by satisfying these risks simultaneously while minimizing the ASN at the same time. Under AQL of p_1 and LQL of p_2, (2.1) can be written as

$$P_1 = \sum_{j=0}^{c} \binom{n}{i} p_1{}^j (1-p_1)^{n-j} \qquad (2.3)$$

$$P_2 = \sum_{j=0}^{c} \binom{n}{i} p_2{}^j (1-p_2)^{n-j} \qquad (2.4)$$

In the case of the single sampling plan, the parameters n and c are determined such that the following inequalities must be satisfied.

$$\sum_{j=0}^{c} \binom{n}{j} p_1{}^j (1-p_1)^{n-j} \geq 1-\alpha \qquad (2.5)$$

$$\sum_{j=0}^{c} \binom{n}{j} p_2{}^j (1-p_2)^{n-j} \leq \beta \qquad (2.6)$$

Similarly, under the conditions of AQL and LQL, the parameters of an SkSP-2 plan namely i, f, n and c will be determined such that the following inequalities are satisfied.

$$P_a(p_1) = \frac{fP_1 + (1-f)P_1^i}{f + (1-f)P_1^i} \geq 1-\alpha \tag{2.7}$$

$$P_a(p_2) = \frac{fP_2 + (1-f)P_2^i}{f + (1-f)P_2^i} \leq \beta \tag{2.8}$$

where P_1 and P_2 are obtained by using (2.3) and (2.4) respectively.

The ASN of the SkSP-2 plan at the quality level of p is given by

$$ASN(p) = \frac{nf}{f + (1-f)P^i} \tag{2.9}$$

When determining the plan parameters, the use of the ASN evaluated at the LQL is recommended because it is larger than the ASN at the AQL.

The optimization procedure to finding the parameters is described below.

Step 1: Find the design parameters (n, c) for the single sampling plan satisfying (2.5) and (2.6) at the same time.

Step 2: Use the combination of (n, c) obtained in Step 1 as guess values and find the design parameters (i, f, n and c) for the SkSP-2 plan by satisfying (2.7) and (2.8).

Step 3: There may exist a number of combinations of design parameters (i, f, n and c) in simulation process so pick up those values for which the ASN at LQL is minimum and i is less than 10.

From the simulation experiment, it is to be pointed out that as the value of i increases, the ASN decreases. At some particular values of i, there would be no effect on the ASN. The design parameters for the SkSP-2 plan and for the single sampling plan are presented in Tables 1 and 2, respectively, when $\alpha = 0.05$ and $\beta = 0.1$.

Table 1
SkSP-2 Plan for Given AQL and LQL

AQL (p_1)	LQL (p_2)	Optimal Parameters						
		i	f	n	c	ASN	$P_a(p_1)$ %	$P_a(p_2)$ %
0.001	0.002	5	0.001	1985	1	1970	95.30	10.00
	0.005	5	0.120	777	1	776	95.02	9.99
	0.010	7	0.060	230	0	230	95.02	9.91
	0.015	5	0.200	153	0	152	95.04	9.91
	0.020	5	0.328	114	0	114	95.01	10.00
	0.030	3	0.629	76	0	76	95.02	9.93
0.005	0.010	5	0.001	396	1	393	95.35	9.98
	0.025	6	0.010	91	0	90	95.06	10.00
	0.050	5	0.096	45	0	45	95.01	9.95
	0.100	4	0.371	22	0	22	95.01	9.86
	0.150	4	0.622	15	0	15	95.00	8.74
0.01	0.020	3	0.002	238	1	225	95.71	9.70
	0.050	5	0.016	45	0	44.97	95.09	10.00
	0.100	4	0.122	22	0	21.99	95.01	9.91
	0.200	2	0.423	11	0	10.89	95.00	9.50
	0.300	2	0.708	7	0	6.98	95.00	8.49
0.05	0.100	5	0.001	39	1	38.80	95.48	9.23
	0.250	3	0.037	9	0	8.90	95.08	8.52
	0.500	2	0.196	4	0	3.94	95.02	7.73
0.100	0.200	4	0.003	19	1	18	95.06	9.75
	0.500	1	0.039	8	0	7	95.09	9.14

Table 2
Single Sampling Plan for Specified AQL and LQL

AQL (p_1)	LQL (p_2)	Optimal parameters	
		n	c
0.001	0.002	12375	18
	0.005	1135	3
	0.010	531	2
	0.015	258	1
	0.020	194	1
	0.030	129	1
0.005	0.010	2478	18
	0.025	266	3
	0.050	105	2
	0.100	38	1
	0.150	25	1
0.01	0.020	1235	18
	0.050	132	3
	0.100	52	2
	0.200	18	1
	0.300	12	1
0.05	0.100	233	17
	0.250	25	3
	0.500	7	1
0.100	0.200	109	16
	0.500	12	3

From Table 1, it is clear that, as p_2 increases (or the quality degrades) for fixed values of p_1, the clearance number of sampling inspection, i, the acceptance number, c and the ASN are decreased in general. We observed the same pattern for the single sampling plans also. For any combinations of (p_1, p_2), the ASN of the SkSP-2 plan is smaller than the ASN of the single sampling plan. For example when $p_1 = 0.01$ and $p_2 = 0.05$, the SkSP-2 requires the ASN of 45 whereas the single sampling plan requires the sample size of 132. Similar tables can be constructed for any others values of AQL and LQL. An Excel program is available with the authors upon request.

Example 1:

Suppose one wants to determine parameters of an SkSP-2 plan from Table 1 according to the conditions given that $p_1 = 0.01$, $p_2 = 0.02$, $\alpha = 0.05$ and $\beta = 0.10$. From this table, one can find the optimal parameters as $i = 3$, $f = 0.002$, $n = 238$ and $c = 1$ corresponding to the above mentioned AQL and LQL values. ASN of this plan at LQL is 225 which is minimum. Based on these parameters, the SkSP-2 plan is operated as follows. The OC curve of this plan is shown in Fig. 1.

Step 1: Start with normal inspection (inspecting every lot), using the single sampling plan (238, 1).

Step 2: When 3 consecutive lots are accepted on normal inspection, switch to skipping inspection. During the skipping inspection, 1 lot out of every 500 lots are inspected.

Step 3: When a lot is rejected on skipping inspection, immediately revert to normal inspection.

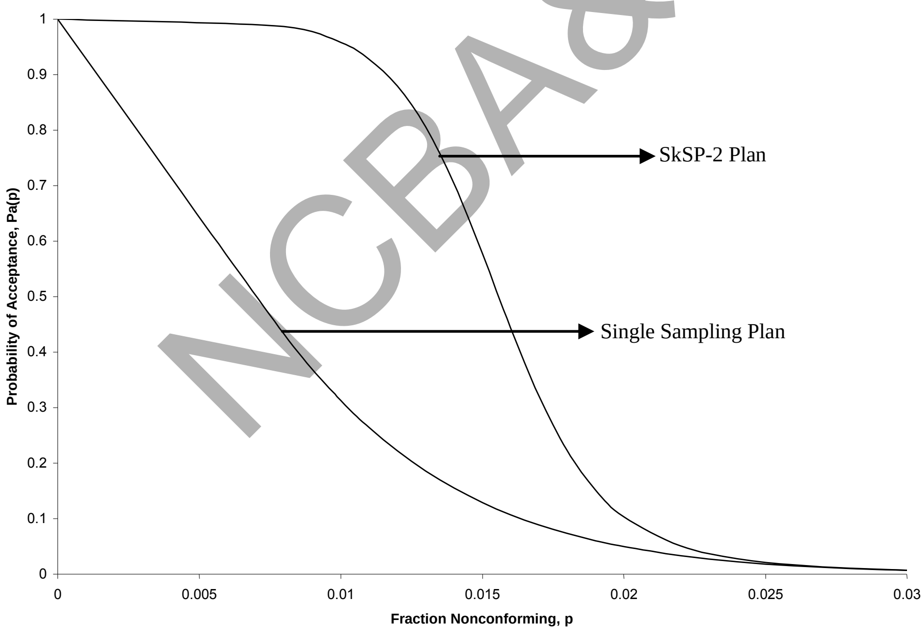


Fig. 1: OC Curves of the Single Sampling Reference Plan and SkSP-2 Plan

3. CONCLUDING REMARKS

In this paper, we have considered the problem of the optimal design of SkSP-2 plans based on two-point approach. Tables have been constructed for easy selection and application of these plans. Sampling plans presented here will have minimum ASN while satisfying the AQL and LQL conditions at the same time. It has been proved that SkSP-2 plans are better than the single sampling plans in achieving the reduced sample size. The proposed approach can be applied to any variants of a skip-lot sampling plan to design a more economical plan.

ACKNOWLEDGEMENTS

The authors would like to thank the reviewers and the Editor for their valuable comments which led to improve the manuscript. The work by Chi-Hyuck Jun was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (Project No. 2009-0072598).

REFERENCES

1. Balamurali, S., Chi-Hyuck Jun, and E. Von Collani (2008). Skip-lot Sampling and Chain Sampling. *Encyclopedia of Statistics in Quality and Reliability* (Eds. F. Ruggeri et al.), John Wiley & Sons.
2. Balamurali, S., Park, H., Jun, C.-H., Kim, K.J. and Lee, J. (2005). Designing of Variables Repetitive Group Sampling Plan Involving Minimum Average Sample Number. *Commun. in Statist.-Simula. and Comput.*, 34, 799-809.
3. Dodge H.F. (1943). A Sampling Inspection Plan for Continuous Production. *Industrial Quality Control*, 14, 264-279. (Also in *Journal of Quality Technology* 1977, 9, 104-119).
4. Dodge H.F. (1955). Skip-lot Sampling Plans. *Industrial Quality Control*, 11, 3-5. (Also in *Journal of Quality Technology* 1977, 9, 143-145).
5. International Organization for Standardization. ISO 2859-3. (2005). *Sampling Procedures for Inspection by Attributes–Part 3*: Skip-Lot Sampling Procedures, International Organization for Standardization, Geneva.
6. Perry R.L. (1973). Skip-lot sampling plans. *Journal of Quality Technology*, 5, 123-130.
7. Schilling, E.G. (1982). *Acceptance Sampling in Quality Control.* Marcel Decker, New York, NY.

AN OPTIMAL DESIGN OF A SKIP LOT SAMPLING PLAN OF TYPE V BY MINIMIZING AVERAGE SAMPLE NUMBER*

Muhammad Aslam[1], **Saminathan Balamurali**[2] **Chi-Hyuck Jun**[3], **Munir Ahmad**[4] and **Mujahid Rasool**[1]

[1] Department of Statistics, Forman Christian College University, Lahore, Pakistan. Email: aslam_ravian@hotmail.com

[2] Department of Mathematics, Kalasalingam University, Krishnankoil 626190, TN, India. Email: sbmurali@rediffmail.com

[3] Department of Industrial and Management Engineering, POSTECH, Pohang 790-784, Republic of Korea. Email: chjun@postech.ac.kr

[4] National College of Business Administration and Economics, Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

In this paper, we propose a designing methodology to find the optimal parameters of skip-lot sampling plan of type V (SkSP-V) in terms of reducing the average sample number. The two-points on the operating characteristic curve approach is used to find the design parameters of the proposed plan and the reference plan as well. The tables are presented and the results are explained using an example. The advantages of the proposed plan over the reference plan is also discussed and proved that the SkSP-V is better than the reference sampling plan in terms of probability of acceptance, average sample number and average total inspection.

KEY WORDS

Binomial sampling; Consumer's risk; Producer's risk; Single sampling; Skip-lot sampling.

1. INTRODUCTION

Dodge (1955a) innovated the concept of continuous sampling and provided mathematical rationale and the rules of operation for the first continuous sampling plan (CSP) familiarly known as CSP-1. Continuous sampling plans can be applied for a product consisting of individual units and manufactured by an essentially continuous process. Later several modifications on continuous sampling plans were proposed and the resultant plans were designated as CSP-2, CSP-3, CSP-F, CSP-T, CSP-V etc. All these plans are available in the US military standard MIL-STD 1235C (1988). For more details about these continuous sampling plans one can refer Stephens (2001).

Dodge (1955b) later presented an extension of continuous sampling plans for individual units to a "skip-lot sampling plan (SkSP) that is applicable to bulk materials or products produced in successive batches or lots", and the plan is designated as "SkSP-1". One of the basic motivations for this extension is stated as "applied to chemical and physical analyses, SkSP-1 sampling plan provides a basis for reducing testing costs".

*Published in Pak. J. Statist. (2012), Vol. 28(1).

So, skip-lot sampling is used for sampling chemical and physical processes in order to bring about substantial savings on inspection of products, which normally conform to specification. This particular sampling plan is useful when the lots are small or where inspection is slow and costly. The operation of the SkSP-1 plan can be seen in Dodge (1955b). Burnett (1967) had presented a Markov chain model for deriving the operating characteristic (OC) function of SkSP-1 plan. Based on the objectives of skip-lot sampling, Perry (1973) formalized the application of skip-lot sampling to the situation in which each lot to be inspected is sampled according to a lot inspection plan, called the reference plan. This plan is designated as SkSP-2 plan. For detailed information about skip-lot sampling plans, one can refer Schilling (1982), ISO 2859-3 (2005) and Balamurali et al (2008). Recently, Aslam et al. (2010) proposed a designing methodology to determine the optimal parameter of a SkSP-2 plan. In this paper, we consider the skip-lot sampling plan of type SkSP-V and propose the designing methodology to determine the optimal parameters using single sampling plan as the reference plan as no such designing methodology is available in the literature . The designing methodology proposed in this paper will satisfy both producer and consumer's risks simultaneously.

2. SkSP-V SKIP LOT SAMPLING PLAN

The continuous sampling plan of type CSP-V is one of the single level continuous sampling plans in which reduced inspection can be achieved by using a smaller clearance interval when reducing the sampling frequency has no advantage upon demonstration of good product quality. Since the skip-lot concept is sound and useful and it is economically advantageous to the skip-lot approach in the design of sampling plans, Balamurali and Jun (2010) developed a new system of skip-lot sampling plan designated as SkSP-V based on the principles of CSP-V plan. The SkSP-V sampling plan is having a provision for reducing a normal inspection. They have also studied the properties of the SkSP-V plan with single sampling plan as the reference plan. According to Balamurali and Jun (2010) the operating procedure of the SkSP-V plan is stated as follows:

(1) At the outset, start with normal inspection using the reference plan. During the normal inspection, lots are inspected one by one in the order of production or in the order of being submitted to inspection.

(2) When i consecutive lots are accepted on normal inspection, discontinue the normal inspection and switch to skipping inspection.

(3) During skipping inspection, inspect only a fraction f of the lots selected at random. Skipping inspection is continued until sampled lot is rejected.

(4) When a lot is rejected on skipping inspection before k consecutively sampled lots are accepted, revert to normal inspection as per (1) above.

(5) When a lot is rejected after k consecutive lots have been accepted revert to normal inspection with reduced clearance number x as per (6) given below.

(6) During normal inspection with clearance number x, lots are inspected one by one in the order of being submitted to inspection and continue the inspection until a lot is rejected or x lots are accepted whichever occurs earlier.

(7) When a lot is rejected, immediately revert to normal inspection with clearance number i as per (1) given above.

(8) When x lots are accepted, discontinue normal inspection and switch to skipping inspection as per (3) above.
(9) Replace or correct all the non-conforming units found with conforming units in the rejected lots.

Associated with this plan are a reference plan and four parameters f, i, k and x. In general, $0 < f < 1$ and i, k and $x(\leq i)$ are positive integers and the plan is designated as SkSP-V (i, f, k, x). The proposed plan is generalization of SkSP-2 plan. When $k = x = i$, the present plan reduces to SkSP-2 sampling plan. It is also important to note that when $f = 1$ the SkSP-V plan reduces to a reference sampling plan.

3. DESIGNING OF SkSP-V PLAN

The probability of accepting a lot based on SkSP-V plan and other performance measures of the SkSP-V sampling plan were derived by Balamurali and Jun (2010) using a Markov chain model. To simplify the number of design parameters, it can be assumed that $k = x$. According to Balamurali and Jun (2010), the probability of acceptance of the SkSP-V plan when $k = x$ is given by

$$P_a(p) = \frac{fP + (1-f)P^i + fP^{k+1}\left(P^i - P^k\right)}{f\left(1 + P^{i+k} - P^{2k}\right) + (1-f)P^i} \tag{3.1}$$

where P is the acceptance probability based on single sampling plan and $Q = 1 - P$. The acceptance probability of a lot under binomial model for the single sampling plan is given by

$$P = \sum_{j=0}^{c} \binom{n}{j} p^j (1-p)^{n-j} \tag{3.2}$$

The two-points on the OC curve approach is considered as a reasonable approach because the lot acceptance probability obtained by one risk may not satisfy the other risk. Further, producer wants that the probability of acceptance should be larger than $1-\alpha$ if the process fraction nonconforming is at the acceptable quality level (AQL) and the consumer demands that the lot acceptance probability should be less than β if the process fraction non-conforming is at the limiting quality level (LQL), see for example Balamurali et al. (2005). According to ANSI/ASQC standard A2 (1987) defines AQL as “the maximum percentage or proportion of variant units in a lot or batch that, for the purpose of acceptance sampling, can be considered as a process average”. Similarly LQL is defined as “the percentage or proportion of variant units in a batch or lot for which, for the purposes of acceptance sampling, the consumer wishes the probability of acceptance to be restricted to a specified low value”. Under the conditions of AQL (p_1) and LQL (p_2), equation (3.2) can be re-written as

$$P_1 = \sum_{j=0}^{c} \binom{n}{i} p_1^{\,j} \left(1 - p_1\right)^{n-j} \tag{3.3}$$

$$P_2 = \sum_{j=0}^{c} \binom{n}{i} p_2^{\,j} \left(1 - p_2\right)^{n-j} \tag{3.4}$$

Under the specified values AQL and LQL, we want to determine the design parameters of the SkSP-V sampling plan (i, f, k, x, n, c) such that the producer's and the consumer's risks should be satisfied simultaneously.

$$\frac{fP_1 + (1-f)P_1^i + fP_1^{k+1}\left(P_1^i - P_1^k\right)}{f\left(1 + P_1^{i+k} - P_1^{2k}\right) + (1-f)P_1^i} \geq 1 - \alpha \tag{3.5}$$

$$\frac{fP_2 + (1-f)P_2^i + fP_2^{k+1}\left(P_2^i - P_2^k\right)}{f\left(1 + P_2^{i+k} - P_2^{2k}\right) + (1-f)P_2^i} \leq \beta \tag{3.6}$$

The values of P_1 and P_2 are determined from (3.3) and (3.4). There may exist multiple solutions, we alternatively determine these parameters to minimize the average sample number at the quality level p_2, which is analogous to minimizing the average sample number (ASN) in a usual single and double sampling plans. Obviously, a sampling plan having smaller ASN would be more desirable. According to Balamurali and Jun (2010) the ASN of the SkSP-V plan at LQL is given as

$$ASN\left(p_2\right) = \frac{nf + nf\left(P_2^{i+k} - P_2^{2k}\right)}{f\left(1 + P_2^{i+k} - P_2^{2k}\right) + (1-f)P_2^i} \tag{3.7}$$

The design parameters of the SkSP-V sampling plan are determined for various combinations of AQL and LQL. Therefore, we consider the following optimization problem to determine parameters of the SkSP-V plan.

$$\text{Minimize } ASN\left(p_2\right) = \frac{nf + nf\left(P_2^{i+k} - P_2^{2k}\right)}{f\left(1 + P_2^{i+k} - P_2^{2k}\right) + (1-f)P_2^i} \tag{3.8}$$

Subject to

$$\frac{fP_1 + (1-f)P_1^i + fP_1^{k+1}\left(P_1^i - P_1^k\right)}{f\left(1 + P_1^{i+k} - P_1^{2k}\right) + (1-f)P_1^i} \geq 1 - \alpha$$

$$\frac{fP_2 + (1-f)P_2^i + fP_2^{k+1}\left(P_2^i - P_2^k\right)}{f\left(1 + P_2^{i+k} - P_2^{2k}\right) + (1-f)P_2^i} \leq \beta$$

$$n > 1, c \geq 0, i, k, x > 1, 0 < f < 1 \tag{3.9}$$

The design parameters values along with the OC values at the producer's risks of 5% and the consumer's risks of 10% are placed in Table 1. The design parameters of the single sampling plan can be obtained by a similar approach, that is shown in Table 2. From Table 1, we can see the various trends in design parameters. For the same value of p_1, as we increase the value of p_2, we noted the decreasing trends in n, c and increasing trend in f. We noted the same behavior in the design parameters of the single sampling plan.

3.1 Example

Suppose one wants to determine parameters of an SkSP-V plan from Table 1 according to the conditions given that $p_1 = 0.005$, $p_2 = 0.025$, $\alpha = 0.05$ and $\beta = 0.10$. From this table, one can find the optimal parameters as $n = 91$, $c = 0$, $i = 6$, $k = 5$, $x = 5$ and $f = 0.01$ corresponding to the above mentioned AQL and LQL values. Based on these parameters, the SkSP-V plan is operated as follows.

Step 1. Start with normal inspection (inspecting every lot) using the single sampling plan (91, 0).

Step 2. When 6 consecutive lots are accepted on normal inspection, discontinue the normal inspection and switch to skipping inspection.

Step 3. During skipping inspection, inspect 1 lot out of every 100 lots selected at random. Skipping inspection is continued until sampled lot is rejected.

Step 4. When a lot is rejected on skipping inspection before 5 consecutively sampled lots are accepted, revert to normal inspection as per (1) above.

Step 5. When a lot is rejected after 5 consecutive lots have been accepted revert to normal inspection with reduced clearance number 5 as per (6) given below.

Step 6. During normal inspection with clearance number 5, all the four lots are inspected one by one in the order of being submitted to inspection and continue the inspection until a lot is rejected or 5 lots are accepted whichever occurs earlier.

Step 7. When a lot is rejected, immediately revert to normal inspection with clearance number 5 as per (1) given above.

Step 8. When 4 consecutive lots are accepted, discontinue normal inspection and switch to skipping inspection as per (3) above.

Step 9. Replace or correct all the non-conforming units found with conforming units in the rejected lots.

Table 1
Parameters of SkSP-V plan for specified AQL and LQL

p_1	p_2	Optimal Parameters							
		i	*k*	*f*	*n*	*c*	ASN	(1−α)%	β%
0.001	0.002	7	6	0.0002	1946	1	1946	95.42	10.00
	0.005	6	5	0.010	460	0	460.00	95.02	9.98
	0.010	4	3	0.119	230	0	229.98	95.00	9.97
	0.015	4	3	0.238	153	0	152.00	95.00	9.93
	0.020	5	4	0.338	114	0	114.00	95.00	10.00
	0.030	3	2	0.644	76	0	75.97	95.00	9.93
0.005	0.010	6	5	0.0005	390	1	390.00	95.09	9.96
	0.025	6	5	0.01	91	0	91.00	95.08	10.00
	0.050	6	5	0.080	45	0	45.00	95.00	9.95
	0.100	3	2	0.414	22	0	21.99	95.00	9.97
	0.150	2	1	0.671	15	0	14.96	95.00	9.08
0.01	0.020	5	4	0.001	198	1	198.00	95.32	9.84
	0.050	5	4	0.016	45	0	45.00	95.13	10.00
	0.100	2	1	0.168	25	0	21.89	95.02	9.50
	0.200	2	1	0.445	11	0	10.95	95.00	9.43
	0.300	2	1	0.720	7	0	6.98	95.00	8.48
0.05	0.100	5	4	0.001	39	1	39.00	95.48	9.23
	0.250	3	2	0.039	9	0	9.00	95.10	8.46
	0.500	2	1	0.218	4	0	3.99	95.01	7.55
0.100	0.200	4	3	0.0003	13	0	13.00	95.00	8.29
	0.500	3	2	0.048	4	0	4.00	95.08	6.70

Table 2
Parameters of Single Sampling Plan for Specified AQL and LQL

p_1	p_2	Optimal Parameters	
		n	c
0.001	0.002	12375	18
	0.005	1135	3
	0.010	531	2
	0.015	258	1
	0.020	194	1
	0.030	129	1
0.005	0.010	2478	18
	0.025	266	3
	0.050	105	2
	0.100	38	1
	0.150	25	1
0.01	0.020	1235	18
	0.050	132	3
	0.100	52	2
	0.200	18	1
	0.300	12	1
0.05	0.100	233	17
	0.250	25	3
	0.500	7	1
0.100	0.200	109	16
	0.500	12	3

4. ADVANTAGES OF THE SKSP-V PLAN

In this section, we discuss the advantages of the SkSP-V sampling over the single sampling plan. For this purpose, we have calculated ASN values of SkSP-V plan and compared with the sample size required for a single sampling plan for different values of p_1 and p_2. Table 3 summarizes the results.

From this table, we can see that for the same values of AQL and LQL, the SkSP-V sampling plan provides much smaller sample size as compared to single sampling plan (or reference sampling plan). For an example, when p_1=0.001 and p_2=0.002, the required sample size $n = 1946$ from Table 1 for SkSP-V plan and it is 12375 when we test the items using the single sampling plan. So, the SkSP-V sampling is more economic than the single sampling in saving the time, cost and the efforts for an experiment.

Table 3
Comparison of Sample Size

p_1	p_2	SkSP-V Plan	Single Sampling Plan
0.001	0.002	1946	12375
	0.005	460	1135
	0.010	230	531
	0.015	153	258
	0.020	114	194
	0.030	76	129
0.005	0.010	390	2478
	0.025	91	266
	0.050	45	105
	0.100	22	38
	0.150	15	25
0.01	0.020	198	1235
	0.050	45	132
	0.100	25	52
	0.200	11	18
	0.300	7	12
0.05	0.100	39	233
	0.250	9	25
	0.500	4	7
0.100	0.200	13	109
	0.500	4	12

In order to show the better efficiency of the SkSP-V plan in terms of probability of acceptance, average sample number and average total inspection (ATI) three figures are provided. Figure 1 gives the OC curves of the SkSP-V plan with parameters $i=6$, $k=3$, $x=3$, $f=0.01$ along with single sampling plan with parameters $N=1000$, $n=50$ and $c=1$ as the reference plan. Figure 2 gives the ASN curves while Figure 3 shows the ATI curves of the above mentioned plans. From Figure 1, it can be observed that the SkSP-V plan increases the probability of acceptance in the region of principal interest, i.e. for good quality levels and maintains the consumer's risk at poor quality levels compared with the single sampling plan. It implies that SkSP-V plan gives comparatively lesser producer's risk while safeguarding the consumer's interest than the single sampling plan. From Figures 2 and 3, it is easily observed that when the lot quality is good, reduction in ASN as well as ATI are achieved through the SkSP-V plan over the single sampling plans. When the lot quality deteriorates, the ASN and ATI of the SkSP-V plan converge with the single sampling plan.

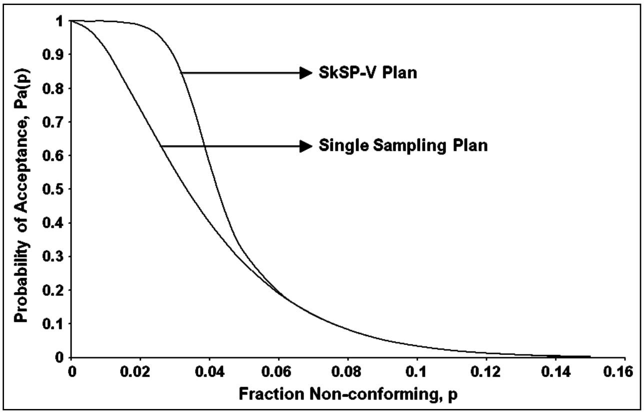


Fig. 1: Operating Characteristic (OC) Curves of SkSP-V & Single Sampling Plans

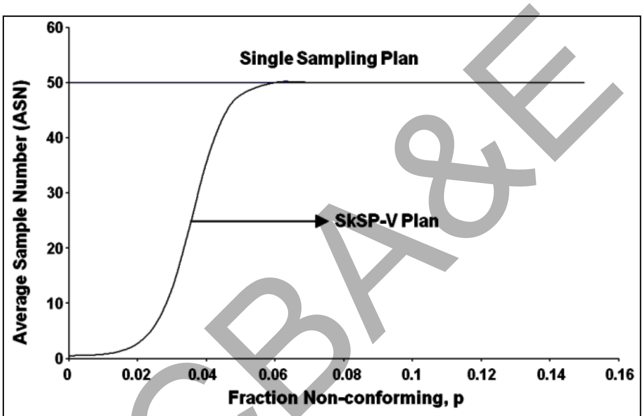


Fig. 2: Average Sample Number (ASN) Curves of SkSP-V & Single Sampling Plans

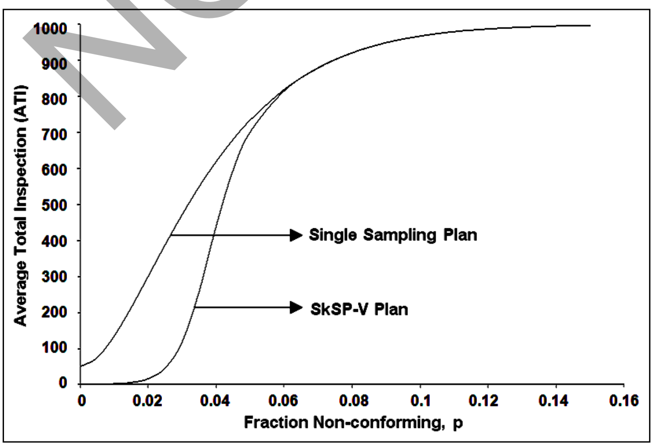


Fig. 3: Average Total Inspection (ATI) Curves of SkSP-V & Single Sampling Plans

5. CONCLUDING REMARKS

In this paper, we have considered the problem of designing the SkSP-V sampling plan. The two points approach is used to find the design parameters of the SkSP-V plan, which is considering the producer's and the consumer's simultaneously. Tables for showing design parameters of both SkSP-V and single sampling plans have been presented and comparison has been made between two plans. The procedure was described to use the proposed methodology in practice. It has been proved that the proposed plan is better than the single sampling plan in terms of the sample size requires.

ACKNOWLEDGEMENTS

The writers are thankful to the two referees and editor for several valuable comments.

REFERENCES

1. American National Standard Institute/American Society for Quality Control (ANSI/ASQC) A2 (1987). *Terms, Symbols and Definitions for Acceptance Sampling*. American Society for Quality Control, Milwaukee, Wisconsin.
2. Aslam, M., S. Balamurali, C.-H. Jun and M. Ahmad (2010). Optimal designing of a skip lot sampling plan by two point method. *Pak. J. Statist.*, 26(4), 585-592.
3. Balamurali, S., Park, H., Jun, C.-H., Kim, K.J. and Lee, J. (2005). Designing of Variables Repetitive Group Sampling Plan Involving Minimum Average Sample Number. *Commun. in Statist.-Simula. and Compu.*, 34, 799-809.
4. Balamurali, S., Chi-Hyuck Jun, and E. Von Collani (2008). Skip-lot Sampling and Chain Sampling. *Encyclopedia of Statistics in Quality and Reliability* (Eds. F. Ruggeri et al.), John Wiley & Sons.
5. Balamurali, S. and Jun, C.-H. (2010). A New System of Skip-lot Sampling Plans Having a Provision for Reducing a Normal Inspection. *Applied Stochastic Models in Business and Industry*, (on line).
   Burnett, T.L. (1967). Markov Chains and Attribute Sampling Plans. IBM Technical Report No. 67-825-2175, Federal Systems Division, Owego, New York.
6. Dodge, H.F. (1955a). A Sampling Inspection Plan for Continuous Production. *Industrial Quality Control*, 14, 3, 264-279. (Also in *Journal of Quality Technology*, 1977, 104-119).
7. Dodge, H.F. (1955b). Skip-lot Sampling Plans. *Industrial Quality Control*, 11, 3-5. (Also in *Journal of Quality Technology* 1977, 9, 143-145).
8. International Organization for Standardization. ISO 2859-3 (2005). *Sampling Procedures for Inspection by Attributes–Part 3*: Skip-Lot Sampling Procedures. International Organization for Standardization, Geneva.
9. Perry, R.L. (1973). Skip-lot sampling plans. *Journal of Quality Technology*, 5, 123-130.
10. Schilling, E.G. (1982). *Acceptance Sampling in Quality Control.* Marcel Decker, New York, NY.
11. Stephens, K.S. (2001). *The Handbook of Applied Acceptance Sampling: Plans, Principles and Procedures*. ASQ Quality Press.

SKSP-V SAMPLING PLAN WITH GROUP SAMPLING PLAN AS REFERENCE BASED ON TRUNCATED LIFE TEST UNDER WEIBULL AND GENERALIZED EXPONENTIAL DISTRIBUTIONS[*]

Muhammad Aslam[1], **S. Balamurali**[2], **Chi-Hyuck Jun**[3] and **Munir Ahmad**[4]

[1] Department of Statistics, Forman Christian College University, Lahore, Pakistan. Email: aslam_ravian@hotmail.com

[2] Departments of Mathematics, Kalasalingam University, Krishnankoil 626190, TN, India. Email: sbmurali@rediffmail.com

[3] Department of Industrial and Management Engineering, POSTECH, Pohang 790-784, Republic of Korea. Email: chjun@postech.ac.kr

[4] National College of Business Administration and Economics, Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

This paper proposes SkSP-V acceptance sampling plans having group sampling plan based on the time truncated life test as the reference plan. The plan is designed for the mean life when the lifetime of the submitted product follows the Weibull distribution or the generalized exponential distribution. The two points on the operating characteristics curve is used to find the plan parameters satisfying the consumer's and the producer's risks while minimizing the average sample number. Also, the advantage of the proposed plan over the single group sampling plan is discussed. The extensive tables are provided and examples are given to adopt the plan in practice.

KEY WORDS

Skip-lot sampling; life tests; group sampling; producer and consumer risks; Weibull and generalized exponential distribution.

1. INTRODUCTION

The skip-lot sampling schemes are widely used to reduce inspection cost when the quality of the lot is relatively good. In the skip-lot sampling operational procedure, only the fraction of a submitted lot is inspected for the acceptance or rejection decision. Dodge (1955) discussed the application of the SkSP-1 sampling plan to bulk material and products produced in successive lots. Perry (1973) discussed the application and advantages of the skip-lot sampling plan by using the single sampling plan as the reference plan. Parker and Kessler (1981) proposed the modified skip-lot sampling plan (MSkSP-1) and discussed its applications. For more detail about the applications of these types of skip-lot sampling plans, reader may refer to Bennett and Callejas (1980), Wilrich (1981), Schneider and Wilrich (1981), Schilling (1982), Cox (1982), Liebesman and Sperstein (1983), Liebesman (1987).ISO 2859-3 (2005) and Balamurali *et al.* (2007).

Recently, Balamurali and Jun (2011) proposed a new system of skip-lot sampling plans having a provision for reducing the normal inspection. This new plan is designated as the SkSP-V sampling plan. They discussed the properties of the new plan and provided

[*]Published in Pak. J. Statist. (2013), Vol. 29(2).

some cost models for the optimal design of the SkSP-V plan. Aslam et al. (2012) determined the plan parameters of the SkSP-V plan based on two-point on the operating characteristic (OC) curve approach at the acceptable quality level (AQL) and the limiting quality level (LQL) and developed tables for the selection of parameters.

Group sampling plans are widely used when the experimenter has the facility to install more than one item in a single tester. A lot of the product is accepted if the total number of failures is less than the specified acceptance number before the end of specified termination time or no failure is recorded before the termination time. As mentioned by Jun et al. (2006), Aslam and Jun (2009), Aslam et al. (2009), and Aslam et al. (2011), group sampling plans are useful in reducing the cost of the experiment than the plan where we inspect/test the item one by one.

According to the best our knowledge, there is no study on the SkSP-V sampling plan by considering the group sampling plan as the reference plan. In this paper, we propose the SkSP-V sampling plan having the group sampling plan based on the truncated life test as the reference plan, designated as SkGSP-V plan and determine the optimal parameters of the proposed plan by considering the consumer's and the producer's risks at the same time while minimizing the average sample number. Further, the proposed plan is applied to Weibull and generalized exponential distributions. The rest of the paper is organized as follows: operating procedure of the proposed plan is given in Section 2. The design of the proposed plan is given in Section 3. In Section 4, the advantage of the proposed plan is discussed. Conclusion of the study is given in the last section.

2. OPERATING PROCEDURE OF SkGSP-V PLAN

Balamurali and Jun (2011) originally proposed the SkSP-V sampling plan using the single sampling plan as the reference. The operating procedure of the proposed SkGSP-V sampling plan using the group sampling plan with group size r can be given similarly as follows.

(1) At the outset, start with the normal inspection using the group sampling plan as reference plan. During the normal inspection, lots are inspected one by one in the order of production or in the order of being submitted to inspection. From each lot under inspection, select a random sample of size n and allocate r items to each of g groups (or testers) so that $n = rg$ and put them on test for the time duration of t_0. Accept the lot if the total number of failures from g groups is smaller than or equal to the acceptance number c. Truncate the test and reject the lot as soon as the total number of failures from g groups exceeds before the time t_0.

(2) When i consecutive lots are accepted on the normal inspection, discontinue the normal inspection and switch to the skipping inspection.

(3) During the skipping inspection, inspect only a fraction f of the lots selected at random. Skipping inspection is continued until sampled lot using the group sampling plan as the reference plan is rejected.

(4) When a lot is rejected on the skipping inspection before k consecutively sampled lots are accepted, revert to the normal inspection as per (1) above.

(5) When a lot is rejected after k consecutive lots have been accepted revert to the normal inspection with the reduced clearance number x as per (6) given below.
(6) During the normal inspection with clearance number x, lots are inspected one by one in the order of being submitted to inspection and continue the inspection until a lot is rejected or x lots are accepted, whichever occurs earlier.
(7) When a lot is rejected, immediately revert to the normal inspection with clearance number i as per (1) given above.
(8) When x lots are accepted, discontinue the normal inspection and switch to the skipping inspection as per (3) above.
(9) Replace or correct all the non-conforming units found with conforming units in the rejected lots.

The proposed plan is characterized by six parameters g,c,i,k,x and f while the number of testers r, and the termination time t_0 and ratio of true median ratio and specified median ratio which plays an important role are regarded as specified parameters. The description of the parameters namely, f,i,k and x involved in the SkSP-V sampling scheme can be found in Balamurali and Jun (2011). In general, $0 < f < 1$ and i, k and $x(\leq i)$ are positive integers. It should be noted that the SkSP-V plan with group sampling plan as the reference plan is the generalization of several sampling plans. For example, when $r = 1$, the proposed plan reduces to the original SkSP-V plan, when $k = x = i$, the proposed plan reduces to SkSP-2 sampling plan having the group sampling plan as the reference plan and when f=1, it reduces to the group sampling plan.

3. DESIGN OF SkGSP-V PLAN

To reduce the number of parameters of the proposed plan, it is assumed that $k = x$. According to Balamurali and Jun (2011), the OC function of the SkSP-V sampling plan when $k = x$ given by

$$P_a(p) = \frac{fP + (1-f)P^i + fP^{k+1}(P^i - P^k)}{f(1+P^{i+k} - P^{2k}) + (1-f)P^i} \tag{1}$$

where P is the lot acceptance probability of the reference plan. The reference plan for the proposed plan is the group sampling plan, so P is given by [see Aslam et al. (2011)].

$$P = \sum_{i=0}^{c} \binom{rg}{i} p^i \left(1-p\right)^{rg-i} \tag{2}$$

where p is the probability that an item fails by time t_0.

It is important to note that, all the acceptance schemes are based on statistical sampling methods. In these techniques items are picked up at random using the simple random sampling technique and put on the test. Statistical sampling may cause the fact that good item is selected from the lot and leads to acceptance the lot where the lot may also constitute the bad items. The chance of committing this error is called the consumer's risk. On the other hand, there is a chance of selecting only the bad items in the test and may lead to the rejection of the lot where there may be good items in the lot. This chance is called the producer's risk. So, there is a need to propose the sampling plan that uses the fraction of items for the inspection using the group sampling plan as the reference plan and also minimizes the risks. Letα be the producer risk and β be the consumer's risk. The plan parameters are determined such that the lot acceptance probability is less than the consumer's risk β at LQL and larger than the producer's confidence level $1-\alpha$ at AQLat the same time by minimizing the average sample number (ASN) at LQL. So, the selection of the plan parameters can be done such that the following two inequalities should be satisfied.

$$P_a(p_1 = AQL) \geq 1-\alpha \tag{3}$$

$$P_a(p_2 = LQL) \leq \beta \tag{4}$$

The ASN of the proposed plan is given [Balamurali and Jun (2011)] as

$$ASN(p_2) = \frac{rgf + rgf\left(P_2^{i+k} - P_2^{2k}\right)}{f\left(1+P_2^{i+k} - P_2^{2k}\right) + (1-f)P_2^{i}} \tag{5}$$

3.1 Proposed Plan under Weibull Distribution

Producers always want to enhance the quality level of the products so that the chance of the rejection of the product at the time of inspection can be minimized. The quality level is always determined through mean or median ratios. The plan parameters obtained by specifying the AQL and LQL do not provide the quality level of the product. So, there is a need to propose SkSP-V sampling plan under the time truncated life tests assuming that the lifetime of the product follows the Weibull distribution using the median life as the quality parameter. The cumulative distribution function (cdf) of the Weibull distribution is given by

$$F(t;\lambda,\beta) = 1-\exp(-(t/\lambda)^{\gamma}),\ t \geq 0 \tag{6}$$

where λ is scale parameter and γ is shape parameter of the Weibull distribution. The q-th percentile of the Weibull distribution is given by

$$\theta_w = \lambda\left(\ln\left(\frac{1}{1-q}\right)\right)^{1/\gamma}$$

Let $t_0 = a\theta_0$ for a constant a, where θ_0 is specified median life and θ_w be the true median life under the Weibull distribution. Then, the probability of failure of an item before experiment time t_0 is obtained as

$$p_w = 1 - exp\left[-a^{\gamma}(\theta_w/\theta_0)^{\gamma} ln\left(\frac{1}{1-q}\right)\right] \tag{7}$$

The plan parameters of the proposed plan when the lifetime of the product follows the Weibull distribution are determined and shown in Tables 1-3 for shape parameter ($\gamma = 1,2,3$) respectively and for median ratio ($\theta_w/\theta_0 = 2,4,6,8,10$) and termination time ratio ($a = 0.5,1.0$). The ASN values at LQL as well as the probability of acceptance values at AQL of the selected plans are also reported in tables.

Table 1
Optimal Parameters for the SkGSP-V Plan under the Weibull Distribution with γ=1

β	θ_w/θ^0	r=5						r=10					
		a=0.5			a=1.0			a=0.5			a=1.0		
		g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$
0.25	2	7,7,2,1,0.2	31.950	0.9548	4,7,2,1,0.15	18.188	0.9595	4,7,2,1,0.1	38.453	0.9517	2,7,2,1,0.15	18.188	0.9595
	4	3,2,2,1,0.15	13.490	0.9789	2,2,2,1,0.10	9.737	0.9730	2,2,2,1,0.10	19.703	0.9683	1,2,2,1,0.10	9.737	0.9730
	6	2,1,2,1,0.2	9.045	0.9766	1,1,2,1,0.35	4.685	0.9641	1,1,2,1,0.2	9.045	0.9766	1,1,2,1,0.10	9.989	0.9512
	8	2,1,2,1,0.2	9.045	0.9863	1,1,2,1,0.35	4.685	0.9788	1,1,2,1,0.2	9.045	0.9863	1,1,2,1,0.10	9.989	0.9739
	10	1,0,2,1,0.25	4.561	0.9531	1,0,2,1,0.10	4.956	0.9533	1,0,2,1,0.10	9.913	0.9533	1,1,2,1,0.10	9.989	0.9832
0.10	2	9,8,2,1,0.10	43.679	0.9563	5,8,2,1,0.10	24.362	0.9527	5,9,2,1,0.1	48.884	0.9601	3,10,2,1,0.10	29.355	0.9556
	4	4,2,2,1,0.10	19.703	0.9683	2,2,2,1,0.10	9.737	0.9730	2,2,2,1,0.10	19.703	0.9683	1,2,2,1,0.10	9.737	0.9730
	6	3,1,2,1,0.10	14.788	0.9729	2,1,2,1,0.10	9.989	0.9516	2,2,2,1,0.10	19.703	0.9889	1,1,2,1,0.10	9.989	0.9516
	8	3,1,2,1,0.10	14.788	0.9847	2,1,2,1,0.10	9.989	0.9739	2,1,2,1,0.10	19.985	0.9724	1,1,2,1,0.10	9.989	0.9739
	10	2,0,2,1,0.10	9.913	0.9533	1,0,2,1,0.10	4.956	0.9533	1,0,2,1,0.10	9.913	0.9533	1,1,2,1,0.10	9.989	0.9832
0.05	2	12,11,2,1,0.15	59.508	0.9504	8,13,2,1,0.10	39.867	0.9608	6,11,2,1,0.15	59.508	0.9504	4,13,2,1,0.10	39.867	0.9608
	4	4,2,2,1,0.15	19.812	0.9535	3,3,2,1,0.10	14.958	0.9726	2,2,2,1,0.15	19.812	0.9535	2,4,2,1,0.10	19.994	0.9732
	6	3,1,2,1,0.15	14.866	0.9600	2,1,2,1,0.10	9.989	0.9516	2,2,2,1,0.15	19.812	0.9835	1,1,2,1,0.10	9.989	0.9516
	8	3,1,2,1,0.15	14.866	0.9772	2,1,2,1,0.10	9.989	0.9739	2,1,2,1,0.10	19.985	0.9724	1,1,2,1,0.10	9.989	0.9739
	10	2,0,2,1,0.10	9.913	0.9533	1,0,2,1,0.10	4.956	0.9533	1,0,2,1,0.10	9.913	0.9533	1,1,2,1,0.10	9.989	0.9832
0.01	2	18,16,2,1,0.10	89.936	0.9618	10,16,2,1,0.10	49.974	0.9573	9,16,2,1,0.10	89.936	0.9618	5,16,2,1,0.10	49.974	0.9573
	4	8,4,2,1,0.10	39.996	0.9669	4,4,2,1,0.10	19.994	0.9732	4,4,2,1,0.10	39.996	0.9669	2,4,2,1,0.10	19.994	0.9732
	6	6,2,2,1,0.10	29.998	0.9667	3,2,2,1,0.10	14.998	0.9699	3,2,2,1,0.10	29.998	0.9667	2,3,2,1,0.10	19.999	0.9792
	8	4,1,2,1,0.10	19.985	0.9724	3,2,2,1,0.10	14.998	0.9858	2,1,2,1,0.10	19.985	0.9724	2,2,2,1,0.10	19.999	0.9683
	10	4,1,2,1,0.10	19.985	0.9823	3,1,2,1,0.10	14.999	0.9601	2,1,2,1,0.10	19.985	0.9823	2,2,2,1,0.10	19.999	0.9824

Table 2
Optimal Parameters for the SkGSP-V Plan under the Weibull Distribution with γ=2

β	θ_w/θ^0	r=5						r=10					
		a=0.5			a=1.0			a=0.5			a=1.0		
		g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$
0.25	2	4,1,2,1,0.15	17.713	0.9593	2,2,2,1,0.10	9.737	0.9730	2,1,2,1,0.15	17.713	0.9593	1,2,2,1,0.10	9.737	0.9730
	4	2,0,2,1,0.25	9.122	0.9717	1,0,2,1,0.10	4.956	0.9746	1,0,2,1,0.25	9.122	0.9717	1,1,2,1,0.15	9.993	0.9897
	6	2,0,2,1,0.25	9.122	0.9879	1,0,2,1,0.10	4.956	0.9898	1,0,2,1,0.25	9.122	0.9878	1,0,2,1,0.10	9.999	0.9780
	8	2,0,2,1,0.40	9.541	0.9891	1,0,2,1,0.20	4.981	0.9889	1,0,2,1,0.40	9.541	0.9891	1,0,2,1,0.10	9.999	0.9885
	10	2,0,2,1,0.60	9.791	0.9896	1,0,2,1,0.30	4.989	0.9895	1,0,2,1,0.60	9.791	0.9896	1,0,2,1,0.15	9.999	0.9893
0.10	2	7,2,2,1,0.15	34.124	0.9652	2,2,2,1,0.10	9.737	0.9730	4,2,2,1,0.10	39.547	0.9659	1,2,2,1,0.10	9.737	0.9730
	4	3,0,2,1,0.20	14.674	0.9646	1,0,2,1,0.10	4.956	0.9746	2,0,2,1,0.10	19.826	0.9746	1,1,2,1,0.15	9.993	0.9897
	6	3,0,2,1,0.20	14.674	0.9851	1,0,2,1,0.10	4.956	0.9898	2,0,2,1,0.10	19.826	0.9898	1,0,2,1,0.10	9.999	0.9780
	8	3,0,2,1,0.25	14.754	0.9897	1,0,2,1,0.20	4.981	0.9889	2,0,2,1,0.20	19.922	0.9889	1,0,2,1,0.10	9.999	0.9885
	10	3,0,2,1,0.40	14.876	0.9896	1,0,2,1,0.30	4.989	0.9895	2,0,2,1,0.30	19.954	0.9895	1,0,2,1,0.15	9.999	0.9893
0.05	2	8,2,2,1,0.10	39.547	0.9659	3,3,2,1,0.10	14.958	0.9726	4,2,2,1,0.10	39.547	0.9659	2,4,2,1,0.10	19.994	0.9732
	4	4,0,2,1,0.10	19.826	0.9746	1,0,2,1,0.10	4.956	0.9746	2,0,2,1,0.10	19.826	0.9746	1,1,2,1,0.15	9.993	0.9897
	6	4,0,2,1,0.10	19.826	0.9898	1,0,2,1,0.10	4.956	0.9898	2,0,2,1,0.10	19.826	0.9898	1,0,2,1,0.10	9.999	0.9780
	8	4,0,2,1,0.20	19.922	0.9889	1,0,2,1,0.20	4.981	0.9889	2,0,2,1,0.20	19.922	0.9889	1,0,2,1,0.10	9.999	0.9885
	10	4,0,2,1,0.30	19.954	0.9895	1,0,2,1,0.30	4.989	0.9895	2,0,2,1,0.30	19.954	0.9895	1,0,2,1,0.150	9.999	0.9893
0.01	2	13,3,2,1,0.10	64.986	0.9523	4,4,2,1,0.10	19.994	0.9732	7,4,2,1,0.10	69.950	0.9779	2,4,2,1,0.10	19.994	0.9732
	4	6,0,2,1,0.10	29.992	0.9573	3,1,2,1,0.10	14.999	0.9847	3,0,2,1,0.10	29.992	0.9573	2,1,2,1,0.10	20.0	0.9724
	6	6,0,2,1,0.10	29.992	0.9842	2,0,2,1,0.10	9.999	0.9780	3,0,2,1,0.10	29.992	0.9842	1,0,2,1,0.10	9.999	0.9780
	8	6,0,2,1,0.15	29.995	0.9873	2,0,2,1,0.10	9.999	0.9885	3,0,2,1,0.15	29.995	0.9873	1,0,2,1,0.10	9.999	0.9885
	10	6,0,2,1,0.20	29.996	0.9894	2,0,2,1,0.15	9.999	0.9893	3,0,2,1,0.20	29.996	0.9894	1,0,2,1,0.15	9.999	0.9893

Table 3
Optimal Parameters for the SkSP-V Plan under the Weibull Distribution with γ=3

β	θ_w/θ^0	r=5						r=10					
		a=0.5			a=1.0			a=0.5			a=1.0		
		g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$
0.25	2	5,0,2,1,0.10	22.329	0.9665	1,1,2,1,0.35	4.685	0.9787	3,0,2,1,0.10	28.572	0.9573	1,1,2,1,0.10	9.989	0.9739
	4	4,0,2,1,0.40	19.082	0.9891	1,0,2,1,0.20	4.981	0.9889	2,0,2,1,0.40	19.082	0.9891	1,0,2,1,0.10	9.999	0.9885
	6	6,0,2,1,0.85	29.971	0.9898	1,0,2,1,0.65	4.997	0.9896	3,0,2,1,0.85	29.971	0.9898	1,0,2,1,0.35	9.999	0.9887
	8	14,0,2,1,0.85	69.999	0.9899	2,0,2,1,0.75	9.999	0.9898	7,0,2,1,0.85	69.999	0.9899	1,0,2,1,0.75	9.999	0.9898
	10	27,0,8,7,0.85	135.0	0.9899	4,0,2,1,0.75	20.0	0.9896	14,0,2,1,0.85	140.0	0.9897	2,0,2,1,0.75	20.0	0.9896
0.10	2	10,1,2,1,0.15	48.543	0.9831	2,1,2,1,0.10	9.989	0.9739	5,1,2,1,0.15	48.543	0.9831	1,1,2,1,0.10	9.989	0.9739
	4	6,0,2,1,0.25	29.508	0.9897	1,0,2,1,0.20	4.981	0.9889	3,0,2,1,0.25	29.508	0.9897	1,0,2,1,0.10	9.999	0.9885
	6	6,0,2,1,0.85	29.971	0.9898	1,0,2,1,0.65	4.997	0.9896	3,0,2,1,0.85	29.971	0.9898	1,0,2,1,0.35	9.999	0.9887
	8	14,0,2,1,0.85	69.999	0.9899	2,0,2,1,0.75	9.999	0.9898	7,0,2,1,0.85	69.999	0.9899	1,0,2,1,0.75	9.999	0.9898
	10	27,0,8,7,0.85	135.0	0.9899	4,0,2,1,0.75	20.0	0.9896	14,0,2,1,0.85	140.0	0.9897	2,0,2,1,0.75	20.0	0.9896
0.05	2	12,1,2,1,0.10	59.325	0.9839	2,1,2,1,0.10	9.989	0.9739	6,1,2,1,0.10	59.325	0.9839	1,1,2,1,0.10	9.989	0.9739
	4	7,0,2,1,0.60	34.946	0.9717	1,0,2,1,0.20	4.981	0.9889	4,0,2,1,0.20	39.844	0.9889	1,0,2,1,0.10	9.999	0.9885
	6	7,0,2,1,0.75	34.973	0.9895	1,0,2,1,0.65	4.997	0.9896	4,0,2,1,0.65	39.979	0.9896	1,0,2,1,0.35	9.999	0.9887
	8	14,0,2,1,0.85	69.999	0.9899	2,0,2,1,0.75	9.999	0.9898	7,0,2,1,0.85	69.999	0.9899	1,0,2,1,0.75	9.999	0.9898
	10	27,0,8,7,0.85	135.0	0.9899	4,0,2,1,0.75	20.0	0.9896	14,0,2,1,0.85	140.0	0.9897	2,0,2,1,0.75	20.0	0.9896
0.01	2	16,1,2,1,0.10	79.953	0.9712	3,2,2,1,0.10	14.998	0.9858	8,1,2,1,0.10	79.953	0.9712	2,2,2,1,0.10	19.999	0.9683
	4	11,0,2,1,0.15	54.977	0.9884	2,0,2,1,0.10	9.999	0.9885	6,0,2,1,0.15	59.989	0.9873	1,0,2,1,0.10	9.999	0.9885
	6	11,0,2,1,0.50	54.996	0.9889	2,0,2,1,0.35	9.999	0.9887	6,0,2,1,0.45	59.998	0.9892	1,0,2,1,0.35	9.999	0.9887
	8	14,0,2,1,0.85	69.999	0.9899	2,0,2,1,0.75	9.999	0.9898	7,0,2,1,0.85	69.999	0.9899	1,0,2,1,0.75	9.999	0.9898
	10	27,0,8,7,0.85	135.0	0.9899	4,0,2,1,0.75	20.0	0.9896	14,0,2,1,0.85	140.0	0.9897	2,0,2,1,0.75	20.0	0.9896

3.2 Proposed Plan under Generalized Exponential Distribution

Gupta and Kundu (2009) originally developed the generalized exponential (GE) distribution with δ as the shape parameter. The cdf of the GE distribution is given by

$$F(t;\delta,\lambda) = \left(1 - exp\left(-\frac{t}{\lambda}\right)\right)^{\delta} \quad (8)$$

where δ is the shape parameter and λ is the scale parameter. The q-th percentile of GE distribution is given by

$$\theta_g = -\lambda \ln\left(1 - q^{1/\delta}\right)$$

Aslam et al. (2010) derived the equation of the probability of failure and is given by

$$p_g = \left[1 - \exp\left(-a \ln(1 - q^{1/\delta}) / (\theta_g / \theta^0)\right)\right]^{\delta} \quad (9)$$

The optimal plan parameters of the proposed plan when the lifetime of the product follows the GE distribution can be determined for two values of the shape parameter $(\delta = 2,3)$ and with other parameters are same as used in Tables 1-3. The plan parameters along with the probability of acceptance values and the ASN values for the GE distribution are reported in Tables 4-5. The probability of acceptance values are computed at AQL and at the same the ASN values are calculated at the LQL.

Table 4
Optimal Parameters for the SkGSP-V Plan under the Generalized Exponential Distribution with δ=2

β	θ_g/θ^0	r=5						r=10					
		a=0.5			a=1.0			a=0.5			a=1.0		
		g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$
0.25	2	4,2,2,1,0.25	18.266	0.9526	2,3,2,1,0.25	9.167	0.9597	2,2,2,1,0.25	18.266	0.9526	1,3,2,1,0.25	9.167	0.9597
	4	2,0,2,1,0.10	9.257	0.9763	1,0,2,1,0.10	4.956	0.9503	1,0,2,1,0.10	9.257	0.9763	1,1,2,1,0.10	9.990	0.9817
	6	2,0,2,1,0.10	9.257	0.9899	1,0,2,1,0.10	4.956	0.9804	1,0,2,1,0.10	9.257	0.9899	1,0,2,1,0.10	9.999	0.9530
	8	2,0,2,1,0.20	9.655	0.9888	1,0,2,1,0.10	4.956	0.9892	1,0,2,1,0.20	9.655	0.9888	1,0,2,1,0.10	9.999	0.9763
	10	2,0,2,1,0.30	9.796	0.9892	1,0,2,1,0.15	4.972	0.9896	1,0,2,1,0.30	9.796	0.9892	1,0,2,1,0.10	9.999	0.9854
0.10	2	7,3,2,1,0.10	34.369	0.9687	3,4,2,1,0.10	14.539	0.9751	4,4,2,1,0.10	38.894	0.9827	2,5,2,1,0.10	19.923	0.9677
	4	3,0,2,1,0.10	14.888	0.9605	1,0,2,1,0.10	4.956	0.9503	2,1,2,1,0.20	19.751	0.9870	1,1,2,1,0.10	9.990	0.9817
	6	3,0,2,1,0.10	14.888	0.9844	1,0,2,1,0.10	4.956	0.9804	2,0,2,1,0.10	19.986	0.9783	1,0,2,1,0.10	9.999	0.9529
	8	2,0,2,1,0.60	9.941	0.9673	1,0,2,1,0.10	4.956	0.9892	1,0,2,1,0.60	9.941	0.9673	1,0,2,1,0.10	9.999	0.9763
	10	2,0,2,1,0.60	9.941	0.9787	1,0,2,1,0.15	4.972	0.9896	1,0,2,1,0.60	9.941	0.9787	1,0,2,1,0.10	9.999	0.9854
0.05	2	9,4,2,1,0.10	44.716	0.9730	4,5,2,1,0.10	19.923	0.9677	5,4,2,1,0.10	49.935	0.9592	2,5,2,1,0.10	19.923	0.9677
	4	3,0,2,1,0.10	14.888	0.9605	1,0,2,1,0.10	4.956	0.9503	3,1,2,1,0.10	29.985	0.9858	1,1,2,1,0.10	9.990	0.9817
	6	3,0,2,1,0.10	14.888	0.9844	1,0,2,1,0.10	4.956	0.9804	2,0,2,1,0.10	19.986	0.9783	1,0,2,1,0.10	9.999	0.9529
	8	3,0,2,1,0.15	14.929	0.9673	1,0,2,1,0.10	4.956	0.9892	2,0,2,1,0.10	19.986	0.9883	1,0,2,1,0.10	9.999	0.9763
	10	3,0,2,1,0.20	14.950	0.9891	1,0,2,1,0.15	4.972	0.9896	2,0,2,1,0.15	19.991	0.9890	1,0,2,1,0.10	9.999	0.9854
0.01	2	12,5,2,1,0.10	59.971	0.9657	5,6,2,1,0.10	24.988	0.9609	6,5,2,1,0.10	59.971	0.9657	3,7,2,1,0.10	29.998	0.9548
	4	6,1,2,1,0.10	29.985	0.9858	3,1,2,1,0.10	14.999	0.9560	3,1,2,1,0.10	29.985	0.9858	2,2,2,1,0.10	19.999	0.9803
	6	4,0,2,1,0.10	19.986	0.9783	2,0,2,1,0.10	9.999	0.9529	2,0,2,1,0.10	19.986	0.9783	1,0,2,1,0.10	9.999	0.9529
	8	4,0,2,1,0.10	19.986	0.9883	2,0,2,1,0.10	9.999	0.9763	2,0,2,1,0.10	19.986	0.9883	1,0,2,1,0.10	9.999	0.9763
	10	4,0,2,1,0.15	19.991	0.9890	2,0,2,1,0.10	9.999	0.9854	2,0,2,1,0.15	19.991	0.9890	1,0,2,1,0.10	9.999	0.9854

Table 5
Optimal Parameters for the SkGSP-V Plan under the Generalized Exponential Distribution with δ=3

β	θ_g/θ^0	r=5						r=10					
		a=0.5			a=1.0			a=0.5			a=1.0		
		g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$	g,c,i,k,f	ASN at p_2	$P_a(p_1)$
0.25	2	4,1,2,1,0.15	17.959	0.9728	2,2,2,1,0.10	9.737	0.9712	2,1,2,1,0.15	17.959	0.9728	1,2,2,1,0.10	9.737	0.9712
	4	2,0,2,1,0.25	9.188	0.9853	1,0,2,1,0.10	4.956	0.9801	1,0,2,1,0.25	9.188	0.9853	1,0,2,1,0.10	9.999	0.9521
	6	2,0,2,1,0.55	9.765	0.9897	1,0,2,1,0.20	4.981	0.9872	1,0,2,1,0.55	9.765	0.9897	1,0,2,1,0.10	9.999	0.9866
	8	3,0,2,1,0.85	14.987	0.9895	1,0,2,1,0.35	4.991	0.9899	2,0,2,1,0.65	19.991	0.9892	1,0,2,1,0.20	9.999	0.9882
	10	6,0,2,1,0.80	29.999	0.9895	1,0,2,1,0.65	4.997	0.9899	3,0,2,1,0.80	29.999	0.9895	1,0,2,1,0.35	9.999	0.9890
0.10	2	5,1,2,1,0.15	24.334	0.9574	2,2,2,1,0.10	9.737	0.9712	3,1,2,1,0.10	29.704	0.9569	1,2,2,1,0.10	9.737	0.9712
	4	3,0,2,1,0.15	14.594	0.9865	1,0,2,1,0.10	4.956	0.9801	2,0,2,1,0.10	19.852	0.9877	1,0,2,1,0.10	9.999	0.9521
	6	3,0,2,1,0.40	14.891	0.9887	1,0,2,1,0.20	4.981	0.9872	2,0,2,1,0.30	19.961	0.9887	1,0,2,1,0.10	9.999	0.9866
	8	3,0,2,1,0.85	14.987	0.9895	1,0,2,1,0.35	4.991	0.9899	2,0,2,1,0.65	19.991	0.9892	1,0,2,1,0.20	9.999	0.9882
	10	6,0,2,1,0.80	29.999	0.9895	1,0,2,1,0.65	4.997	0.9899	3,0,2,1,0.80	29.999	0.9895	1,0,2,1,0.35	9.999	0.9890
0.05	2	6,1,2,1,0.10	29.704	0.9569	3,3,2,1,0.10	14.958	0.9704	3,1,2,1,0.10	29.704	0.9569	2,4,2,1,0.10	19.994	0.9708
	4	4,0,2,1,0.10	19.852	0.9877	1,0,2,1,0.10	4.956	0.9801	2,0,2,1,0.10	19.852	0.9877	1,0,2,1,0.10	9.999	0.9521
	6	4,0,2,1,0.30	19.961	0.9886	1,0,2,1,0.20	4.981	0.9872	2,0,2,1,0.30	19.961	0.9886	1,0,2,1,0.10	9.999	0.9866
	8	4,0,2,1,0.65	19.991	0.9892	1,0,2,1,0.35	4.991	0.9899	2,0,2,1,0.65	19.991	0.9892	1,0,2,1,0.20	9.999	0.9882
	10	6,0,2,1,0.80	29.999	0.9895	1,0,2,1,0.65	4.997	0.9899	3,0,2,1,0.80	29.999	0.9895	1,0,2,1,0.35	9.999	0.9890
0.01	2	10,2,2,1,0.10	49.971	0.9637	4,4,2,1,0.10	19.994	0.9708	5,2,2,1,0.10	49.971	0.9637	2,4,2,1,0.10	19.994	0.9708
	4	6,0,2,1,0.10	29.994	0.9806	2,0,2,1,0.10	9.999	0.9521	3,0,2,1,0.10	29.994	0.9806	1,0,2,1,0.10	9.999	0.9521
	6	6,0,2,1,0.20	29.997	0.9885	2,0,2,1,0.10	9.999	0.9866	3,0,2,1,0.20	29.997	0.9885	1,0,2,1,0.10	9.999	0.9866
	8	6,0,2,1,0.45	29.999	0.9889	2,0,2,1,0.20	9.999	0.9882	3,0,2,1,0.45	29.999	0.9889	1,0,2,1,0.20	9.999	0.9882
	10	6,0,2,1,0.80	29.999	0.9895	2,0,2,1,0.35	9.999	0.9890	3,0,2,1,0.80	29.999	0.9895	1,0,2,1,0.35	9.999	0.9890

4. COMPARISONS OF PLANS

In order to show the efficiency of the proposed plan compared with the single group sampling plan when the lifetime of the product follows the Weibull distribution and GE distribution, we determine the sample size for reference plan by using $r = 5, a = 0.5$, $\beta = 0.25$ and shape parameter 2 and placed in Table 6.

Table 6
Comparison of Plans in sample size when $r = 5, a = 0.5, \beta = 0.25$

Median Ratio	Proposed Plan (ASN)	Weibullwith γ=2 (n)	GE with δ=2 (n)
2	18.266	40	35
4	9.257	20	15
6	9.257	10	15
8	9.655	10	10
10	9.796	10	10

From Table 6, we can see that for the entire median ratio, the proposed SkGSP-V sampling plan provides the lesser ASN than the single group sampling plan under both the Weibull distribution and GE distribution. This shows that the SkGSP-V sampling plan is more economical than the single group sampling plan.

5. APPLICATION OF PROPOSED PLAN

This section discusses the applications of the proposed plan under both Weibull and GE distributions.

5.1 Weibull Distribution

Suppose that a manufacturer of the energy saver bulbs would like to adopt an SkGSP-V plan to decide whether to accept or reject the lot of submitted products. The minimum mean life required for the product is $\theta_0 = 8000$ hrs. Thus a lot should be accepted if there is enough evidence that the true mean life of a product exceeds 8000 hrs. The producer's risk is opted as $\alpha = 5\%$ when the true mean life is 16000 and the consumer's risk is chosen as β=0.25when the true mean life is 8000. Now $\theta_w / \theta^0 = 2$. A time truncated life test using the capacity of 5 items at a tester is conducted. The test time duration is limited by 4000 hrs. That is $a = 0.5$. The life time the product is assumed to follow a Weibull distribution. The failure data of 10 products are obtained from previous lots as 507, 720, 892, 949, 1031, 1175, 1206, 1428, 1538, 1983[Aslam et al. (2011)]. The maximum likelihood estimator of the shape parameter is obtained as $\hat{\gamma} = 2.87$ and we assume that γ=3. So for the specified requirements such as $r = 5$, $\gamma = 3$, $a = 0.5$, $\theta_w / \theta^0 = 2$, $\alpha = 5\%$ and $\beta = 10\%$, the optimal parameters of the SkGSP-V plan are determined from Table 3 as $g = 5$, $c = 0$, $i = 2$, $k = 1$ and $f = 0.1$.

5.2 Generalized Exponential Distribution

Suppose that an experimenter wants to implement the group sampling plan with $r=5$ to make a decision about acceptance or rejection of the submitted products. The specified median life of the product is $\theta^0=1000$ and the test duration is 1000 hrs. The producer's risk is 5% at $\theta_g/\theta^0=2$ and the consumer's risk is $\beta=10\%$. We consider the data provided by Wood (1996) which represents the failure time of software in hours. The failure time represents the time from the starting of the execution of the software until the software is failed. The data are 519, 968, 1430, 1893, 2490, 3058, 3625, 4422 and 5218. Aslam et al. (2010) have shown that the generalized exponential distribution is a good fit to the abovementioned data. The maximum likelihood estimator of the parameters δ and λ are 2.65 and 0.6547 respectively. Now let us assume that $\hat{\delta}=3$. For specified $r=5$, $a=1$, the optimal parameters of the proposed plan can be determined from Table 5 as $g=2$, $c=2$, $i=2$, $k=1$ and $f=0.1$. For this obtained plan parameters, the SkGSP-V plan is operated as follows.

(1) At the outset, start with normal inspection using the group sampling plan with parameters (2, 2) as the reference plan. During the normal inspection, lots are inspected one by one in the order of production or in the order of being submitted to inspection. From each lot under inspection, select a random sample of size $n=10$ and allocate 5 items to each of 2 groups (or testers) and put them on test for the time duration of 1000 hrs. Accept the lot if the total number of failures from 2 groups is smaller than or equal to 2. Truncate the test and reject the lot as soon as the total number of failures reaches 2 by the time 1000 hrs.

(2) When 2 consecutive lots are accepted on normal inspection, discontinue the normal inspection and switch to skipping inspection.

(3) During skipping inspection, inspect only a fraction $1/10$ of the lots selected at random. Skipping inspection is continued until sampled lot (using single group sampling plan as the reference plan) is rejected.

(4) When a lot is rejected on skipping inspection before 1 sampled lot is accepted, revert to normal inspection as per (1) above.

(5) When a lot is rejected after 1 lot has been accepted revert to normal inspection with reduced clearance number 1 as per (6) given below.

(6) During normal inspection with clearance number 1, lots are inspected one by one in the order of being submitted to inspection and continue the inspection until a lot is rejected or 1 lot is accepted whichever occurs earlier.

(7) When a lot is rejected, immediately revert to normal inspection with clearance number 2 as per (1) given above.

(8) When 1 lot is accepted, discontinue normal inspection and switch to skipping inspection as per (3) above.

(9) Replace or correct all the non-conforming units found with conforming units in the rejected lots.

6. CONCLUSIONS

In this paper, we have proposed an SkSP-V sampling plan with group sampling plan as the reference plan and the new plan is designated as SkGSP-V. Tables have also been constructed for the easy selection of the optimal plan parameters when the lifetime of the product follows the Weibull distribution and GE distribution. The proposed plan performs better than the reference plan in terms of sample size. The tables given in the paper can be used to select the plan parameters of SkGSP-V plan for testing of electronic products. The proposed plan can be extended by using two stage group sampling plan as the reference plan as future research.

ACKNOWLEDGMENTS

The authors are deeply thankful to the three reviewers and the editor for their valuable suggestions to improve the quality of the manuscript.

REFERENCES

1. Aslam, M. and Jun, C.-H. (2009). A group acceptance sampling plan for truncated life test having Weibull distribution. *J. App. Statist.*, 39(9), 1021-1027.
2. Aslam, M., Jun, C.-H., Lee, H., Ahmad, M. and Rasool, M. (2011). Improved group sampling plans based on truncated life tests. *The Chilean J. Statist.*, 2(1), 85-97.
3. Aslam, M., Jun, C.-H, and Ahmad, M. (2009). A group sampling plan based on truncated life tests for gamma distributed items. *Pak. J. Statist.*, 25(3), 333-340.
4. Aslam, M., Kundu, D. and Ahmad, M. (2010).Time truncated acceptance sampling plan for generalized exponential distribution. *J. App. Statist.*, 37(4), 555-566.
5. Aslam, M., Balamurali, S., Jun, C.-H and Ahmad, M. and Rasool, M. (2012). An optimal design of a Skip lot sampling plans of type V by minimizing average sample number. *Pak. J. Statist.*, 28(1), 131-140.
6. Balamurali, S. and Jun, C.-H. (2011). A New System of Skip-lot Sampling Plans Having a Provision for Reducing a Normal Inspection. *Applied Stochastic Models in Business and Industry*, 27(3), 348-363
7. Balamurali, S., Jun, C.-H. and Von Collani, E. (2007). Skip-lot sampling and chain sampling. *Encyclopaedia of Statistics in Quality and Reliability*. Wiley: Berlin.
8. Bennett, G.K. and Callejas, C.J. (1980). The economic design of skip-lot sampling plans. *Transactions on Spring Annual Conference*, 349-355.
9. Carr, W.E. (1982). Sampling plan adjustment for inspection error and skip-lot Plan. *Journal of Quality Technology*, 14(1), 10-18.
10. Cox, D.C. (1982). Skip-lot sampling plans. *Quality*, 21(8), 26-27.
11. Dodge, H.F. (1995). Skip-lot sampling plan. *Industrial Quality Control*, 11(5), 3-5. (Also reprinted in *Journal of Quality Technology* 9(3), 143-145).
12. International Organization for Standardization (2005). ISO 2859-3. Sampling procedures for inspection by attributes–part 3: Skip-lot sampling procedures, International Organization for Standardization, Geneva.
13. Jun, C.-H., Balamurali, S. and Lee, S.-H. (2006). Variables sampling plans for Weibull distributed lifetimes under sudden death testing. *IEEE Transactions on Reliability*, 55(1), 53-58.

14. Liebesman, B.S. (1987). The Development of an Attribute Skip-lot Sampling Standard. *Frontiers in Statistical Quality Control*, 3, 3-33. (Edited by Lenz HJ *et al*.).
15. Liebesman, B.S. and Saperstein, B. (1983). A proposed attribute skip-lot sampling program. *Journal of Quality Technology*, 15(3), 130-139.
16. Parker, R.D. and Kessler, L. (1981). A modified skip-lot sampling plan. *Journal of Quality Technology*, 13(1), 31-35.
17. Perry, R.L. (1973). Skip-lot sampling plans. *Journal of Quality Technology*, 5(3), 123-130.
18. Schilling, E.G. (1982). *Acceptance Sampling in Quality Control*. Marcel Decker: New York.
19. Schneider, H. and Wilrich, P-Th. (1981). *The Efficiency of the Switching Rules of the Military Standard 105D and the Skip-Lot Sampling System–Some Results of a Simulation Study*. Disk.-Arb. 5/81, Inst. f. Quant. Ok. U. Statistik, Freie Universitat Berlin, Berlin.
20. Wood, A. (1996). Predicting software reliability. *IEEE Trans. Softw. Eng.*, 22, 69-77.

A REPETITIVE TYPE OF GROUP ACCEPTANCE SAMPLING PLAN FOR ASSURING THE PERCENTILE LIFE*

Muhammad Aslam[1], **Chi-Hyuck Jun**[2] and **Munir Ahmad**[3]

[1] Department of Statistics, Forman Christian College University, Lahore, Pakistan. Email: aslam_ravian@hotmail.com

[2] Department of Industrial and Management Engineering, POSTECH, Pohang 790-784, Republic of Korea. Email: chjun@postech.ac.kr

[3] National College of Business Administration & Economics, Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

In this paper, repetitive types of group acceptance sampling plans are proposed when the lifetime of the product follows the Weibull distribution or the generalized exponential distribution. Quality characteristics are considered in terms of percentile lifetimes. Plan parameters are found by satisfying the producer's risk and consumer's risk at the same time while minimizing the average sample number. Extensive tables are given for practical use. Two examples are given to illustrate the proposed plan in real world.

KEY WORDS

Repetitive acceptance sampling plan; the Weibull distribution; the generalized exponential distribution; group sampling plan.

1. INTRODUCTION

With the availability of latest machinery to manufacture the products, various quality activities are required in industries. Quality assurance is to guarantee the high quality of the products during the manufacturing the products. Quality management systems are organizational efforts including policies, plans and supporting infrastructure for quality management. Hazard analysis and critical control points have been used in food industry to minimize the dangers. Six sigma techniques are used to reduce the causes of defects. Even control charts are also available to help the producer and consumer to maintain the quality of products according to specifications. The methods described above are not helpful to producers when he want to inspect every incoming part to ensure quality. Producers want to avoid the heavy losses in time, cost and reputation in market. At this stage, when the final products are ready, only the acceptance sampling tools help the producer to ensure the quality of the product and maintain the competitiveness in the market.

Producers normally manufacture the product under the same environment and machinery, so the quantity of products built up under the same conditions is known as a lot of the product. For inspection of the product a random sample is selected from this lot. As the decision is made on the basis of a random sample selected from the lot, there is a chance that a bad lot is accepted or a good lot is rejected. These probabilities are known as type-1 error (producer's risk) and type-2 error (consumer's risk), respectively. More

*Published in Pak. J. Statist. (2013), Vol. 29(4).

efficient sampling plans not only reduce the cost of the inspection but provide the protection to consumer and producer by minimizing these risks. It is very true that many plans are available in the literature including single sampling plans, double sampling plans and sequential plans but there is still need to study plans which can reduce the sample size further for the inspection purposes. A repetitive sampling plan is proposed by Sherman (1965), who argued that this plan provides the optimal sample size as compared to single sampling plan and its operation is similar to the sequential sampling plans. Later, Balamurali and Jun (2006) developed the variable repetitive plans for the normal distribution. They verified the Sherman's argument of efficiency.

As mentioned earlier, the main purpose of an acceptance sampling plan is to provide the optimal parameters such as sample size to save the cost and time of the experiment. In single plans, a single item is installed to a single tester, which requires the much time, efforts and increased cost of the experiment. On the other hand, group plans are implemented when the experimenter has the facility to put multiple items in a single tester. So, these plans can save the cost and time of the experiment than the ordinary sampling plans. Group acceptance sampling plans have been proposed by Aslam and Jun (2009), Aslam et al. (2010). Lio et al. (2010) proposed the ordinary acceptance sampling plan using the Burr type XII percentiles. Recently, Aslam et al. (2011) proposed the improved single and two stages group plans for the Weibull distribution. Other applications of group sampling plans can be seen in Aslam et al. (2012) and Aslam et al. (2013).

The Weibull distribution is widely used in the area of acceptance sampling plans and reliability analysis. Recently, Aslam and Jun (2009) proposed the group acceptance sampling plans using the Weibull distribution. Also, the generalized exponential distribution is a possible alternative of the Weibull and gamma distributions. Gupta and Kundu (1999) originally derived the generalized exponential distribution with two parameters. More recently, Aslam et al. (2010) developed the ordinary acceptance sampling plans using the generalized exponential distributions. Aslam et al. (2011) proposed the two group plans using the generalized exponential distribution. Properties of the generalized exponential distribution can be seen in Gupta and Kundu (2007).

According to author's best knowledge, no attempt has been made to study improved group sampling plans using the repetitive scheme for the time truncated experiment for assuring the percentile life as quality parameter. According to Lio et al. (2010) sampling plans based on the population mean may not catch the specific percentile of product lifetime required for engineering design considerations. When the quality of interest is a low percentile, a sampling plan based on the mean could accept the lot having the low percentile below the pre-specified standard required by the customer. In this paper, we will propose a repetitive group sampling plan for the Weibull and the generalized exponential distributions using the percentiles life as quality parameter. In summary, the main objective of this paper is two-fold: one is to improve a group sampling plan by incorporating the repetitive scheme and the other is to develop a sampling plan for assuring the percentile life. The rest of the paper is set as: The design of the proposed plan under the two distributions is given in Section 2. Some examples are given in Section 3. In the last section, concluding remarks are given.

2. THE REPETITIVE TYPE OF GROUP ACCEPTANCE SAMPLING PLANS

To design the proposed plan we will use the percentile life including the median as the quality parameter. According to Gupta (1962), for the skewed distribution the median represents a better quality parameter than the mean. Let θ be the quality parameter of interest (such as p-th percentile life) of a certain product. We want to formulate the null and alternative hypotheses $H_0: \theta \geq \theta^0$ and $H_1: \theta < \theta^0$. Here θ^0 is the specified value. The submitted lot of products is considered to be good if null hypothesis is accepted on the basis of information obtained from the sample selected from the lot and rejected if the sample information does not support it. These hypotheses can be tested using the following proposed plan:

Step 1 Take a random sample of size n from a lot and distribute r items to g groups so that $n = r \times g$ and put them on life test for a fixed experiment time t_0.

Step 2 Accept the lot if the total number of failures from all groups, D, is smaller than or equal to c_1. Truncate the experiment and reject the lot as soon as the number of failures exceeds c_2.

Step 3 If the total number of failures from all groups is between c_1 and c_2 ($c_1 < D \leq c_2$), then go to Step-1 and repeat the experiment.

There are three parameters of the above mentioned plan namely, g, c_1 and c_2. Here, r is the prespecified value, which depends on the type of testers. The experiment time t_0 is also specified in terms of the multiple of the target percentile life. The proposed plan is the extension of some existing acceptance sampling plans. This plan reduces to the attribute repetitive sampling plan considered by Sherman (1965) if $r = 1$ and it reduces to the group acceptance sampling plan if $c_1 = c_2$.

Let F be the cumulative distribution function (cdf) of the underlying distribution. Then the probability that an item fails before the end of the experiment time t_0 is given by

$$p = F(t_0) \tag{1}$$

For a particular distribution such as Weibull or generalized exponential distribution, p can be expressed by the quality characteristic of interest like p-th percentile in our case. The experiment duration will be expressed by the multiple of the specified percentile value.

The operating characteristics function for the repetitive group plan given by Sherman (1965) is as

$$P_A(p) = \frac{P_a}{P_a + P_r}, 0 < p < 1. \tag{2}$$

In Eq. (2), P_a is the probability of lot acceptance based on a single sample and P_r is the probability of lot rejection on the basis of a single sample. These probabilities are given as

$$P_a = \sum_{i=0}^{c_1} \binom{rg}{i} p^i \, (1 - p^i)^{rg-1} \tag{3a}$$

$$P_r = 1 - \sum_{i=0}^{c_2} \binom{rg}{i} p^i \,(1-p^i)^{rg-1} \tag{3b}$$

As mentioned above, three parameters are involved in the proposed plan. To find these three parameters we will consider two points on operating characteristics (OC) curve. We will specify the producer's and consumer's risks such that lot acceptance probabilities at these two points equal to these specified risks.

Producer desires that the lot acceptance probability should be larger than his confidence level $1-\alpha$ and consumer's demands that lot acceptance probability should be less than or equal to his risk β. Let the acceptable quality level (AQL) be p_1 and the limiting quality level (LQL) be p_2, which are the specified values of failure probability of an item corresponding to producer's and consumer's risks, respectively.

It is important to note that we may have several combinations which satisfy the above mentioned conditions. So, we should select the combination of the plan parameters which leads to the smallest average sample number (ASN) among all the existing plan parameters. We expect that the ASN at the LQL is larger than the ASN at the AQL, so it would be reasonable to minimize the ASN at the LQL. The ASN for our proposed repetitive type of group acceptance sampling plan is given by

$$ASN = \frac{rg}{1 - \sum_{i=0}^{c_2} \binom{rg}{i} p^i (1-p)^{rg-i} + \sum_{i=0}^{c_1} \binom{rg}{i} p^i (1-p)^{rg-i}} \tag{4}$$

Then, for specified values of AQL and LQL, the design parameters (g, c_1, c_2) of the proposed plan can be found using the following optimization problem.

Minimize *ASN*

Subject to

$$P_A(p_1) \geq 1-\alpha \tag{5a}$$

$$P_A(p_2) \leq \beta \tag{5b}$$

$$g > 1 \tag{5c}$$

2.1 Proposed Plan under the Weibull Distribution

Suppose that life time of a submitted product follows the Weibull distribution with the following cumulative distribution function (cdf).

$$F(t) = 1 - \exp(-(t/\lambda)^{\gamma}),\ t \geq 0 \tag{6}$$

where γ is the known shape parameter and λ is the unknown scale parameter. If the shape parameters are unknown it can be estimated from the previous failure time data. It is important to note that the cdf of the Weibull distribution depends on λ only through t/λ. The p-th percentile of the Weibull distribution is given by

$$\theta_p^W = \lambda \left(\ln\left(\frac{1}{1-p}\right)\right)^{1/\gamma} \quad (7)$$

The failure probability in Eq. (1) for the Weibull distribution is obtained by

$$p_W = 1 - \exp\left(-\left(t_0 / \lambda\right)^{\gamma}\right) \quad (8)$$

It is more convenient to express the experiment time t_0 as the multiple of specified percentile life θ^0 and to express λ in terms of θ^0. Here $t_0 = a\theta^0$ is the experiment time. So, the failure probability in Eq. (8) becomes

$$p_W = 1 - \exp\left[-a^{\gamma}\left(\theta_p^W/\theta^0\right)^{-\gamma} \ln\left(\frac{1}{1-p}\right)\right] \quad (9)$$

Consumer wants the percentile life of the submitted product to be longer than the specified life θ^0. So, the consumer's risk may be evaluated at $\theta_p^W = \theta^0$. As acceptance sampling plans asserts pressure on producer to enhance the quality level of his product, producer wants the plan which not only provides him the protection from rejecting good lots but also tell him the probability of acceptance at various quality levels. So we will use various percentile ratios (that is, θ_p^W / θ^0) when considering the producer's risk.

Tables 1-3 are constructed under the Weibull distributions with γ =2 for 10%, 20% and 50% percentiles. Many combinations are considered: two cases for the group size (r=5, 10), two cases for the termination ratio (a=0.5, 1.0) and two cases for the consumer's risk (β =0.25, 0.10). Also, three cases of the percentile ratio are considered (θ_p^W / θ^0 =2, 4, 6). We chose the ratio like this just for convenience. Any value of parameter ratio can be selected to find the plan parameters of the proposed plan. Ratio=1 means that the true percentile life is just same as the target percentile life to be assured. So, for lower ratio cases the acceptance conditions (acceptance numbers and sample size) become larger. An EXCEL program is available from the authors upon request. The producer's risk is fixed as 0.05.

Table 1: Proposed plans under the Weibull (γ =2) for 10% percentiles

β	θ_p^W / θ^0	$r=5$				$r=10$			
		$a=0.5$		$a=1.0$		$a=0.5$		$a=1.0$	
		(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN
0.25	2	0,2,17	174.5	1,2,6	38.8	0,2,9	176.6	2,3,5	58.0
	4	0,1,14	99.4	0,1,4	27.4	0,1,7	99.4	0,1,2	27.4
	6	↑	↑	↑	↑	↑	↑	↑	↑
0.10	2	1,3,34	239.4	1,3,9	62.2	1,3,17	239.4	2,4,6	76.7
	4	0,1,20	123.7	0,1,5	31.2	0,1,10	123.7	0,1,3	34.9
	6	↑	↑	↑	↑	↑	↑	↑	↑

(note) (↑) shows the same values of plan parameters apply as in the above cell.

Table 2: Proposed plans for the Weibull (γ =2) for percentiles 20%

β	θ_p^W / θ^0	$r=5$				$r=10$			
		$a=0.5$		$a=1.0$		$a=0.5$		$a=1.0$	
		(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN
0.25	2	0,2,8	83.7	0,2,2	23.3	0,2,4	83.7	0,2,1	23.3
	4	0,1,7	48.9	0,1,2	13.7	0,1,4	53.1	0,1,1	13.7
	6	↑	↑	↑	↑	↑	↑	↑	↑
0.10	2	1,3,16	114.1	0,3,3	38.8	1,3,8	114.1	1,4,3	39.7
	4	0,1,10	60.7	0,1,3	17.3	0,1,5	60.7	0,1,2	21.2
	6	↑	↑	↑	↑	↑	↑	↑	↑

(note) (↑) shows the same values of plan parameters apply as in the above cell.

Table 3: Proposed plans for the Weibull (γ =2) for percentiles 50%

β	θ_p^W / θ^0	$r=5$				$r=10$			
		$a=0.5$		$a=1.0$		$a=0.5$		$a=1.0$	
		(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN
0.25	2	0,2,3	29.4	0,3,1	22.9	0,3,2	46.7	2,5,1	23.2
	4	0,1,3	19.0	0,1,1	5.9	0,1,2	22.7	1,2,1	10.5
	6	↑	↑	↑	↑	↑	↑	0,1,1	↑
0.10	2	0,3,4	46.7	1,5,2	25.8	0,3,2	46.7	1,5,1	25.8
	4	0,1,3	19.0	0,1,1	5.9	0,1,2	22.7	1,2,1	10.5
	6	↑	↑	↑	↑	↑	↑	0,1,1	↑

(note) (↑) shows the same values of plan parameters apply as in the above cell.

From these tables we can observe the following trends: as the percentile changes from 10% to 50% while other conditions remain the same, the number of groups reduces. For example, when β =0.25, θ_p^W / θ^0 =2, a=0.5 and r=5, the number of groups required reduces from 17 to 8 as the percentile changes from 10% to 20%.

Similarly, Tables 4-6 are constructed under the Weibull distributions with γ =3 for 10%, 20% and 50% percentiles.

Table 4: Proposed plans for the Weibull (γ =3) for percentiles 10%

β	θ_p^W / θ^0	$r=5$				$r=10$			
		$a=0.5$		$a=1.0$		$a=0.5$		$a=1.0$	
		(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN
0.25	2	0,1,27	193.5	0,1,4	27.4	0,1,14	198.2	0,1,2	27.4
	4	↑	193.5	↑	27.4	↑	↑	↑	↑
	6	↑	↑	↑	↑	↑	↑	↑	↑
0.10	2	0,2,43	362.1	1,2,8	46.6	0,2,22	363.3	1,2,4	46.6
	4	0,1,39	243.2	0,1,5	31.2	0,1,20	247.0	0,1,3	34.9
	6	↑	↑	↑	↑	↑	↑	↑	↑

(note) (↑) shows the same values of plan parameters apply as in the above cell.

Table 5: Proposed plans for the Weibull (γ =3) for percentiles 20%

β	θ_p^W/θ^0	$r=5$				$r=10$			
		$a=0.5$		$a=1.0$		$a=0.5$		$a=1.0$	
		(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN
0.25	2	0,1,13	92.9	0,1,2	13.7	0,1,7	97.4	0,1,1	13.7
	4	↑	↑	↑	↑	↑	↑	↑	↑
	6	↑	↑	↑	↑	↑	↑	↑	↑
0.10	2	0,2,21	172.8	1,2,4	23.2	1,2,15	173.7	1,2,2	23.2
	4	0,1,19	117.3	0,1,3	17.3	0,1,10	121.0	0,1,2	21.2
	6	↑	↑	↑	↑	↑	↑	↑	↑

(note) (↑) shows the same values of plan parameters apply as in the above cell.

Table 6: Proposed plans for the Weibull (γ =3) for percentiles 50%

β	θ_p^W/θ^0	$r=5$				$r=10$			
		$a=0.5$		$a=1.0$		$a=0.5$		$a=1.0$	
		(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN
0.25	2	0,1,5	33.8	0,2,1	9.4	1,2,4	50.0	1,2,3	11.3
	4	↑	33.8	0,1,1	5.9	0,1,3	37.6	0,1,1	10.1
	6	↑	↑	↑	↑	↑	↑	↑	↑
0.10	2	0,2,7	57.1	0,2,1	9.4	1,2,5	57.6	1,2,3	11.3
	4	0,1,6	37.6	0,1,1	5.9	0,1,3	37.6	0,1,1	10.1
	6	↑	↑	↑	↑	↑	↑	↑	↑

(note) (↑) shows the same values of plan parameters apply as in the above cell.

From these tables, it can be seen that as the shape parameter increases from 2 to 3, the number of groups as well as the ASN increase when all other values remain the same. It is also observed that as the termination ratio increases from 0.5 to 1.0, we noted the decreasing trend in number of groups and ASN. It is also observed that as the percentile ratio increases from 2 to 10, the number of groups and the ASN decrease. It is also noted that when the percentile ratio is larger than two for all cases of consumer's risk and termination ratios, c_1=0 and c_2=1.

2.2 Under the Generalized Exponential Distribution

The cdf of the generalized exponential distribution is given by

$$F(t;\lambda,\delta)=\left(1-\exp(-t/\lambda)\right)^{\delta} \tag{10}$$

where δ is the known shape parameter and λ is the scale parameter of the generalized exponential distribution. Note that the p-th percentile of the generalized exponential distribution is given by

$$\theta_p^G=-\lambda\ln\left(1-p^{1/\delta}\right) \tag{11}$$

Again, it is more convenient to express the experiment time t_0 as the multiple of the specified percentile life θ^0 and to express λ in terms of θ^0. So, the probability of failure p in the generalized exponential distribution is given as

$$p_G = \left[1 - \exp\left(a \ln\left(1 - p^{1/\delta}\right) / \left(\theta_p^G / \theta^0\right)\right)\right]^{\delta} \qquad (12)$$

Tables 7-9 are constructed under the generalized exponential distribution with δ =2 for 10%, 20% and 50% percentiles. The table setting is just same as for the Weibull distributions.

Table 7: Proposed plans for the GED (δ =2) for percentiles 10%

β	θ_p^W / θ^0	$r=5$				$r=10$			
		$a=0.5$		$a=1.0$		$a=0.5$		$a=1.0$	
		(c_1, c_2, g)	ASN	(c_1, c_2, g)	ASN	(c_1, c_2, g)	ASN	(c_1, c_2, g)	ASN
0.25	2	0,2,15	152.1	1,3,7	59.2	2,3,15	180.6	1,3,4	60.9
	4	0,1,12	85.6	0,1,4	27.4	0,1,6	85.6	0,1,2	27.4
	6	↑	↑	↑	↑	↑	↑	↑	↑
0.10	2	1,3,29	206.9	2,4,12	76.7	0,3,11	241.8	2,4,6	76.7
	4	0,1,17	106.0	0,1,5	31.2	0,1,9	109.7	0,1,3	34.9
	6	↑	↑	↑	↑	↑	↑	↑	↑

(note) (↑) shows the same values of plan parameters apply as in the above cell.

Table 8: Proposed plans for the GED (δ =2) for percentiles 20%

β	θ_p^W / θ^0	$r=5$				$r=10$			
		$a=0.5$		$a=1.0$		$a=0.5$		$a=1.0$	
		(c_1, c_2, g)	ASN	(c_1, c_2, g)	ASN	(c_1, c_2, g)	ASN	(c_1, c_2, g)	ASN
0.25	2	2,3,13	79.6	0,2,2	23.3	2,3,7	83.6	0,2,1	23.3
	4	0,1,6	41.3	0,1,2	13.7	0,1,3	41.3	0,1,1	13.7
	6	↑	41.3	↑	13.7	↑	41.3	↑	13.7
0.10	2	0,3,13	110.4	0,3,3	38.8	0,3,5	110.4	0,4,2	52.4
	4	0,1,8	49.1	0,1,3	17.3	0,1,4	49.1	1,2,2	23.2
	6	↑	49.1	↑	17.3	↑	49.1	0,1,2	21.2

(note) (↑) shows the same values of plan parameters apply as in the above cell.

Table 9: Proposed plans for the GED (δ =2) for percentiles 50%

β	θ_p^W / θ^0	$r=5$				$r=10$			
		$a=0.5$		$a=1.0$		$a=0.5$		$a=1.0$	
		(c_1, c_2, g)	ASN	(c_1, c_2, g)	ASN	(c_1, c_2, g)	ASN	(c_1, c_2, g)	ASN
0.25	2	0,3,3	35.7	0,3,1	22.9	0,4,2	47.0	2,6,1	44.1
	4	0,1,2	13.3	0,2,1	9.4	0,1,1	13.3	2,3,1	11.3
	6	↑	↑	0,1,1	5.9	↑	↑	1,2,1	10.5
0.10	2	0,3,3	35.7	1,6,2	54.8	0,4,2	47.0	1,6,1	54.8
	4	0,1,3	17.0	0,2,1	9.4	1,2,2	22.7	1,2,3	11.3
	6	↑	↑	0,1,1	5.9	0,1,2	21.0	1,1,2	10.5

(note) (↑) shows the same values of plan parameters apply as in the above cell.

From these tables for the generalized exponential distribution with shape parameter 2 we can observe that the number of groups reduces when the percentile changes from 10% to 50%.

Similarly, Tables 10-12 are constructed under the generalized exponential distributions with δ =3 for 10%, 20% and 50% percentiles.

Table 10: Proposed plans for the GED (δ =3) for percentiles 10%

β	θ_p^W / θ^0	$r=5$				$r=10$			
		$a=0.5$		$a=1.0$		$a=0.5$		$a=1.0$	
		(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN
0.25	2	0,1,19	134.6	1,2,6	38.8	0,1,10	139.1	1,2,3	38.8
	4	↑	134.6	0,1,4	27.4	↑	↑	0,1,2	27.4
	6	↑	↑	↑	↑	↑	↑	↑	↑
0.10	2	0,2,30	247.0	1,2,8	46.6	0,2,15	247.0	1,2,4	46.6
	4	0,1,27	167.1	0,1,5	31.2	0,1,14	170.9	0,1,3	34.9
	6	↑	↑	↑	↑	↑	↑	↑	↑

(note) (↑) shows the same values of plan parameters apply as in the above cell.

Table 11: Proposed plans for the GED (δ =3) for percentiles 20%

β	θ_p^W / θ^0	$r=5$				$r=10$			
		$a=0.5$		$a=1.0$		$a=0.5$		$a=1.0$	
		(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN
0.25	2	0,1,8	57.1	1,2,3	19.5	0,1,4	57.1	0,2,1	23.3
	4	↑	57.1	0,1,2	13.7	↑	57.1	0,1,1	13.7
	6	↑	↑	↑	↑	↑	↑	↑	↑
0.10	2	1,2,18	104.8	1,3,5	31.5	1,2,9	104.8	0,3,2	33.3
	4	0,1,12	73.0	0,1,3	17.3	0,1,6	73.0	0,1,2	21.2
	6	↑	↑	↑	↑	↑	↑	↑	↑

(note) (↑) shows the same values of plan parameters apply as in the above cell.

Table 12: Proposed plans for the GED (δ =3) for percentiles 50%

β	θ_p^W / θ^0	$r=5$				$r=10$			
		$a=0.5$		$a=1.0$		$a=0.5$		$a=1.0$	
		(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN	(c_1,c_2,g)	ASN
0.25	2	1,2,4	25.2	0,3,1	22.9	1,3,3	38.7	1,2,5	23.2
	4	0,1,3	18.8	0,1,1	5.9	0,1,2	22.5	1,2,1	10.5
	6	↑	↑	↑	↑	↑	↑	0,1,1	10.1
0.10	2	1,3,6	38.7	1,5,2	25.8	1,3,3	38.7	1,5,1	25.8
	4	0,1,3	18.8	0,1,1	5.9	0,1,2	22.5	1,2,1	10.5
	6	↑	↑	↑	↑	↑	↑	0,1,1	10.1

(note) (↑) shows the same values of plan parameters apply as in the above cell.

If the shape parameter changes from 2 to 3, we note increasing trend in the number of groups for the generalized distribution case. It is observed that there is a decreasing trend in the number of groups and ASN as the termination time increases or the percentile ratio increases by keeping all the other values at the constant level. It is also noted that the plan parameters are determined by c_1 = 0 and c_2 = 1 at a larger level of percentile ratio for all cases of consumer's risk and termination time. It is interesting to note that the number of groups required decreases when the underlying distribution changes from the Weibull to generalized exponential distribution.

3. ILLUSTRATIVE EXAMPLES

In this section, we will give some examples to use the proposed plans in industry.

Example-1

Suppose that manufacturer wants to adopt the proposed group plan with $r = 5$ for assuring that the median life of the submitted products is at least 1000 hours when $\beta^* = 0.25$ and $\alpha = 0.05$ at the median ratio = 4. He wants to run this experiment 500 hours. The lifetime of the product is known to follow the Weibull distribution with unknown shape parameter. To estimate the shape parameter of the Weibull distribution we collected the failure data from 10 products of the previous lots as follows: 507, 720, 892, 949, 1031, 1175, 1206, 1428, 1538, 2083. Then, the maximum likelihood estimate (MLE) of the shape parameter is obtained by $\hat{\gamma}$ = 2.883 [Aslam et al. 2010]. So, let us assume that $\hat{\gamma}$ = 3.

From Table 6, plan parameters in case of the proposed group sampling are c_1=0, c_2=1 and g=5. This plan is implemented as: select 25 items from the lot and make 5 groups. Accept the lot if no failure occurs before 500 hours and reject if the total numbers of failures from 5 groups is larger than 1.The procedure is repeated if the number of failures is 1.

Example-2

Suppose that an experimenter wants to adopt the proposed sampling plan to decide about the acceptance or the rejection of the submitted lot of products. The specified 20% percentiles life of the product is θ^0 = 1000 hr and the test duration is 500 hours. The producer's risk is α = 0.05 at θ_p^G / θ^0 = 2 and the consumer's risk is β = 0.25. We have the following values; 519, 968, 1430, 1893, 2490, 3058, 3625, 4422, 5218. Let us assume that product under inspection follows the generalized exponential distribution with δ = 3 [Aslam et al. 2010].

For all above specified values, suppose that experimenter wants to adopt the proposed group plan when there is facility to install 5 items in single group. From Table 11, plan parameters in case of group sampling are c_1= 0, c_2= 1 and g = 8. This plan is implemented as: select 40 items from the lot and put them on test for 500 hours accept the lot if no failure occurs before 500 hours. If the number of the failures exceeds 1, then reject the lot. Take another sample of size 40 and proceed the procedure again when the number of failures is 1.

4. CONCLUDING REMARKS

We proposed the repetitive type of attributes group sampling plans for the Weibull distribution and generalized exponential distribution. Extensive tables are provided for practical point of view. It is concluded that the proposed plan are useful to save the cost and time of the experiment than the single acceptance sampling plans. We compared the results of the Weibull distribution and generalized exponential distribution for various values of the shape parameter. The proposed plans can be used for inspection of electronic product such as the computer devices, mobile devises, automobiles devices and software. The present approach can be extended for some other lifetime distribution as a future research. Developing of attribute repetitive plans using the cost model is fruitful area for future research.

ACKNOWLEDGMENTS

The authors are deeply thankful to the editor and the reviewer for their valuable suggestions to improve the quality of the manuscript. The work by Chi-Hyuck Jun was financially supported by Korea Ministry of Environment (MOE) as Eco design Human Resource Development Project.

REFERENCES

1. Aslam, M., Azam, M. and Jun, C.-H. (2012). Tightened-normal-tightened group acceptance sampling plan for assuring percentile life. *Industrial Engineering & Management Systems*, 11(4), 390-396.
2. Aslam, M., Balamurali, S., Jun, C.-H. and Ahmad, M. (2013). SkSP-V Sampling Plan with Group Sampling Plan as Reference based on Truncated Life Test under Weibull and Generalized Exponential Distributions. *Pak. J. Statist.*, 29(2), 217-230.
3. Aslam, M. and Jun, C.-H. (2009). A group acceptance sampling plan for truncated life test having Weibull distribution. *J. App. Statist.*, 36(9), 1021-1027.
4. Aslam, M., Ahmad, M. and Razzaque, A. (2010). Group Acceptance Sampling Plan for Lifetimes Having Generalized Pareto Distribution. *Pakistan Journal of Commerce and Social Sciences*, 4(2), 185-193.
5. Aslam, M., Jun, C.-H., Lee, H., Ahmad, M., and Rasool, M. (2011). Improved group sampling plans based on truncated life tests. *The Chilean J. Statist.*, 2(1), 85-97.
6. Aslam, M., Kundu, D. and Ahmad, M. (2010). Time truncated acceptance sampling plan for generalized exponential distribution. *J. App. Statist.*, 37, 555-566.
7. Aslam, M., Kundu, D. Jun, C.-H and Ahmad, M. (2011). Time Truncated Improved Group Acceptance Sampling Plans for Generalized Exponential Distribution. *Journal of Testing and Evaluation*, 39 (4), 671-677.
8. Balamurali, S. and Jun, C.-H. (2006). Repetitive group sampling procedure for variables inspection. *J. App. Statist.*, 33(3), 327-338.
9. Gupta, R.D. and Kundu, D. (1999). Generalized exponential distribution. *Aust. & N.Z. J. Statist.*, 41, 173-188.
10. Gupta, R.D. and Kundu, D. (2007). Generalized exponential distribution: existing methods and recent developments. *J. Statist. Plann. and Infer.*, 137, 3537-3547.
11. Gupta, S.S. (1962). Life test plans for normal and log-normal distributions. *Technometrics*, 4, 151-160.

12. Lio, Y.L., Tzong-Ru Tsai and Shuo-Jye Wu. (2010). Acceptance sampling plans from truncated life tests based on the Burr type XII percentiles. *Journal of the Chinese Institute of Industrial Engineers*, 27(4), 270-280.
13. Sherman, R.E. (1965). Design and evaluation of repetitive group sampling plan.*Technometrics*, 7, 11-21.
14. Suresh, K.K. and Divya, P.R. (2009). Selection of single sampling plan through decision region.*Int. J. App. Math. and Statist.*, 14(09), 66-78.
15. Suresh, K.K. and Saminathan, R. (2007). Selection of multiple repetitive group sampling plan involving maximum allowable percent defective and maximum allowable average outgoing quality. *International Journal of Statistics and Management Systems*, 2(1-2), 22-30.

CONTRIBUTION OF PAKISTAN JOURNAL OF STATISTICS IN 'STATISTICS AND PROBABILITY' LITERATURE: AN ANALYSIS BASED ON THE JOURNAL CITATION REPORT (2010)*

Faisal Qadeer[1] and **Munir Ahmad**[2]
National College of Business Administration and Economics, Lahore, Pakistan
Email: [1]mfaisalqr@hotmail.com; [2]drmunir@ncbae.edu.pk

ABSTRACT

Only twenty countries have impact factor journals in 'Statistics and Probability'. In this paper, we trace contributions of Pakistan Journal Statistics in publishing research papers in comparison to other countries. The analysis is based on Journals Citation Reports 2010 edition issued by the Thomson's Institute of Scientific Information. The paper provides country and publishing group level comparisons of the impact factors of journals in this subject. Of 20 countries, only two Islamic countries i.e. Pakistan and Turkey have one journal each in the 'Statistics and Probability' literature.

KEYWORDS

Journal Citation Reports, Impact Factor, Statistics and Probability, Pakistan.

1. INTRODUCTION

Pakistan Journal of Statistics (PJS) is now an Impact Factor (IF) journal. The years of hard work and teamwork of the journal officials have finally received recognition from the Institute of Scientific Information (ISI), now owned by Thomson Reuter Corporation. This institute maintains citation data basis like Science Citation Index (CSI) and Social Science Citation Index (SSCI). ISI publishes several reports. Journal Citation Reports (JCR) publishes annually and is an internationally recognized document widely used for ranking of journals in various subject categories. The 2010 editions of JCR provide citation information for 10,804 journals indexed in Thomson's SCI and SSCI.

The JCR database has been a regular source of generating information for analyzing various interesting issues. For example some authors describe submission in a particular journal from various parts of the world (Konradsen and Munk-Jørgensen 2007) and others evaluate contribution of different world regions in research production and quality in particular field (Bliziotis et al. 2005; Sorrentino et al,. 2000) or make cross field comparisons (Althouse et al,. 2009). Some other seems interested in studying the divide between First and Third world (Wishart and Davies 1998). The basic purpose of the paper is to highlight the contributions of Pakistan Journal of Statistics. It also describes the contributions of various countries and publishing houses in the subject category of 'Statistics and Probability'. The analyses have been based on JCR 2010.

Different methods and approaches are used for making comparisons among journals and countries. The critique in making comparisons of various methods and approaches in the usual biblometric academic debates is beyond the scope of this paper. Providing a

*Published in Pak. J. Statist. (2012), Vol. 28(3).

simple description of the subject category 'Statistics and Probability' and enabling the statistical scientists to have an overview of the academic contributors in field, making readily available list of IF journals along with courtiers of origin and publishers are also aims of this study.

2. IMPACT FACTOR BY COUNTRIES

A list of the country originating IF journals is presented in Table 1, Only 20 countries in the world have IF journals in the subject category of 'Statistics and Probability'. There are 110 journals in this category (Appendix A, B). An overwhelming majority (102) of journals is in English language and the remaining eight journals are multi-language journals. PJS, Pakistan is one of the journals that have also entered in the list of journals having IF. There is only one other Islamic country i.e. Turkey in the list. Off the 110 IF journals, 90 (about 82 per cent) originates from only four countries namely USA, England, Netherland and Germany having 37, 27, 14 and 12 journals respectively.

Table 1
Country of Origin of the Journals and Language
(JCR 2010-Statistics & Probability)

S#	Country of Origin	Language		Total	
		English	Multi-Language	No.	%
1	Australia	1	0	1	0.91
2	Belgium	2	0	2	1.82
3	Canada	1	1	2	1.82
4	Colombia	1	0	1	0.91
5	England	24	3	27	24.55
6	France	0	1	1	0.91
7	Germany	12	0	12	10.91
8	India	1	0	1	0.91
9	Japan	1	0	1	0.91
10	Netherlands	13	1	14	12.73
11	Pakistan	1	0	1	0.91
12	Portugal	1	0	1	0.91
13	Russia	1	0	1	0.91
14	Singapore	2	0	2	1.82
15	South Korea	1	0	1	0.91
16	Spain	2	0	2	1.82
17	Sweden	1	0	1	0.91
18	Taiwan	1	0	1	0.91
19	Turkey	1	0	1	0.91
20	USA	35	2	37	33.64
	Total	**102**	**8**	**110**	**100.00**

Figure 1 shows the number of journals for various countries. The countries having only one journal are grouped together in 'Other Countries' category including 'Islamic countries' (i.e. Pakistan and Turkey). There are only 7 journals originating from the countries of eastern hemisphere i.e. India, Japan, Pakistan, Russia, Singapore, South Korea and Taiwan. All the remaining journals originate from the western nations. As discussed above, in fact only 4 countries are dominant in this subject category. This would be interesting to make such a comparison in all subjects of Science and Social Sciences and see whether the position is similar to the subject of 'Statistics and Probability'. In the last part of this study this comparison is also reported.

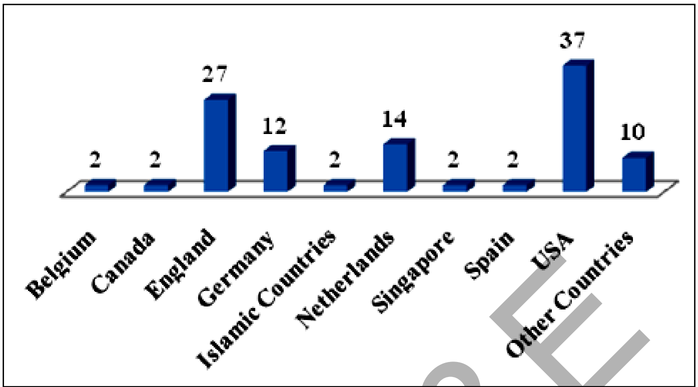


Fig. 1: Number of Journals by Country of Origin (JCR 2010-Statistics & Probability)

3. ANALYSIS OF JOURNAL RANKING

There are several metrics to rank journals. This study is based on 3-year IF of journals, which is the most widely used metric for evaluating the impact of peer-reviewed journals. The metric is defined as 'the ratio of citations in 1 calendar year to the number of citable items published in the previous 2 years (Bankhead 2010). Garfield (1955), the founder of ISI, proposed the bibliographic system for scientific literature. This ratio was then used to select the journals for inclusion in the SCI (Garfield, 1999). This bibliometric measure is a leading indicator of measuring journal influence on the academic literature. There has been strong criticism on the possible biases in calculating IF (see for example, Dong et al., 2005). Impact factor, as a valid indicator of measuring the quality of journals, has been regularly discussed (Campanario 2011; Gomez-Sancho & Mancebon-Torrubia, 2009; Rossner et al., 2008; and Yu et al. 2010). However, these issues are beyond the scope of this paper.

Figure 2 shows the comparison of various countries/regions of the world in terms of originating IF journal. The dominance of the 4 countries, with respect to journal IF, is very much evident. The mean IF of the journals originating from England, USA, Netherland and Germany are 1.28, 1.26, 0.88 and 0.66 respectively followed by Canada (0.65), Spain (0.65) and Japan (0.64). The means IF for the 10 countries grouped as 'Other Countries' is 0.53. Mean IF for the Islamic Countries is only 0.28 i.e. the lowest among all other rations. Islamic countries (Pakistan and Turkey) have recently entered this list. The chances of reaching to the international audience have been very few for the journals originating from these countries. This may be one of the possibilities for this low IF. However, after a few years this would be more appropriate to analyze this point for a more rational finding of this explanation. For the time being this may be inferred that the two journals originating from Islamic Countries are far less cited in ISI journals of the world.

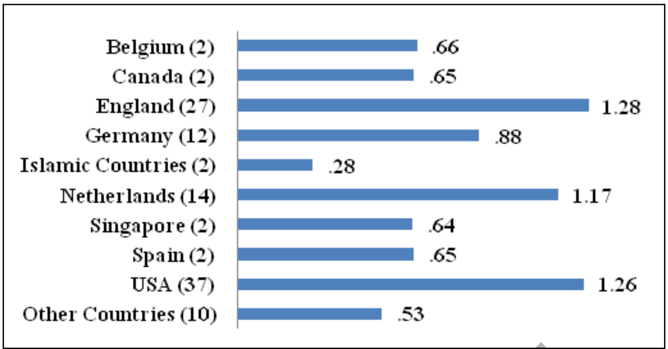


Fig. 2: Mean IF of Journals by Country of Origin (JCR 2010-Statistics & Probability)

Publishing groups usually handle online availability, sales/marketing of the research papers of various journals to the international audience. Each publisher prefers to have a variety of journals on all important disciplines. Editing an academic journal is a concentrated and time-consuming job that an editor could not pay attention to professional handling of marketing of the journals. The publishing business like all other businesses has become very competitive, dynamically changing, creative and capital intensive. A journal published by leading publishing house is likely to increase the standardization, worth of the journal and reach to all parts of the world.

The cross-tabulation of the publishing group by country of origin of the journals in ‘Statistics and Probability’ is presented in Table 2. There are twelve publishing groups that are publishing at least two IF journals in this discipline. The leading publishing groups in this regard are Wiley-Blackwell, Springer, Taylors and Francis, Elsevier, Institute of Mathematical Statistics, and American Statistical Association.

There may be several reasons for publishing groups to have publishing rights of large number of journals. However, this may not be related to the mean IF of all journals of that publisher. The top three publishers of ‘Statistics and Probability’ with highest mean IF are Oxford University Press (2.30), Institute of Mathematical Statistics (1.65), and American Statistical Association (1.50). This may be relevant to mention here that these are academic organizations dedicated to Statistics rather than publishing groups. The top four publishers in terms of the number of journals are Willey-Blackwell (20), Springer (20), Taylor and Francis (14) and Elsevier (7). These are mainly publishing groups and not statistical specific publishes. This may be interesting to note here that none of the publishers with highest number of journals is included in the top three publishers in term of mean IF. The complete comparative analysis is provided in Figure 3. There are 21 Publishing houses which publish only one IF journal. These are converged under the heading ‘Other Publishers’.

Table 2
Publishing Group and Country of Origin of the Journals
(JCR 2010-Statistics & Probability)

Publishing Group (IF)	Country of Origin										Total
	Belgium	Canada	England	Germany	Islamic Countries	Netherlands	Singapore	Spain	USA	Other Countries	
1. American Statistical Association									5		**5**
2. Applied Probability Trust			2								**2**
3. Cambridge University Press									3		**3**
4. Elsevier			1			7			1		**9**
5. Institute of Mathematical Statistics									6	1	**7**
6. Oxford University Press			2								**2**
7. Sage Publications Ltd			2								**2**
8. Springer	1			10		4		1	3	1	**20**
9. Taylor & Francis			5	1					8		**14**
10. University of Washington									2		**2**
11. Wiley-Blackwell			15	1		1			2	1	**20**
12. World Scientific Publication Company						1	2				**3**
Others (1 each)	1	2			2	1		1	7	7	**21**
Total	**2**	**2**	**27**	**12**	**2**	**14**	**2**	**2**	**37**	**10**	**110**

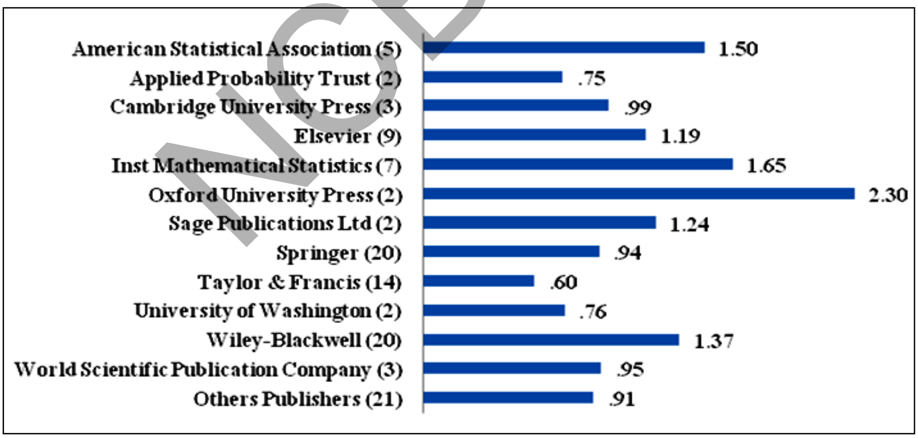


Fig. 3: Mean IF of Journals by Publishers

As discussed above that all IF journals in the discipline of 'Statistics and Probability' originates from only 20 countries. It would be interesting to compare the overall IF journals originating from these countries in other Sciences and Social Sciences Subjects to that of ‘Statistics and Probability’. For subjects in *Social Science*, there are 2,731 IF journals originating from 52 countries. About 91% (2,494) of these journals originates from the 20 countries which originates IF journals in ‘Statistics and Probability’. For subjects in *Science*, there are 8,073 IF journals originating from 84 countries. About 83% (6,702) of these journals originates from the 20 countries. For both Social Sciences and Sciences, there are 10,804 IF journals. About 85% (9,196) of these journals originates from the 20 countries (Table 3).

Table 3
A Comparison of Contribution of the Twenty Countries in ‘Statistics and Probability’ and other Subject (JCR 2010)

Country of Origin	Social Science Subjects		Science Subjects (SS)		All Subjects		Statistics & Probability (SP)		SP/SS
	Freq	%	Freq	%	Freq	%	Freq	%	
Australia	85	3.1	132	1.6	217	2.0	1	0.9	0.76
Belgium	8	0.3	21	0.3	29	0.3	2	1.8	9.52
Canada	26	1.0	94	1.2	120	1.1	2	1.8	2.13
Colombia	6	0.2	15	0.2	21	0.2	1	0.9	6.67
England	720	26.4	1,570	19.4	2,290	21.2	27	24.6	1.72
France	25	0.9	189	2.3	214	2.0	1	0.9	0.53
Germany	110	4.0	545	6.8	655	6.1	12	10.9	2.20
India	5	0.2	94	1.2	99	0.9	1	0.9	1.06
Japan	8	0.3	207	2.6	215	2.0	1	0.9	0.48
Netherlands	175	6.4	655	8.1	830	7.7	14	12.7	2.14
Pakistan	0	0.0	11	0.1	11	0.1	1	0.9	9.09
Portugal	2	0.1	5	0.1	7	0.1	1	0.9	20.00
Russia	6	0.2	147	1.8	153	1.4	1	0.9	0.68
Singapore	5	0.2	51	0.6	56	0.5	2	1.8	3.92
South Korea	12	0.4	75	0.9	87	0.8	1	0.9	1.33
Spain	52	1.9	73	0.9	125	1.2	2	1.8	2.74
Sweden	5	0.2	14	0.2	19	0.2	1	0.9	7.14
Taiwan	3	0.1	31	0.4	34	0.3	1	0.9	3.23
Turkey	12	0.4	49	0.6	61	0.6	1	0.9	2.04
USA	1,229	45.0	2,724	33.7	3,953	36.6	37	33.6	1.36
Sub-Total	2,494	91.3	6,702	83.0	9,196	85.1	110	100	1.64
Other Countries	237	8.7	1,371	17.0	1,608	14.9	-	-	-
Total	**2,731**	**100**	**8,073**	**100**	**10,804**	**100**	**110**	**100**	1.02

The total IF journals in ‘Statistics and Probability’ are about 1.0 percent of IF journals in all subjects of Social Sciences and Sciences (Table 3). The four countries (England, Germany, Netherland and USA) producing highest number of IF journals in ‘Statistics and Probability’, are also producing highest number of IF journals in other subjects. The percentage of IF journals in ‘Statistics and Probability’ to the IF journals in Science Subject for England, Germany, Netherland and USA are 1.72, 2.20, 2.14, and 1.36 respectively. Pakistan is contributing one journal out of the total 11 IF journals.

Therefore, the percentage of IF journal in ‘Statistics and Probability’ originating from Pakistan (9.09 %). This parentage is relatively better than the overall trend. This is due to the sole effort of PJS that can be regarded as an achievement. This does not mean that PJS has achieved its destination. A lot more organized effort is needed to take this journal to a much higher position and gain more respect among the statistical scientists.

According to JCR 2010, PJS is one of the eleven IF journals originating from Pakistan. The remaining 10 journals are related to subjects like Biology, Chemistry, Medical and Agriculture (Appendix C). In the local context of Pakistan, these subjects more popularly known as pre-medical group. Thus Pakistan seems to be academically made a mark in this group of subjects. PJS is clearly a distinct subject from this group. This might have been relatively difficult for PJS to produce an IF journal other than academically established subjects.

4. CONCLUSIONS

Before, concluding remarks, this would be realistic to frankly reveal the limitations of this study. More discussions on the results might have been expected. The paper is too short for such a worldwide comparison, which actually requires more exhaustive comparisons. However, confined by our scope i.e. to highlight contributions of PJS, this has been done intentionally. The results about some of the countries showing a low number of journals but high ratios should be cautiously compared with other countries. This may lead to distorted interpretations. The values of 9.09 for Pakistan and 20.2 for Portugal (Table 3) are not readily comparable to the values of nations which cover a large number of subject categories. The result may signal a specialization, or a distinctive competence in the field by the county. However, futures studies should include the number of all subject categories per nation for more realistic comparisons among various countries of the world. The use of a concentration measure, like Gini Index, with respect to how the journals are distributed over countries/publishing house may also be considered.

In the end this may be proposed that, Islamic countries have to make a lot of effort to make their academic importance felt in the academic world. Producing a high quality work is required. The Islamic countries have to start *collaborative researches* with the academically advanced nations. This will also improve their chances of participating in *fundamental research* so difficult to be successfully carried out in these countries due to shortage of trained personals. This type of research often receives high citations. Research organizations and universities should *pool resources* to access the digital research data indispensably needed for a sound review of literature without which the production of citable documents is out of question. This needs greater cooperation to collaborate and co-ordinate the purchase, housing of and *subscription to international literature*. PJS may particularly attempt to make international availability of the journal possible through collaborating with *leading publishing houses* of the top statistical association on one hand and help out other journals of the Islamic countries struggling for even the first impact factor. Only these kinds of *positivism* by the journals like PJS having a low IF will work for *continuous improvement* in its IF. Negative approaches like unnecessary self citations and manipulation of the editorial policies to improve IF should be avoided.

REFERENCES

1. Althouse, B.M., West, J.D., Bergstrom, C.T. and Bergstrom, T. (2009). Differences in impact factor across fields and over time. *J. Amer. Soc. for Infor. Sci. and Tech.*, 60(1), 27-34.
2. Bankhead, C. (2010). Impact factor may not be best for assessing journals. *Urology Times* (August).
3. Bliziotis, I.A., Paraschakis, K., Vergidis, P.I., Karavasiou, A.I., Falagas, M.E. (2005). Worldwide trends in quantity and quality of published articles in the field of infectious diseases. *BMC Infectious Diseases*, 5(16), 1-8.
4. Campanario, J.M. (2011). Large increases and decreases in journal Impact Factors in only one year: the effect of journal self-citations. *Journal of the American Society for Information Science and Technology*, 62(2), 230–235.
5. Dong, P, Loh, M. and Mondry, A. (2005). The "impact factor" revisited. *Biomedical Digital Libraries*, 2(7), 1-8
6. Garfield E. (1955). Citation indexes for science; a new dimension in documentation through association of ideas. *Science*, 122, 108-111.
7. Garfield E. (1999). Journal Impact factor, a brief review. *Can. Med. Assoc. Journal*, 161(8), 979-980
8. Gomez-Sancho, J.M. and Mancebon-Torrubia, M.J. (2009). The evaluation of scientific production: Towards a neutral impact factor. *Scientometrics*, 81(2), 435-458.
9. Konradsen, J. and Munk-Jørgensen, A. (2007). The destinies of the low- and middle-income country submissions. *Acta Psychiatr Scand*, 115, 331-334.
10. Mallick, S.K., Krichel, T. and Novarese, M. (2010). Subscription to digital libraries and corresponding journal impact: A value-based approach to demand for digital research data. *Journal of Digital Asset Management*, 6(6), 336-343.
11. Rossner, M., Van Epps, H., and Hill, E. (2008). Irreproducible results: a response to Thomson Scientific. *Journal of Cell Biology*, 180 (2), 254-255.
12. Sorrentino, D., Biase, F., Trevisi, A. and Bartoli, E. (2000). Scientific Publications in Gastroenterology and Hepatology in Western Europe, USA and Japan in the Years 1992–1996: A Global Survey. *Digestion*, 61, 77-83. *Indian Journal of Dermatology, Venereology and Leprology*, 77, 1-2.
13. Yu, G., Yang, D.H., and Liang, W. (2010). Reliability-based citation impact factor and the manipulation of impact factor. *Scientometrics*, 83(1), 259-270.
14. Wishart, M.J. and Davies, B.R. (1998). The increasing divide between First and Third Worlds: science, collaboration and conservation of Third World aquatic ecosystems. *Freshwater Biology*, 39, 557-567.

APPENDIX A

List of Impact Factor Journals in Subject Category 'Statistics and Probability' (JCR 2010)

S#	Full Name of Journals	3-Year IF	Country
1	Advances in Applied Probability	0.72	England
2	Advances in Data Analysis and Classification	0.58	Germany
3	American Statistician	0.98	USA
4	Annals de L Institute Henri Poincare-Probabilites Et Statistiques	0.76	France
5	Annals of Applied Probability	1.12	USA
6	Annals of Applied Statistics	1.75	USA
7	Annals of Probability	1.47	USA
8	Annals of Statistics	2.94	USA
9	Annals of the Institute of Statistical Mathematics	0.97	Japan
10	Applied Stochastic Models in Business and Industry	0.83	England
11	AStA-Advances in Statistical Analysis	0.69	Germany
12	Astin Bulletin	0.71	Belgium
13	Australian and New Zealand Journal of Statistics	0.62	Australia
14	Bayesian Analysis	1.21	USA
15	Bernoulli	1.00	Netherlands
16	Biometrical Journal	1.44	Germany
17	Biometrics	1.76	USA
18	Biometrika	1.83	England
19	Biostatistics	2.77	England
20	British Journal of Mathematical and Statistical Psychology	1.42	England
21	Canadian Journal of Statistics-Revue Canadienne de Statistique	0.69	USA
22	Chemometrics and Intelligent Laboratory Systems	2.22	Netherlands
23	Combinatorics Probability and Computing	0.99	USA
24	Communications in Statistics-Simulation and Computation	0.34	USA
25	Communications in Statistics-Theory and Methods	0.35	USA
26	Computational Statistics	0.50	Germany
27	Computational Statistics and Data Analysis	1.09	Netherlands
28	Econometric Reviews	1.09	USA
29	Econometric Theory	1.02	USA
30	Econometrica	3.19	England
31	Econometrics Journal	0.69	England
32	Electronic Communications in Probability	0.56	USA
33	Electronic Journal of Probability	0.95	USA
34	Electronic Journal of Statistics	1.03	USA
35	Environmental and Ecological Statistics	1.65	Netherlands
36	Environmetrics	0.75	England
37	Extremes	1.05	USA
38	Finance and Stochastics	1.33	Germany
39	Fuzzy Sets and Systems	1.88	Netherlands

S#	Full Name of Journals	3-Year IF	Country
40	Hacettepe Journal of Mathematics and Statistics	0.39	Turkey
41	IEEE-ACM Transactions on Computational Biology and Bioinformatics	1.66	USA
42	Infinite Dimensional Analysis Quantum Probability and Related Topics	0.57	Singapore
43	Insurance Mathematics and Economics	1.18	Netherlands
44	International Journal of Agricultural and Statistical Sciences	0.04	India
45	International Journal of Game Theory	0.59	Germany
46	International Statistics Review	0.86	England
47	Journal of Agricultural Biological and Environmental Statistics	0.72	USA
48	Journal of Applied Probability	0.77	England
49	Journal of Applied Statistics	0.31	England
50	Journal of Biopharmaceutical Statistics	1.07	USA
51	Journal of Business and Economic Statistics	1.69	USA
52	Journal of Chemometrics	1.38	England
53	Journal of Computational and Graphical Statistics	1.21	USA
54	Journal of Computational Biology	1.60	USA
55	Journal of Multivariate Analysis	1.01	USA
56	Journal of Nonparametric Statistics	0.46	England
57	Journal of Official Statistics	0.49	Sweden
58	Journal of Quality Technology	1.38	USA
59	Journal of Statistical Computation and Simulation	0.47	England
60	Journal of Statistical Planning and Inference	0.69	Netherlands
61	Journal of Statistical Software	2.65	USA
62	Journal of the American Statistical Association	2.06	USA
63	Journal of the Korean Statistical Society	0.33	South Korea
64	Journal of the Royal Statistical Society Series A-Statistics in Society	2.57	England
65	Journal of the Royal Statistical Society Series B-Statistical Methodology	3.50	England
66	Journal of the Royal Statistical Society Series C-Applied Statistics	0.65	England
67	Journal of Theoretical Probability	0.60	Belgium
68	Journal of Time Series Analysis	0.68	England
69	Lifetime Data Analysis	0.87	Netherlands
70	Mathematical Population Studies	0.59	USA
71	Methodology and Computing in Applied Probability	0.77	USA
72	Metrika	0.58	Germany
73	Multivariate Behavioral Research	1.29	USA
74	Open Systems and Information Dynamics	1.57	Netherlands
75	Oxford Bulletin of Economics and Statistics	1.18	England
76	Pakistan Journal of Statistics	0.16	Pakistan
77	Pharmaceutical Statistics	1.63	England

S#	Full Name of Journals	3-Year IF	Country
78	Probabilistic Engineering Mechanics	1.25	England
79	Probability in the Engineering and Informational Sciences	0.97	USA
80	Probability Theory and Related Fields	1.59	Germany
81	Quality and Quantity	0.69	Netherlands
82	Revista Colombiana de Estadistica	0.06	Colombia
83	REVSTAT-Statistical Journal	0.73	Portugal
84	Scandinavian Actuarial Journal	0.61	England
85	Scandinavian Journal of Statistics	0.84	England
86	SORT-Statistics and Operations Research Transactions	0.25	Spain
87	Stata Journal	2.00	USA
88	Statistica Neerlandica	0.32	Netherlands
89	Statistica Sincia	0.96	Taiwan
90	Statistical Applications in Genetics and Molecular Biology	1.84	USA
91	Statistical Methods and Applications	0.37	Germany
92	Statistical Methods in Medical Research	1.77	England
93	Statistical Modeling	0.71	England
94	Statistical Papers	0.60	Germany
95	Statistical Science	2.48	USA
96	Statistics	0.52	Germany
97	Statistics and Computing	1.85	Netherlands
98	Statistics and Probability Letters	0.44	Netherlands
99	Statistics in Medicine	2.33	England
100	Stochastic Analysis and Applications	0.42	USA
101	Stochastic Environmental Research and Risk Assessment	1.78	Germany
102	Stochastic Models	0.45	USA
103	Stochastic Processes and their Applications	0.95	Netherlands
104	Stochastics and Dynamics	0.71	Singapore
105	Stochastics-An International Journal of Probability and Stochastic Processes	0.37	England
106	Survey Methodology	0.55	Canada
107	Technometrics	1.56	USA
108	Test	1.04	Spain
109	Theory of Probability and Its Applications	0.32	Russia
110	Utilitas Mathematica	0.74	Canada

APPENDIX B

Ranking of the Impact Factor Journals of the World as per 2010 JCR Science Edition

Rank	Full Name of Journals	IF
1	Journal of the Royal Statistical Society Series B-Statistical Methodology	3.50
2	Econometrica	3.19
3	Annals of Statistics	2.94
4	Biostatistics	2.77
5	Journal of Statistical Software	2.65
6	Journal of the Royal Statistical Society Series A-Statistics in Society	2.57
7	Statistical Science	2.48
8	Statistics in Medicine	2.33
9	Chemometrics and Intelligent Laboratory Systems	2.22
10	Journal of the American Statistical Association	2.06
11	Stata Journal	2.00
12	Fuzzy Sets and Systems	1.88
13	Statistics and Computing	1.85
14	Statistical Applications in Genetics and Molecular Biology	1.84
15	Biometrika	1.83
16	Stochastic Environmental Research and Risk Assessment	1.78
17	Statistical Methods in Medical Research	1.77
18	Biometrics	1.76
19	Annals of Applied Statistics	1.75
20	Journal of Business and Economic Statistics	1.69
21	IEEE-ACM Transactions on Computational Biology and Bioinformatics	1.66
22	Environmental and Ecological Statistics	1.65
23	Pharmaceutical Statistics	1.63
24	Journal of Computational Biology	1.60
25	Probability Theory and Related Fields	1.59
26	Open Systems and Information Dynamics	1.57
27	Technometrics	1.56
28	Annals of Probability	1.47
29	Biometrical Journal	1.44
30	British Journal of Mathematical and Statistical Psychology	1.42
31	Journal of Chemometrics	1.38
32	Journal of Quality Technology	1.38
33	Finance and Stochastics	1.33
34	Multivariate Behavioral Research	1.29
35	Probabilistic Engineering Mechanics	1.25
36	Bayesian Analysis	1.21
37	Journal of Computational and Graphical Statistics	1.21
38	Oxford Bulletin of Economics and Statistics	1.18
39	Insurance Mathematics and Economics	1.18
40	Annals of Applied Probability	1.12
41	Computational Statistics and Data Analysis	1.09

Rank	Full Name of Journals	IF
42	Econometric Reviews	1.09
43	Journal of Biopharmaceutical Statistics	1.07
44	Extremes	1.05
45	Test	1.04
46	Electronic Journal of Statistics	1.03
47	Econometric Theory	1.02
48	Journal of Multivariate Analysis	1.01
49	Bernoulli	1.00
50	Combinatorics Probability and Computing	0.99
51	American Statistician	0.98
52	Probability in the Engineering and Informational Sciences	0.97
53	Annals of the Institute of Statistical Mathematics	0.97
54	Statistica Sincia	0.96
55	Stochastic Processes and their Applications	0.95
56	Electronic Journal of Probability	0.95
57	Lifetime Data Analysis	0.87
58	International Statistics Review	0.86
59	Scandinavian Journal of Statistics	0.84
60	Applied Stochastic Models in Business and Industry	0.83
61	Methodology and Computing in Applied Probability	0.77
62	Journal of Applied Probability	0.77
63	Annals de L Institute Henri Poincare-Probabilites Et Statistiques	0.76
64	Environmetrics	0.75
65	Utilitas Mathematica	0.74
66	REVSTAT-Statistical Journal	0.73
67	Journal of Agricultural Biological and Environmental Statistics	0.72
68	Advances in Applied Probability	0.72
69	Statistical Modeling	0.71
70	Stochastics and Dynamics	0.71
71	Astin Bulletin	0.71
72	Econometrics Journal	0.69
73	Journal of Statistical Planning and Inference	0.69
74	Canadian Journal of Statistics-Revue Canadienne de Statistique	0.69
75	Quality and Quantity	0.69
76	AStA-Advances in Statistical Analysis	0.69
77	Journal of Time Series Analysis	0.68
78	Journal of the Royal Statistical Society Series C-Applied Statistics	0.65
79	Australian and New Zealand Journal of Statistics	0.62
80	Scandinavian Actuarial Journal	0.61
81	Journal of Theoretical Probability	0.60
82	Statistical Papers	0.60
83	International Journal of Game Theory	0.59
84	Mathematical Population Studies	0.59
85	Metrika	0.58

Rank	Full Name of Journals	IF
86	Advances in Data Analysis and Classification	0.58
87	Infinite Dimensional Analysis Quantum Probability and Related Topics	0.57
88	Electronic Communications in Probability	0.56
89	Survey Methodology	0.55
90	Statistics	0.52
91	Computational Statistics	0.50
92	Journal of Official Statistics	0.49
93	Journal of Statistical Computation and Simulation	0.47
94	Journal of Nonparametric Statistics	0.46
95	Stochastic Models	0.45
96	Statistics and Probability Letters	0.44
97	Stochastic Analysis and Applications	0.42
98	Hacettepe Journal of Mathematics and Statistics	0.39
99	Stochastics-An International Journal of Probability and Stochastic Processes	0.37
100	Statistical Methods and Applications	0.37
101	Communications in Statistics-Theory and Methods	0.35
102	Communications in Statistics-Simulation and Computation	0.34
103	Journal of the Korean Statistical Society	0.33
104	Statistica Neerlandica	0.32
105	Theory of Probability and Its Applications	0.32
106	Journal of Applied Statistics	0.31
107	SORT-Statistics and Operations Research Transactions	0.25
108	Pakistan Journal of Statistics	0.16
109	Revista Colombiana de Estadistica	0.06
110	International Journal of Agricultural and Statistical Sciences	0.04

APPENDIX C

Ranking of the Impact Factor Journals of Pakistan as per 2010 JCR Science Edition

Rank	Name of Journal	3-Year IF
1	Pakistan Journal of Botany	0.947
2	Pakistan Journal of Pharmaceutical Sciences	0.728
3	Pakistan Veterinary Journal	0.707
4	JCPSP-Journal of the College of Physicians and Surgeons Pakistan	0.342
5	Journal of Animal and Veterinary Advances	0.292
6	Journal of Animal and Plant Sciences	0.250
7	Asian Journal of Animal and Veterinary Advances	0.235
8	Journal of the Chemical Society of Pakistan	0.194
9	Pakistan Journal of Medical Sciences	0.166
10	Pakistan Journal of Statistics	0.156
11	Pakistan Journal of Zoology	0.145

A NOTE ON THE IMPACT FACTOR JOURNALS OF 'STATISTICS AND PROBABILITY'*

Faisal Qadeer[1] and **Munir Ahmad**[2]
National College of Business Administration and Economics, Lahore, Pakistan
Email: [1]mfaisalqr@hotmail.com; [2]drmunir@ncbae.edu.pk

ABSTRACT

This brief communications aims at sharing the list of impact factor journals of 'Statistics & Probability' as per the Journal Citation Report 2011 released by Institute of Scientific Information. This would update the academia and practitioner on their information about the ranking of the journals. The study makes country and regions wise comparisons of the changes in no of journals and their mean impact factors reported in the previous and current reports.

KEYWORDS

Journal Citation Reports, Impact Factor, Statistics and Probability, Pakistan.

Institute of Scientific Information (ISI) announced the first impact factor (IF) of Pakistan Journal of Statistics (PJS) in the Journal Citation Report (JCR) 2010. At that event, PJS traced its contributions in comparison with the other courtiers (Qadeer& Ahmad, 2012), the study includes a complete list of 110 IF journals for the subject category of 'Statistics & Probability'. The academic audience of PJS appreciated that information very much. Many of them proposed that the list of IF journals may be updated every year.

Now after the release of JCR (2011) this is high time to update the list. The list of 3-year IF journals of in the subject category 'Statistics & Probability' in the Science Citation Index is now being presented here (Appendix A). As per this report there are 116 IF journals as against 110 in the previous list. Therefore, 6 more journals qualified to enter the list. Two of these new entrants (Quality & Quantity & Statistics in Biopharmaceutical Research) originate from USA; 1 each from Germany (ALEA-Latin American Journal of Probability), Brazil (Brazilian Journal of Probability and Statistics), France (ESAIM-Probability and Statistics) and Taiwan (Quality Engineering).

*Published in Pak. J. Statist. (2013), Vol. 29(2).

Table 1
Impact Factor Change by Country of Origin: JCR 2010 VS 2011

Country of Origin	Impact Factor Change (no of journals)				Total	Mean IF		
	Lower	Same	Higher	New		2010	2011	Change
England	12	1	14	0	27	1.278	1.300	0.022
Germany	9	0	3	1	13	0.880	0.758	(0.122)
Netherlands	6	0	8	0	14	1.171	1.097	(0.074)
USA	20	0	17	2	39	1.263	1.275	0.013
Others (17)	8	1	11	3	23	0.548	0.490	(0.059)
Total	**55**	**2**	**53**	**6**	**116**	**1.083**	**1.046**	**(0.037)**

There are 21 countries now that have at least one IF Journal, please recall as per JCR 2010, there were 20 countries. The change in the number of IF of the journals for the four major countries is summarized in Table 1. As 6 journals are new, therefore the comparison of IF of 110 journals can be made. Off the 110, the IF of 55 journals is relatively lower in JCR 2011 than JCR 2010, 53 journals have higher IF and the remaining 2 journals have IF in the two reports. The mean IF change of the journals from England and USA is positive, this change in mean IF of Netherland, Germany and 17 other countries is negative. This may also be noted from Table 1 that the mean IF change of all the journals in this subject category is in negative direction.

Table 2
Impact Factor Change by Regions of the World: JCR 2010 VS 2011

Country of Origin	Impact Factor Change (No. of Journals)				Total	Mean IF		
	Lower	Same	Higher	New		2010	2011	Change
Asia	3	0	5	1	9	0.514	0.524	0.010
Australia	1	0	0	0	1	0.618	0.436	(0.182)
Europe	30	1	30	2	63	1.088	1.032	(0.055)
N. America	21	0	18	2	41	1.231	1.239	0.008
S. America	0	1	0	1	2	0.056	0.158	0.102
Total	**55**	**2**	**53**	**6**	**116**	**1.083**	**1.046**	**(0.037)**

The change in the number of IF of the journals for the five regions of the world originating IF journals in the subject category of ‘Statistics & Probability’ has been summarized in Table 2. The mean IF change of the journals from Asia, N. America and S. America is positive. This change in mean IF of Australia and Europe is negative.

Table 3
Impact Factor Journals of Pakistan

S#	Name of Journal	Subject Category	2010	2011
			3-Year IF	
1	Asian Journal of Animal and Veterinary Advances	Veterinary Sciences	0.24	0.87
2	International Journal of Agriculture and Biology	Agriculture, Multidisciplinary	-	0.94
3	International Journal of Pharmacology	Pharmacology & Pharmacy	-	1.50
4	Journal of Animal and Plant Sciences	Agriculture, Multidisciplinary	0.25	0.59
5	Journal of Animal and Veterinary Advances	Veterinary Sciences	0.29	0.39
6	Journal of the Chemical Society of Pakistan	Chemistry, Multidisciplinary	0.19	1.38
7	JCPSP-Journal of the College of Physicians and Surgeons Pakistan	Medicine, General & Internal	0.34	0.34
8	Pakistan Journal of Botany	Plant Sciences	0.95	0.91
9	Pakistan Journal of Medical Sciences	Medicine, General & Internal	0.17	0.16
10	Pakistan Journal of Pharmaceutical Sciences	Pharmacology & Pharmacy	0.73	1.10
11	Pakistan Journal of Statistics	Statistics and Probability	0.16	0.29
12	Pakistan Journal of Zoology	Zoology	0.15	0.34
13	Pakistan Veterinary Journal	Veterinary Sciences	0.71	1.26

According to JCR 2011, there are 13 IF journals of Pakistan (Table 3) as against 11 IF journals as per JCR 2010. The mean IF change of all the journals from Pakistan is positive except for the two journals for which the change is in negative direction by a very small number. These journals fall in eight subject categories in the Science Citation Index. For a realistic comparison, the details of the number of journals and the mean IFs for these subject categories are presented in Table 4.

Table 4
Mean IF of Selected Subject Categories

S #	Subject Category	Journals	Mean IF 2011
1	Agriculture, Multidisciplinary	57	0.78
2	Chemistry, Multidisciplinary	154	3.00
3	Medicine, General & Internal	155	2.53
4	Pharmacology & Pharmacy	261	5.71
5	Plant Sciences	190	1.96
6	Statistics and Probability	116	1.05
7	Veterinary Sciences	145	0.94
8	Zoology	146	1.30

From Table 5, this may be noted that making comparison among various subjects is not a realistic yard stick in ranking of the journals. For example there are 261 journals in the subject category ‘Pharmacology & Pharmacy’ is 5.71, whereas such there are only 4 journals in ‘Statistics and Probability’ that has an IF more than 3. Therefore, we should be cautious in making comparison among the journals falling in different subject categorizers. A more realistic approach would be to compare the IF of a journal with the mean IF journal of its own subject category. Future research may further analyze this point.

REFERENCES

1. Journal Citations Report (2010), *ISI Web of Knowledge*, (June, 2011).
2. Journal Citations Report (2011), *ISI Web of Knowledge*, (June, 2012).
3. Qadeer, F. and Ahmad, M. (2012). Contribution of Pakistan Journal of Statistics in ‘Statistics and Probability’ Literature: An analysis based on the Journal Citation Report (2010). *Pak. J. Statist.*, 28(3), 409-423.

APPENDIX A

List of Impact Factor Journals in the Subject Category 'Statistics and Probability'

S#	Name of Journal	3-Years Impact Factor 2010	3-Years Impact Factor 2011	Country
1	Advances in Applied Probability*	0.72	0.68	England
2	Advances in Data Analysis and Classification	0.58	0.56	Germany
3	*ALEA-Latin American Journal of Probability and Mathematical Statistics*	-	0.38	Germany
4	American Statistician	0.98	0.96	USA
5	Annals de L Institute Henri Poincare-Probabilites Et Statistiques*	0.76	0.90	France
6	Annals of Applied Probability	1.12	1.09	USA
7	Annals of Applied Statistics	1.75	1.58	USA
8	Annals of Probability	1.47	1.79	USA
9	Annals of Statistics	2.94	3.03	USA
10	Annals of The Institute of Statistical Mathematics	0.97	0.86	Japan
11	Applied Stochastic Models in Business and Industry	0.83	0.69	England
12	AStA-Advances in Statistical Analysis	0.69	0.44	Germany
13	Astin Bulletin	0.71	0.49	Belgium
14	Australian & New Zealand Journal of Statistics	0.62	0.44	Australia
15	Bayesian Analysis	1.21	1.65	USA
16	Bernoulli	1.00	1.05	Netherlands
17	Biometrical Journal	1.44	1.25	Germany
18	Biometrics*	1.76	1.83	USA
19	Biometrika	1.83	1.91	England
20	Biostatistics	2.77	2.15	England
21	*Brazilian Journal of Probability and Statistics*	-	0.26	Brazil
22	British Journal of Mathematical & Statistical Psychology	1.42	1.31	England
23	Canadian Journal of Statistics-Revue Canadienne de Statistique*	0.69	0.67	USA
24	Chemometrics and Intelligent Laboratory Systems	2.22	1.92	Netherlands
25	Combinatorics Probability & Computing	0.99	0.78	USA
26	Communications in Statistics-Simulation and Computation	0.34	0.39	USA
27	Communications in Statistics-Theory and Methods	0.35	0.27	USA
28	Computational Statistics	0.50	0.28	Germany

S#	Name of Journal	3-Years Impact Factor		Country
		2010	2011	
29	Computational Statistics & Data Analysis	1.09	1.03	Netherlands
30	Econometric Reviews	1.09	0.78	USA
31	Econometric Theory	1.02	0.86	USA
32	Econometrica	3.19	2.98	England
33	Econometrics Journal	0.69	0.87	England
34	Electronic Communications in Probability	0.56	0.53	USA
35	Electronic Journal of Probability	0.95	0.71	USA
36	Electronic Journal of Statistics	1.03	1.15	USA
37	Environmental and Ecological Statistics	1.65	1.31	Netherlands
38	Environmetrics	0.75	1.06	England
39	*ESAIM-Probability and Statistics*	-	0.27	France
40	Extremes	1.05	1.26	USA
41	Finance and Stochastics	1.33	1.20	Germany
42	Fuzzy Sets and Systems	1.88	1.76	Netherlands
43	Hacettepe Journal of Mathematics and Statistics	0.39	0.35	Turkey
44	IEEE-ACM Transactions on Computational Biology and Bioinformatics	1.66	1.54	USA
45	Infinite Dimensional Analysis Quantum Probability and Related Topics	0.57	0.70	Singapore
46	Insurance Mathematics & Economics*	1.18	1.29	Netherlands
47	International Journal of Agricultural and Statistical Sciences	0.04	0.01	India
48	International Journal of Game Theory	0.59	0.30	Germany
49	International Statistics Review*	0.86	0.54	England
50	Journal of Agricultural Biological and Environmental Statistics	0.72	1.21	USA
51	Journal of American Statistical Association	2.06	1.99	USA
52	Journal of Applied Probability*	0.77	0.63	England
53	Journal of Applied Statistics	0.31	0.41	England
54	Journal of Biopharmaceutical Statistics	1.07	1.34	USA
55	Journal of Business & Economic Statistics	1.69	1.78	USA
56	Journal of Chemometrics	1.38	1.95	England
57	Journal of Computational and Graphical Statistics	1.21	1.06	USA
58	Journal of Computational Biology	1.60	1.55	USA
59	Journal of Multivariate Analysis	1.01	0.88	USA

S#	Name of Journal	3-Years Impact Factor		Country
		2010	2011	
60	Journal of Nonparametric Statistics	0.46	0.46	England
61	Journal of Official Statistics	0.49	0.33	Sweden
62	Journal of Quality Technology	1.38	1.56	USA
63	Journal of Statistical Computation and Simulation	0.47	0.50	England
64	Journal of Statistical Planning and Inference	0.69	0.72	Netherlands
65	Journal of Statistical Software	2.65	4.01	USA
66	Journal of the Korean Statistical Society	0.33	0.47	South Korea
67	Journal of the Royal Statistical Society Series B-Statistical Methodology	3.50	3.65	England
68	Journal of the Royal Statistical Society Series A-Statistics in Society	2.57	2.11	England
69	Journal of the Royal Statistical Society Series C-Applied Statistics	0.65	0.83	England
70	Journal of Theoretical Probability	0.60	0.68	Belgium
71	Journal of Time Series Analysis	0.68	0.76	England
72	Lifetime Data Analysis	0.87	0.92	Netherlands
73	Mathematical Population Studies	0.59	0.24	USA
74	Methodology and Computing in Applied Probability	0.77	0.75	USA
75	Metrika	0.58	0.67	Germany
76	Multivariate Behavioral Research	1.29	1.41	USA
77	Open Systems & Information Dynamics	1.57	1.17	Netherlands
78	Oxford Bulletin of Economics and Statistics	1.18	1.00	England
79	Pakistan Journal of Statistics	0.16	0.29	Pakistan
80	Pharmaceutical Statistics	1.63	2.07	England
81	Probabilistic Engineering Mechanics	1.25	1.25	England
82	Probability in The Engineering and Informational Sciences	0.97	0.64	USA
83	Probability Theory and Related Fields	1.59	1.53	Germany
84	*Quality & Quantity*	0.69	0.77	Netherlands
85	*Quality Engineering*	-	0.75	USA
86	Quality Technology and Quantitative Management	-	0.28	Taiwan
87	RevistaColombiana de Estadistica	0.06	0.06	Colombia
88	REVSTAT-Statistical Journal	0.73	0.13	Portugal
89	Scandinavian Actuarial Journal	0.61	0.50	England

S#	Name of Journal	3-Years Impact Factor		Country
		2010	2011	
90	Scandinavian Journal of Statistics	0.84	1.12	England
91	SORT-Statistics and Operations Research Transactions	0.25	0.43	Spain
92	Stata Journal	2.00	2.22	USA
93	StatisticaNeerlandica	0.32	0.50	Netherlands
94	StatisticaSincia	0.96	1.02	Taiwan
95	Statistical Applications in Genetics and Molecular Biology	1.84	1.52	USA
96	Statistical Methods and Applications	0.37	0.41	Germany
97	Statistical Methods in Medical Research	1.77	2.44	England
98	Statistical Modeling	0.71	0.90	England
99	Statistical Papers	0.60	0.59	Germany
100	Statistical Science	2.48	3.04	USA
101	Statistics	0.52	0.72	Germany
102	Statistics & Probability Letters	0.44	0.50	Netherlands
103	Statistics and Computing	1.85	1.43	Netherlands
104	*Statistics in Biopharmaceutical Research*	-	0.54	USA
105	Statistics in Medicine	2.33	1.88	England
106	Stochastic Analysis and Applications	0.42	0.46	USA
107	Stochastic Environmental Research and Risk Assessment	1.78	1.52	Germany
108	Stochastic Models	0.45	0.67	USA
109	Stochastic Processes and their Applications	0.95	1.01	Netherlands
110	Stochastics and Dynamics	0.71	0.75	Singapore
111	Stochastics-An International Journal of Probability and Stochastic Processes	0.37	0.48	England
112	Survey Methodology	0.55	0.93	Canada
113	Technometrics	1.56	1.25	USA
114	Tests	1.04	1.13	Spain
115	Theory of Probability And Its Applications	0.32	0.40	Russia
116	Utilitas Mathematica*	0.74	0.14	Canada

* A Multi-Language Journal (all other journals are in English Language)

MEASURING HR-LINE RELATIONSHIP QUALITY: A CONSTRUCT AND ITS VALIDATION*

Rashid Rehman[1], **Faisal Qadeer**[2], **Zahida Habib**[3] and **Munir Ahmad**[2]

[1] Al-Ghurair University, Dubai, United Arab Emirates.
Email: drrashid.rehman@gmail.com

[2] National College of Business Administration and Economics,
Lahore, Pakistan.
Email: drqadeer@ncbae.edu.pk; drmunir@ncbae.edu.pk

[3] University of Education, Lahore, Pakistan.
Email: zahidatariq_77@yahoo.com

ABSTRACT

Despite wide recognition of the importance of relationship between HR professionals and line managers (HR-line relationship) in strategic HRM literature, no measurement construct is available that could grasp the complicated concept. We introduce a variable, HR-line relationship quality and propose a measurement construct for the variable. The first dimension of the variable covers the traditional elements - often measured in psychology and marketing. The second and third dimension covers HR specific elements based on the attitudes of HR professionals towards the line managers and vice versa. The proposed construct is validated through an empirical survey from a sample of HRM specialists and line managers. The paper concludes that HR-line relationship quality can be measured from a 34 item construct. This construct can be used to quantify HR-line relationship quality, which would be helpful to understand and manage the relationship for the success of strategic HRM.

KEY WORDS

HR-Line Relationship Quality; Line Managers; Devolvement; SHRM; Measurement Construct.

1. INTRODUCTION

The importance of HR-line relationship has been widely recognized in strategic HRM literature (Qadeer et al., 2011a). However, the concept needs to be translated into measurement construct. Theoretically, the concept has passed the exploratory stages. However, description of the concept particularly involving empirical work has rarely been attempted in HRM literature. For developing a construct that could grasp the concept of HR-line relationship, identification of its dimensions and elements is required. This paper attempts to bridge this gap by developing a construct for measuring the quality of HR-line relationship and then fine tuning the construct through an empirical survey. There is a strong justification for carrying out this study, because, keeping an eye on the development of new HRM constructs is 'expected to remain the core concept in HRM research' (Dorenbosch & Veldhoven, 2006).

*Published in Pak. J. Statist. (2013), Vol. 29(2).

The study introduces a variable HR-line relationship quality (HLRQ) that represents the quality of relationship between HR professionals and line managers in an organization. Before moving further, some questions needs to be answered here. What is the operational meaning of HR-line relationship? Why it has become important in HRM? What may be the possible dimensions and elements of HLRQ? Why HR specific attitudes needs to be included in the measurement construct.

The first two issues are brief in nature and are discussed in this section. The third and fourth questions are covered in the next section. The answers to all these questions would help us identify the dimensions of the proposed construct for measuring HLRQ. For validation of the proposed construct an empirical survey has been conducted form HRM professionals and line managers.

HRM is different from personnel management in many ways. One of the major difference between the two is that line manager plays a key role in HRM in coordinating resources, which is not the case under personnel management (Legge, 1989). Similarly, the transition from traditional HRM towards strategic HRM brings many changes. In order to keep the initiative of strategic HRM fast, proactive and integrated rather than slow, reactive and fragmented as in case of HRM, one of the main changes is that, much of the HRM responsibility devolves down to line mangers rather than to HR specialists (Truss and Gratton, 1994; Mello, 2007). 'The relationship that exists between HR professionals and line managers in management of employees of an organization is referred to as *HR-line Relationship*' (Qadeer et al., 2011a).

HR-line relationship has been become important in strategic HRM. Qadeer et al. (2011a) discusses many reasons - like the evolutionary trends in the field of HRM; the transition towards SHRM; devolvement (line involvement in HRM); successful implementation of HRM; an obvious requirement for fulfillment of other concepts in HRM and in general management – that contributes in enhancing the importance of HR-line relationship.

The nature of relationship between HR managers and line managers –along with other contextual variables- are very much dependent upon the attitudes and behaviors of the two groups towards each other. Therefore, some HR specific attitudes of both the parties also need to be involved in measuring the quality of relationship between HR professionals and line managers. Relationship quality often measured in marketing and psychology does not include HR specific attitudes. Thus a comprehensive concept is required that not only include the traditional elements of relationship quality, but also HR specific elements that covers the attitudes of both HR managers and line managers. The measurement construct of HLRQ, therefore, should not only include the traditional elements for measuring relationship quality but also include HRM specific elements that are relevant in understanding the relations between HR specialists and line managers. The next section focuses on the identification of the dimensions of HLRQ, this would help us propose a construct for measuring this variable.

2. LITERATURE REVIEW

Dimensions of HR-line relationship quality

Many expressions are available in literature for describing the nature of relationship between HRM professional and line managers, for example consensual, conflictual, collaborative, partnership, trade-off. However, these prefix lacks measurability aspect. The concept of HLRQ represents the strength of the relationship which ultimately depends upon the a) relationships quality and b) positive attitudes of HR managers and line managers towards each other with reference to their perceived role and HRM performance.

The strength of relationship can be measured on the basis of previous research in psychology, marketing and general management research. For example, relationship quality has been measured in research pertaining to leader-member relations (Scandura & Graen, 1984), supervisor-worker relations (Game, 2008), and information inquirer-information source relations (Tan & Zhao 2003). In addition, interpersonal relations (Anderson & Williams 1996; Lewicki et al., 1998; Mcknight et al., 1998; Settoon & Mossholde 2002; Murphy et al., 2003) and friendship research (Morrey & Kito 2009) also involve this variable. In marketing literature relationship quality is measured for better management of firm-customer relationship (De Wolf et al., 2001; Forrester & Maute 2001; Beatson et al., 2008), exporters-importers relation (Ural, 2007), buyers-sellers relation (Chang 2005; Naude et al., 2007), manufacturer-retailer relation (Kim et al., 2004), and inter-firm relations (Arino et al., 2001; Johnson et al., 2004). It is also the case with, inter-functional relation (Prinslloo et al., 2007) and owner- managers and supervisors relationship (Chell & Tracey, 2005).

The traditional elements of relationship quality are satisfaction, trust, commitment, operational relations, role clarity, stability and security (Arino et al., 2001; De Wolf et al., 2001; Forrester and Maute 2001; Johnson et al., 2004; Kim et al., 2004; Chell and Tracey, 2005; Beatson et al., 2008; Game, 2008). The measures for different sub-dimensions of relationship quality are available and can be used with some modification in the construct for measuring the first dimension i.e. 'relationship quality' of HLRQ.

However, for the second dimension (i.e. positive attitudes of HR managers and line managers towards each other with reference to their role in HRM), further sub-dimensions needs to be identified on the basis of the attitudes and behaviors of both HR professional and line managers towards each other and HRM. Renwick (2000) concludes that HR and line managers exercise their power, expertise and strategic positions to engage in both conflictual and consensual relations, and are emerged in a dialogue on reconfiguring HR work between them.

Attitudes of HR professionals towards line managers

Hyman and Cunningham (1996) observe that HR specialists are 'sceptical', point deficiencies and have 'serious doubts' about abilities of line managers in HRM. They frequently state that line managers try to avoid these tasks. There are complaints from HR specialists that line managers either do not take advantage of preparatory training and development opportunities or acquire general management skills rather than specialist employee relations responsibilities. Papalexandris and Panayotopoulou (2005) find that

HR managers fear that their influence in the organization may reduce; fear of replacement; and there are difficulties for them to train line managers to HRM work properly.

In the absence of HR professionals taking the initiative, an HR-line partnership is unlikely to develop, as line managers are generally reluctant to ask HR professionals for help (Bond & Wise 2003). Lack of HR focus of line managers makes it difficult for HR professionals to collaborate with them. The belief that line managers do not give HRM work the 'priority' because they consider it amongst their other tasks and view HR work as an 'illegitimate' part of their job (McGovern et al., 1997). However, there are evidences that line managers are keen and serious, relatively happy in some HR work and are considerate of employee needs and wishes (Renwick, 2003). The attitudes of HR professionals about the HR focus of the line managers may shape their HR specific attitudes.

Negative interpersonal affect renders task competence virtually irrelevant in a person's choice of a partner for task interactions but that positive interpersonal affect increases a person's reliance on competence as a criterion for choosing task partners, facilitating access to organizational resources relevant to the task (Casciaro & Lobo, 2008). The efforts to improve multiple competencies of HR professionals or line managers would not be effective until the two have a high quality relationship. HR professionals' attitude of co-ordination towards line managers improves the chances of high quality relationship. HR specialists may feel that line managers pursue objectives which are often incompatible with HR department (Chimhanzi, 2004), act in isolation in making HR related decisions (Ulrich, 1997) and are reluctant to approach them for help (Kulik & Perry, 2008). This means that there is poor co-ordination between the two. Similarly, HR professionals' feeling that line managers dislike monitoring from HR (Larson & Brewster, 2003) and often argue over HR duties (Renwick, 2003) also shows lack of co-ordination.

Lawler III and Mohrman (2003) find that the use of joint HR-line teams to develop HR systems and policies strongly relates to HR being a strategic partner. Joint line/HR task teams improve business understanding of HR professionals and combine their expertise with the expertise of the line. In this way, knowledge barriers on both the sides minimize. If HR professionals take line managers as their team partner then they are likely to believe that the line managers are involved in a supportive relationship and working 'as a team' with HR (Renwick, 2000; 2003). In such a situation HR's perceptions of their unit's reputation among line managers is likely to be high. Good team partners should have a positive image of each other.

Attitudes of line towards HRM and HR professionals

Line managers can serve as the central bridging mechanism reconciling the pressures of external control and the requirement for internal motivation to sustain performance (Harney & Jordan, 2008). Exploring line management behavior is a promising avenue for more extensive research in the field of HRM. On the basis of line managers' HR experience in doing HR work, Renwick (2003) finds that they see HR as positive helpers in HR work; are taking this responsibility and accountability; are already managing large employees. It may be argued here that recognition of the contribution of HR professionals by the line managers helps improve their relationship. Extensive participation between HR and line managers can create mutual benefit for both as they jointly contribute to solve business problems (Gennard & Kelly, 1997). The value-adding contribution of HR is through business partnership roles by providing strategic advice to line (Galang, 1999; Gennard & Kelly, 1997; McConville & Holden, 1999). When in the opinion of line, HR professionals are behaving exactly as per their expectations (Teo & Rodwell, 2007), co-operating well to get the job done (Patterson et al., 2005) and offering the necessary support and advice to tackle HR issues (Renwick, 2003); there is a recognition of HR contribution and better mutual relations.

On the other hand, line managers may feel that HR managers do not understand the real business of the organization and only serve to create a distraction rather than add value to the bottom-line (Gubbins et al., 2006). Line managers do not want be distracted particularly when they regard themselves competent in hard HRM- 'common-sense backed by experience' (Hyman & Cunningham, 1996). They fail to understand the importance of some basic HR activities. For example, Siddique (2004) finds that line managers consider job analysis to be unnecessary paperwork and employees present it as a discrete performance evaluation mechanism that management might use as a justification to get rid of certain employees. These views are clearly detrimental to developing a close partnership between line managers, HR professionals and employees. Bond and McCracken (2005) find that except for extraordinary situations there is a little reference from line managers to HR specialists while making decisions about employee requests for time-off at short notice. Even in absence of these perceptions line managers may believe that HR is fearful of losing influence if they do the HR work (Papalexandris & Panayotopoulou, 2005).

The team partner dimensions discussed in the attitudes of HR professionals is also relevant in the discussions of attitudes of line managers. Mitsuhashi et al. (2000) finds that line managers do not perceive HR to be a strategic partner.

On the basis of this review, we propose that the second dimension of HLRQ is positive-ness of the perceptions of HR professional towards line managers and vice versa. The positive feelings of HR managers about their line counterpart possesses *HR focus,* have *co-ordination* and working as their *team partner* may improve the quality of their relationship. Same would be the result of the positive feeling of line managers that HR professionals are making *contribution,* working as a *team partner* and not creating *distraction.* In order to develop a measurement construct HR-line relationship quality and ascertain the impact of the above identified factors on it an empirical survey has been conducted.

3. METHODS AND MEASURES

Variables

The main variable of the study is *HR-line relationship quality* (HLRQ). This variable has three dimensions a) *relationship quality, b) positive-ness of HR towards line* and c) *positive-ness of line towards HR.* The fist dimension covers four sub-dimensions namely s*atisfaction, trust, commitment, and operational relations.* The second dimension represents positive behavior or impression of HR professionals towards line managers. This dimension covers three sub-dimensions namely *HR focus, co-ordination* and *team* *partner*. The third dimension represents positive behavior or impression of line managers towards HRM and HR professionals is named as *positive-ness of line towards HR*. This dimension further covers three sub-dimensions namely *contribution*, *distraction* and *team partner*. This means that the sub-dimension of team partner is common for the two types of positive-ness.

Measurements

For measuring HLRQ, a construct of 38 items is proposed. The present study adopts 16 items from the existing research (Arino et al., 2001; De Wolf et al., 2001; Chimhanzi, 2004; Johnson et al., 2004; Kim et al., 2004; Chell and Tracey, 2005; and Beatson et al., 2008) to measure *relationship quality*. In order to harmonize these items, suitable changes in wording, order and style seems logical. Starting with a common statement, both HR professionals and line managers rate their relations with their respective counterparts on numeric scale of 1-7, for the two ends, strongly disagree - strongly agree respectively.

Positive-ness of HR towards line measures through 15 items rated by HR professionals on the 1-7 numeric scale. These items (4-adopted and 11-fresh) are outcome of review of the prevailing research (McGovern et al., 1997; Ulrich, 1997; Renwick, 2000, 2003; Larson & Brewster, 2003; Chimhanzi, 2004; Watson et al., 2007; and Kulik & Perry, 2008).

Positive-ness of line towards HR measures through 12 items rated by line managers on 1-7 numeric scale. These items (3-adopted and 9-fresh) are also outcome of review of the prevailing research (Gennard & Kelly, 1997; Ulrich, 1997; Renwick, 2000, 2003; Chimhanzi, 2004; Papalexandris & Panayotopoulou, 2005; Patterson et al., 2005; Gubbins et al., 2006; Teo & Rodwell, 2007; and Kulik & Perry, 2008).

In nutshell, 21 are common for HR professionals and line managers; 10 are to be responded by HR professionals only and 7 by line managers only. The responses of HR professionals and line managers are combined to quantify HLRQ. Table 1 present the summary.

Table 1
Measures of HR-line Relationship Quality

Dimensions	Sub-dimensions	No of items	Remarks
Relationship Quality	Satisfaction	3	13 adopted measures with very minor changes and 3 new measures
	Trust	3	
	Commitment	3	
	Operational relations	7	
Positive-ness of HR towards line	HR Focus	5	4 adopted and 18 new measures *(5 are common)*
	Coordination	5	
	Team Partner	*5*	
Positive-ness of Line towards HR	Contribution	4	
	Distraction	3	
	Team Partner	*5*	
Total Measures for HR-line relationship quality		38	17 adopted and 21 new measures

The sample and Data Collection

Any effort to measure HR-line relationship quality is relevant in those organizations which have formal HR departments. This criterion was applied to fifty-four higher education institutes of Punjab-the largest province of Pakistan to select the sample of 8 universities that have a formal HR department. Head of academic departments (HODs) are the line managers because they are associated with the achievement of the primary purpose of universities. Therefore, the term line managers have been replaced with HODs in this survey.

A sample of 32 managers – 14 HR professionals and 18 HODs completed the questionnaire. Two HR professional for each of the eight university including HR head respond in this survey. HR head is indispensable to be part of this survey due to two reasons. First, technically, she/he is the head of HRM and measurement of the HLRQ requires his/her vital input. Second, the size of HR departments is small in the universities; HR heads are dealing with HODs most of the time and are fully aware about HRM of the organization.

Table 2
Number of Academic Departments and Employees

Name of University	**Academic Departments**	**Number of Employees**		
		Faculty	**Others**	**Total**
U1	9	120	156	276
U2	6	124	200	324
U3	8	062	260	322
U4	5	041	100	141
U5	6	150	540	690
U6	10	289	823	1112
U7	7	080	60	140
U8	13	110	370	480
Total	64	976	2509	3485

On the other hand, two HODs are randomly selected out of the 64 HODs, except U6 and U8 for which three HODs are selected randomly; this is due to relatively large number of academic departments in the two universities as compared to the other universities. For secrecy, universities have been renamed as U1, U2 and so on up-to U8. Table 2 presents the summary of information about the number of academic departments and number of employees in the eight universities.

The cross-sectional survey uses two instruments for the two types of respondents i.e. for HR professionals and HoDs. The questionnaires have been reviewed several times. It has been vetted or filled from experienced individuals of universities, experts of questionnaire making and fellow researchers. After distribution of the questionnaires to HR professionals and to HoDs follow up were made directly by the authors and through officials of HR departments nominated by the university, co-coordinators and volunteer teachers.

4. ANALYSIS AND INTERPRETATIONS

The response rate for HR professionals is 87.5 % and that of HODs is 66.7 %. Over all there are 78 % male respondents in the survey. The percentage of females representing HR professionals is 36 % and the percentage of females representing HODs is only 11 %. The mean working experience of HODs (7.17 years) is about twice of those of the HR Professionals (3.39 years). In this sample HODs are experienced individuals, highly qualified in their discipline, whereas HR professionals are relatively less qualified and less experienced.

Keeping in view the purpose of the survey i.e. to validate the proposed construct for measurement of HLRQ, items that show a low factor loading have to be eliminated for fine tuning of a construct. Factor analysis (principal component analysis with varimax rotation) is performed on all multiple scale items to determine item retention (Coyle-Shapiro et al., 2004). The examination of the factor loadings for each of the construct reveals that single factor emerges for most of the constructs. For the unidimensionality of each construct, this study includes appropriate items that loaded at least 0.70 on their respective component. De Wolf et al., (2001) also use similar method with the item inclusion loading level of 0.65. The minimum eigen value is 1 in all the factor analysis.

Out of 16 proposed items, there is an elimination of 1 item for the sub-dimension operational relations (Table 3). The factor loadings of each of items for the sub-dimensions *satisfaction, trust and commitment* are in the acceptable range. Out of the 22 items there is an elimination of 1 item of *HR-focus* and 2 items of *co-ordination* have acceptable loadings. The factor loading for *team partner*, c*ontribution* and *distraction* shows acceptable loadings. Therefore, 34 items now retains in the measurement construct for HLRQ.

Reliability Analysis

The reliability of an instrument is its ability to give nearly identical results in repeated measurement under identical conditions. This study is conducted on multi-point numeric scales, so the Chronnbach’s Alpha is used which is a suitable test for testing reliability of the measure. The minimum acceptable Alpha in social science is 0.70 (Hair et al., 1998). The reliability for all the scales after eliminating the items is within the acceptable range. The Alpha value for s*atisfaction, trust, commitment*, *operational relations, HR focus, co-ordination, team partner, contribution and distraction* are 0.95, 0.89, 0.82, 0.94, 0.83, 0.75, 0.88, 0.90 and 0.73 respectively.

Table 3
Factor Loadings for Sub-dimensions HR-line Relationship Quality

Dimensions	Sub-Dimension	Principal Components Analysis	Factor Loading
Relationship Quality	Satisfaction	satisfaction with the relationship	0.963
		happy with the efforts they are making in this relationship	0.963
		satisfied with their method of support	0.943
	Trust	have trustworthy impression	0.932
		trust all kinds of information being provided	0.899
		While making decisions they consider our welfare as well as their own	0.892
	Commitment	committed to develop a quality relationship with us	0.919
		feel a strong attachment	0.915
		willing "to go the extra mile" to maintain good relations	0.737
	Operational Relations	are quick to respond for operational adjustments	0.782
		Their behavior always matches with our original expectations	0.903
		Mutual conflicts are resolved amicably and fairly	0.866
		Our relations are stable	0.819
		feel secure in maintaining the relations	0.934
		These relationships is quite steady	0.893
		want changes in terms of our working relationship	*(0.489)**
Positive-ness of HR towards Line	HR Focus	are keen to take part in doing HR work	0.776
		are serious in doing HR work	0.777
		feel secure in knowing that HR experts can be called on if needed	0.818
		give HR work the priority it needs	0.881
		view work belonging to HR as an illegitimate part of their job	*0.623**
	Co-ordination	pursue objectives which are often incompatible	0.835
		act in isolation in making HR related decisions	0.864
		dislike monitoring from HR professionals	0.776
		often argue over (who or when to complete) HR duties	*0.605**
		are reluctant to approach HR for help	*0.622**
	Team Partner	are involved in a supportive relationships	0.830
		are working "as a team" with HR	0.871
		have a positive impression of the HR staff	0.851
		view HR as a business partner	0.873
		see HR staff as rigid and inflexible	0.713

Dimensions	Sub-Dimension	Principal Components Analysis	Factor Loading
Positive-ness of Line towards HR	Contribution	contribute to solve business problems	0.875
		are behaving exactly as per my expectations	0.914
		co-operate well to get the job done	0.925
		offer the necessary support and advice to tackle HR issues	0.778
	Distraction	fear of reduced influence if HR work is done by me	0.716
		only serve to create a distraction rather than add value	0.907
		do not understand the real business of the organization	0.805
	Team Partner	involved in a supportive relationships	0.830
		working "as a team" with HR	0.871
		positive towards solving problems	0.851
		behave like a business partner	0.873
		are rigid and inflexible	0.713

*** Items eliminated**

CONCLUSION

Data has been collected from multiple sources which is always better than a single informant approach. Keeping in view the unique nature of higher education sector, it is not fully justified to generalize the results for the other sectors of Pakistan. Greater numbers of respondents and survey in diversified sectors would have further strengthened this research. Given these limitations, it may be concluded that HR-line relationship quality may be measured from 34 items for the three dimensions: a) *relationship quality, b) positive-ness of HR towards line* and c) *positive-ness of line towards HR*. Twenty of the items are common for both HR professionals and line managers. These items measure four sub-dimensions (i.e. satisfaction, trust, commitment and operational relations) of *relationship quality* and one common sub-dimension (team partner) for *positive-ness* of both HR professional and line managers. Seven of the items are for HR professionals that measure two sub-dimensions (i.e. HR focus and co-ordination) of their positive-ness towards line managers. Moreover, 7 items are for line managers that measure two sub-dimensions (i.e. contribution and distraction) of their positive-ness towards HR.

According to Qadeer et al. (2011b) the nature of ownership (public or private) of a university does not make significant difference on the HRM patterns of Pakistan. And HRM in universities are more influenced by the culture of the country. Therefore, the above mentioned construct may also be used in any other university as well.

The paper contributes theoretically and empirically by further developing the concept HR-line relationship quality, positive-ness of HR towards line managers and positive-ness of line towards HR (Qadeer et al., 2011a). It identifies various dimensions and sub-dimensions of all these variables. It proposes items as first step and collects data through an empirical survey for fine-tuning of the constructs as the second step. Future research should test the construct for HLRQ in diversified sectors. The measurement and validation of generic construct HR-line relationship quality require inclusion of many HR professionals and line managers from various types of organizations. On the contrary, the diagnosis of HR-line relationship quality in a particular organization to improve the quality of relationship requires in depth analysis of that particular organization. Therefore, conducting case studies may also be helpful in future.

ACKNOWLEDGEMENTS

We thank Chris Brewster for his stimulating, incisive and detailed comments on an earlier draft of this paper. We are also grateful to the two anonymous reviewers for their valuable input.

REFERENCES

1. Anderson, S.E. and Williams, L.J. (1996). Interpersonal, job and individual factor related to helping processes at work. *Journal of Applied Psychology,* 81(6), 282-296.
2. Arino, A., Jose, T. and Ring, P.S. (2001). Relational quality: managing trust in corporate alliances. *California Management Review*, 44(1), 109-131.
3. Beatson, A., Lings, I. and Gudergan, S. (2008). Employee behaviour and relationship quality: Impact on customers. *The Service Industries Journal*, 28(2), 211-223.
4. Bond, S. and McCracken, M. (2005). The importance of training in operational sing HR policy. *Journal of European Industrial Training*, 29(3), 246-260.
5. Bond, S. and Wise, S. (2003). Family leave policies and devolution to the line. *Personnel Review*, (32)1, 58-73.
6. Casciaro, T. and Lobo, M.S. (2008).When competence is irrelevant: The role of interpersonal affect in task-related ties. *Administrative Science Quarterly*, 53(4) 655-684.
7. Chang, K. (2005). Relationship quality and negotiation interdependence: The case study of international defect claim. *Total Quality Management*, 16(7), 903-914.
8. Chell, E. and Tracey, P. (2005). Relationship building in small firms: The development of a model. *Human Relations*, 58(5), 577-616.
9. Chimhanzi, J. (2004). The Impact of integration mechanisms on marketing/HR dynamics. *Journal of Marketing Management*, 20(7/8), 713-740.
10. Coyle-Shapiro, J. A.M., Kessler, I. and Purcell, J. (2004). Exploring organizationally directed citizenship behaviour: Reciprocity or ‘it’s my job’? *Journal of Management Studies*, 41(1), 85-106.
11. De Wolf, K., Schroder, G.O. and Iacobucci, D. (2001). Investments in consumer relationships: A cross-country and cross-industry exploration. *Journal of Marketing*, 65(5), 33-50.
12. Dorenbosch, L. and Veldhoven, M. (2006). Rethinking HRM practices measurement: A classification of choices for researchers Presented at: *EIASM 21st Workshop Strategic Human Resource Management.* Aston University, Birmingham, UK.
13. Forrester, W.R. and Maute, M.F. (2001). The impact of relationship satisfaction on attributions, emotions, and behaviors following service failure. *Journal of Applied Business Research*, 17(1), 1-14.

14. Galang, M.C. (1999). Stakeholders in high-performance work systems. *International Journal of Human Resource Management*, 10(2), 287-305.
15. Game, A.M. (2008). Negative emotions in supervisory relationships: The role of relational models. *Human Relations*, 61(3), 355-393.
16. Gennard, J. and Kelly, J. (1997). The unimportance of labels: The diffusion of the personnel/HRM function. *Industrial Relations Journal*, 28(1), 27-42.
17. Gubbins, C., Garavan, T., Hogan, C. and Woodlock, M. (2006). Enhancing the role of the HRD function: The case of a health services organization. *The Irish Journal of Management*, 22(2), 171-206.
18. Hair, J.F., Anderson, R.E., Tatham, R.L. and Black, W.C. (1998). Multivariate Data Analysis. NJ: Prentice Hall.
19. Harney, B and Jordan, C. (2008). Unlocking the black box: Line managers and HRM-Performance in a call centre context International. *Journal of Productivity and Performance Management*, 57(4), 275-296.
20. Hyman, J. and Cunningham, I. (1996). Change in employee relations-can line managers deliver? *Management Research News*, 19(4/5), 35-37.
21. Johnson, J.L., Sohi, R.S. and Grewal, R. (2004). The role of relational knowledge stores in inter firm partnering. *Journal of Marketing*, 68(3), 21-36.
22. Kim, J.W., Choi, J., Qualls, W. and Park, J. (2004). The impact of CRM on firm - and relationship level performance in distribution networks. *Communications of the Association for Information Systems*, 14, 632-652.
23. Kulik, C.T. and Perry, E.L. (2008). When less is more: The effect of devolution on HR's strategic role and construed image. *Human Resource Management*, 47(3), 541-558.
24. Larsen, H.H. and Brewster, C. (2003). Line management responsibility for HRM: What is happening in Europe? *Employee Relations*, 25(3), 228-244.
25. Lawler, III, E.E. and Mohrman (2003). HR as a strategic partner, what does it take to make is happen. *Human Resource Planning*, 26(5), 15-37.
26. Legge, K. (1989). Human resource management: A critical analysis In Storey, J. (Ed.), *New Perspectives on Human Resource Management*. Routledge, London.
27. Lewicki, R.J., McAllister, D.J. and Bies, R.J. (1998). Trust and distrust: New relationships and realities. *Academy of Management Review*, 23(3), 438-458.
28. McConville, T. and Holden, L. (1999). The filling in the sandwich: Managers in the health sector. *Personnel Review*, 28(5/6), 406-424.
29. McGovern, F., Gratton, L., Hope-Hailey, V., Stiles, P. and Truss, C. (1997). Human resource management on the line? *Human Resource Management Journal*, 7(4), 12-29.
30. Mcknight, D.H., Cummings, L.L. and Chervany, N.L. (1998). Initial trust formation in new organizational relationships. *Academy of Management Review*, 23(3), 473-490.
31. Mello, J.A. (2007). *Strategic Human Resource Management*, Baba Barkha Nath Printers, New Dehli.
32. Mitsuhashi, H.J., Park, H., Wright, P.M. and Chua, R.S. (2000). Line and HR executives' perceptions of HR effectiveness in firms in the People's Republic of China. *The International Journal of Human Resource Management*, 11(2), 197-216.

33. Morrey, M.M. and Kito, M. (2009). Relational-interdependent self-construal as a predictor of relationship quality: the mediating roles of one's own behaviors and perceptions of the fulfillment of friendship functions. *The Journal of Social Psychology*, 149(3), 205-222.
34. Murphy, S.M., Wayne, S.J., Liden, R.C. and Erdogan, B. (2003). Understanding social loafing: The role of justice perceptions and exchange relationships. *Human Relations*, 56(1), 61-84.
35. Naude, P., Ashnal, B., Chaharsooghi, K. and Perzon, H. (2007). An analysis of B2B relationship quality among Iranian managers: A comparison between Iranian and English managers. *Total Quality Management*, 18(8), 861-874.
36. Papalexandris, N. and Panayotopoulou, L. (2005). Exploring the partnership between line managers and HRM in Greece. *Journal of European Industrial Training*, 29(4), 281-291.
37. Patterson, M.G., A. West, M.A., Shackleton, V.J., Dawson, J.F. Lawthom, R., Maitlis, S., Robinson, D.L. and Wallace, L.M. (2005). Validating the organizational climate measure: Links to managerial practices, productivity and innovation. *Journal of Organizational Behavior*, 26(4), 379-408.
38. Prinslloo, M., Bckstrm, L. and Sangari, E.S. (2007). The impact of incentives on interfunctional relationship quality: Views from a South African firm. *Total Quality Management and Business Excellence*, 18(8), 901-913.
39. Qadeer, F., Shafique, M. and Rehman, R. (2011a). An overview of HR-line relationship and its future directions. *African Journal of Business Management*, 5(7), 2512-2523.
40. Qadeer, F., Shafique, M., Ahmad, M. and Rehman, R. (2011b). Does ownership of higher education institute influence its HRM patterns? The case of Pakistan. *International Journal of Business and Management*, 6(10), 230-241.
41. Renwick, D. (2000). HR-line work relations: Review, pilot case and research agenda. *Employee Relations*, 22(2), 179-201.
42. Renwick, D. (2003). Line manager involvement in HRM: An inside view. *Employee Relations*, 25(3), 262-280.
43. Scandura, T.A. and Graen, G.B. (1984). Moderating effects of initial leader-member exchange status on the effects of a leadership intervention. *Journal of Applied Psychology*, 69(3), 428-436.
44. Settoon, R.P. and Mossholde, K.W. (2002). Relationship quality and relationship context as antecedents of person- and task-focused interpersonal citizenship behavior. *Journal of Applied Psychology*, 87(2), 255-267.
45. Siddique, C.M. (2004). Job analysis: A strategic human resource management practice. *The International Journal of Human Resource Management*, 15(1), 219-244.
46. Tan, H.H. and Zhao, B. (2003). Individual-and perceived contextual-level antecedents of individual technical information inquiry in organizations. *Journal of Psychology*, 137(6), 597-621.
47. Teo, S.T.T. and Rodwell, J.J. (2007). To be strategic in the new public sector, HR must remember its operational activities. *Human Resource Management*, 46(2), 265-284.

48. Truss, C. and Gratton, L. (1994). Strategic human resource management: A conceptual approach. *The International Journal of Human Resource Management,* 5(3), 663-686.
49. Ulrich, D. (1997). *Human Resource Champions*, Harvard University Press, Boston.
50. Ural, T. (2007). The antecedents and consequences of relationship quality according to stages of the relationship between exporters and importers. *Problems and Perspectives in Management*, 5(3), 111-132.

Dr. Munir Ahmad born in Sialkot, Pakistan, had early education at Murray College Sialkot, Master in Statistics from Punjab University, Lahore, and Postgraduate Diploma from Aberdeen University, Aberdeen, Scotland and PhD in Statistics from Iowa State University, Ames Iowa, USA.

He was Lecturer, Punjab University, Lahore; Associate Professor at Karachi University, Karachi; Associate Professor, Department of Mathematics, Michigan Tech. University, Houghton, USA; Associate Professor, Tripoli University, Tripoli, Libya and Professor at King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.

He joined The University of Lahore as Dean, Institute of Mathematical Sciences and later joined National College of Business Administration and Economics (NCBA&E), Lahore as Rector.

He is a Member, International Statistical Institute, Netherlands (1980 till now), Fellow, Royal Statistical Society, UK (1965 till now), Member, American Statistical Association, USA (1965-2002), Founding President and Patron, Islamic Countries Society of Statistical Sciences, Pakistan, Chief Editor, Pakistan Journal of Statistics (1985 till now), Editor, Journal of Applied Probability and Statistics (2006 till now), Editor-in-Chief, Journal of Islamic Countries Society of Statistical Sciences (2015 till now), Contributing Editor, Current Index to Statistics, American Statistical Association, USA (1989-2003), President, Pakistan Statistical Association, Karachi Chapter (1970-1972), Member, Pakistan Statistical Association, since 1954, Member, Board of Governors, National Institute of Population Studies, Government of Pakistan, (1995-1997), Member, National Statistical Council, Government of Pakistan, (2000-____), Member, Editorial Board, Nigerian Journal of Risk and Insurance (2009-____), Patron, Pakistan Journal of Commerce and Social Sciences (2009-____), Associate Editor, Chilean Journal of Statistics (2009-_____), Member, Advisory Board, Türkiye Klinikleri Journal of Biostatistics (2009) and Member, Board of Editors, Brazilian Journal of Statistics, Brazil.

He supervised M.S., M.Phil., Ph.D. and Post-Doc students in NCBA&E and various Universities.